Title: A correspondence between quantum error correcting codes and quantum reference frames

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A CORRESPONDENCE BETWEEN

QUANTUM ERROR CORRECTING CODES AND QUANTUM REFERENCE FRAMES

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QISS | THE QUANTUM INFORMATION STRUCTURE OF SPACETIME

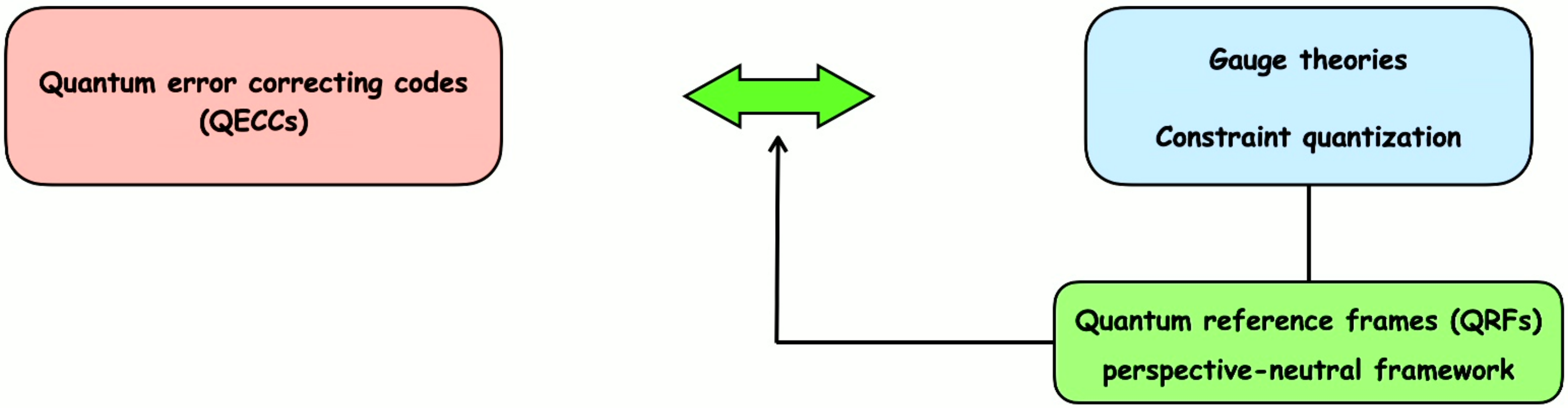
Based on work in collaboration with S. Carrozza, A. Chatwin-Davies, and P. A. Höhn

2411.xxxxx + follow up

QG Seminar
Perimeter Institute
Nov 7, 2024

A more accurate title would have been:

Quantum error correcting codes / gauge theories correspondence via quantum reference frames



QECCs have been usefully invoked in HEP and QG before

- decode information from Hawking radiation [Hayden, Preskill '07; Yoshida, Kitaev '17;...]
- holography [Almheiri, Dong, Harlow '14; Pastawski, Yoshida, Harlow, Preskil '15; Harlow '17;...]
- new angle on renormalization and coarse-graining [Furuya, Lashkari, Ouseph '20; Furuya, Lashkari, Moosa '22]

Our aim is not specific applications of QEC in HEP, but rather developing a fundamental correspondence

SIMILARITIES BETWEEN GAUGE THEORIES AND QUANTUM ERROR CORRECTING CODES

REDUNDANCY

Gauge theories describe physics redundantly, i.e. not all of the defining DoF are physically relevant (gauge symmetry does not exist without redundancy)

QECCs protect quant. info. from local errors by redundantly encoding logical states into a larger physical space (otherwise any uncontrolled interaction will result in a logical error in the computation)

MULTIPLE CHOICES OF REDUNDANT DoF

A plethora of ways to gauge fix or, equivalently, gauge-invariantly dress the bare observables

A given code can correct a plethora of errors

GAUGE TRANSFORMATIONS & STABILIZER GROUP

In gauge theories, the physical information commutes with gauge transformations

In stabilizer codes, the logical information to be protected from errors commutes with stabilizer group

Is this analogy just a coincidence? Is there a deeper structural relationship between QEC and gauge theories?

Can we in general understand gauge theories and the distribution of gauge-invariant (quantum) information in spacetime as a QECC?

Can we meaningfully think of a QECC as a gauge theory?

If so, what are the foundational and practical insights to be gained from such a correspondence?

Our aim is to start setting the stage for answering such questions by developing a **QECC/gauge theory correspondence**

Beyond the analogy ⟶ **Identify and give meaning to encodings and errors on the gauge theory side** (usually not considered)

Here is where **QRFs** come into the picture

PN-framework: choice of QRF = choice of split between redundant (the QRF) and non-redundant info

On the QEC side, this corresponds to a decoding and is connected with sets of correctable errors

QECC/QRF correspondence in a nutshell

Every QRF setup according to the perspective-neutral formulation gives rise to a QECC and, conversely, every (group-based) QECC defines a QRF setup according to the perspective-neutral framework.

In particular, every encoding is equivalent to a choice of QRF, and every maximal set of correctable errors is in one-to-one correspondence with a QRF, constituted by those DoF on which these errors act exclusively.

What may be the use?

- Quantum simulations of gauge theories (lot of interest in particle physics community, IBM roadmap,...)
- There are some initial steps in quantum simulations of quantum gravity aspects
- Code design
- Systematic foundational understanding of QEC and, conversely, of gauge theories and gravity (where QRFs necessarily arise)

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PLAN OF THE TALK

- Structure of QECCs (mainly Pauli stabilizer codes)
- Structure of perspective-neutral framework to QRFs
- Building the QECC/QRF dictionary
- Towards gauge theories: surface code with boundary (if time allows)
- Outlook

STRUCTURE OF QECCs

[Nielsen-Chuang book; Gottesman '97, '04, '09; Girvin '19; ...]

(a) **Physical space** $\mathcal{H}_{\text{physical}}$ = space of all possible states of the physical system used to incarnate the quantum info of interest

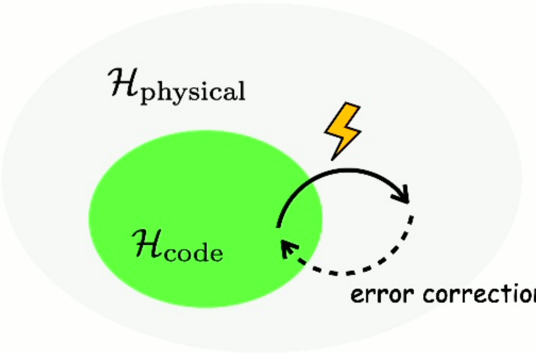
n physical qubits: $\mathcal{H}_{\text{physical}} \simeq (\mathbb{C}^2)^{\otimes n}$

(b) **Code (sub)space** $\mathcal{H}_{\text{code}}$ = space of states actually needed to incarnate the desired quantum information

Pauli stabilizer codes: invariant subspace under the stabilizer group $\mathcal{S} \simeq \mathbb{Z}_2^{\times(n-k)}$ ⇨ $\mathcal{H}_{\text{code}} \simeq (\mathbb{C}^2)^{\otimes k}$

(c) **Logical space** $\mathcal{H}_{\text{logical}}$ = abstract code/states one wishes to implement physically

k logical qubits: $\mathcal{H}_{\text{logical}} \simeq (\mathbb{C}^2)^{\otimes k}$



(d) **Error spaces** $\mathcal{H}_i \subset \mathcal{H}_{\text{physical}}$ = target spaces of errors acting on the code

STRUCTURE OF QECCs

[Nielsen-Chuang book; Gottesman '97, '04, '09; Girvin '19; ...]

$\mathcal{H}_{\text{physical}} \xrightarrow{\;\Pi_{\text{code}} = \frac{1}{|\mathcal{S}|}\prod_i (I + S_i)\;} \mathcal{H}_{\text{code}}$

decoding $\mathcal{D} = \mathcal{N}^\dagger$: $\mathcal{H}_{\text{code}} \to \mathcal{H}_{\text{logical}}$

encoding : $\mathcal{H}_{\text{logical}} \to \mathcal{H}_{\text{code}}$

$\mathcal{H}_{\text{physical}} \cup \mathcal{H}_i$

errors E_i : $\mathcal{H}_{\text{code}} \to \mathcal{H}_i$

$$\mathcal{N}(\rho_L) = N^\dagger(\rho_L \otimes |0\rangle\langle 0| \otimes \cdots \otimes |0\rangle\langle 0|)N$$

ancilla ready states

$n \times n$ unitary

Error channels $\tilde{\mathcal{E}}(\rho) = \sum_k \tilde{E}_k\, \rho\, \tilde{E}_k^\dagger$

$$\sum_k \tilde{E}_k^\dagger\, \tilde{E}_k \leq I$$

Purpose of QEC: find ways to correct errors

Error-correction operations $\mathcal{O}$ such that $\mathcal{O} \circ \tilde{\mathcal{E}} \circ \mathcal{N} = \mathcal{N}$

KL condition

[Knill-Laflamme '97]

$$\mathcal{E} = \{E_0, E_1, \ldots, E_m\} \quad \Leftrightarrow \quad \Pi_{\text{code}}\, E_i^\dagger\, E_j\, \Pi_{\text{code}} = C_{ij}\Pi_{\text{code}}$$

set of correctable errors

EXAMPLE: [[3,1]] CODE

$k = 1$ logical qubit encoded into $n = 3$ physical qubits $\mathcal{H}_{\text{code}} = \text{span}\{|000\rangle, |111\rangle\} \simeq \mathbb{C}^2 \subset (\mathbb{C}^2)^{\otimes 3} \simeq \mathcal{H}_{\text{physical}}$

Stabilizer group $\mathcal{S} = \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$ $\Pi_{\text{code}} = \frac{1}{4}(I + Z_1Z_2 + Z_2Z_3 + Z_1Z_3)$

coherent group-averaging projector

Multiple sets of correctable errors e.g. $\mathcal{E} = \{I, X_1, X_2, X_3\}$ $\mathcal{E}' = \{I, X_1, X_2, X_1X_2\}$

Error correction: Detection (measuring a pair of stabilizer generators)
(alternatively, projective measurement channel $\sum_i P_i \rho P_i$) + Recovery (act with $E_i^\dagger$)

Note: X_3 and X_1X_2 have the same syndrome measurement $\longrightarrow$ what we consequently do to recover determines the error set that we correct for

STRUCTURE OF GAUGE THEORIES/CONSTRAINT QUANTIZATION

(a) **Kinematical space** $\mathcal{H}_{\rm kin}$ with unitary representation of gauge group $\mathcal{G}$ **stabilizer group**

Physical space → Encompasses all the defining (gauge-variant and -invariant) DoF of the theory

(b) **Gauge-invariant space** $\mathcal{H}_{\rm pn}$ → Physically predictive content encoded in gauge-invariant states $U^g|\psi_{\rm pn}\rangle = |\psi_{\rm pn}\rangle \quad , \quad \forall\, g \in \mathcal{G}$

Code space → $\mathcal{H}_{\rm pn} = \mathcal{H}_1$ carries the trivial representation of $\mathcal{G}$

(c) **???** **Gauge theory counterpart of the logical space?**

What about encodings, decodings, etc.?

(d) **Non-trivially charged sectors** $\mathcal{H}_{i\neq 1}$ carrying non-trivial irreps of $\mathcal{G}$

Error spaces

PERSPECTIVE-NEUTRAL FRAMEWORK TO QRFs

[Vanrietvelde, Höhn, Giacomini, Castro-Ruiz '18;
Krumm, Höhn, Müller '20, '21; Höhn, Smith, Lock '21;
de la Hamette, Galley, Höhn, Loveridge, Müller '22;
Höhn, Kotecha, **FMM** '23; Chataigner, Höhn, Lock, **FMM** '24;
Carrozza, Höhn '21; Carrozza, Eccles, Höhn '22;
De Vuyst, Eccles, Höhn, Kirklin '24]

- Built using structures of gauge theory and constraint quantization
- QRFs arise whenever constructing observables in gauge systems
- Broader applicability than gauge theories and gravity → framework agnostic as to whether the symmetry group is an actual gauge symmetry or corresponds to some operational restriction

Aim of PN-program: develop a purely internal description, i.e. to describe the composite quantum system from within

Wipe out any external frame info → gauge-invariance = external frame-independence

Framework applies to: simulation of gauge theories (external frame = Lab, but physically meaningful states mimicking gauge-inv.)

communication scenarios without shared external frames

actual gauge theories (external frame and its transformations are fictitious)

WHAT IS AN INTERNAL QRF & WHAT IS A CHOICE OF QRF?

[Vanrietvelde, Höhn, Giacomini, Castro-Ruiz '18;
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QRF R $(\mathcal{H}_R\ ,\ U_R)$ with U_R a non-trivial (possibly projective) unitary representation of $\mathcal{G}$

Frame orientation states coherent state system $\{U_R, |g\rangle_R\}$ s.t. $U_R^g|g'\rangle_R = |gg'\rangle_R \quad , \quad \forall\, g, g' \in \mathcal{G}$ (many, for same R)

Complete QRF U_R regular action $\longrightarrow$ R can be used to completely parametrize $\mathcal{G}$ -orbits, i.e. can absorb redundancy entirely

Ideal QRF perfectly distinguishable orientation states $\langle g|g'\rangle_R = \delta_{g,g'}$ (property of seed state and representation)

Choice of QRF choice of TPS $\mathcal{H}_{\text{kin}} \simeq \mathcal{H}_R \otimes \mathcal{H}_S$ $U^g = U_R^g \otimes U_S^g$ (not necessary but enough for Pauli stabilizer codes)

↙ redundant ↘ non-redundant

Frame orientation observables covariant POVM $E_g = |g\rangle\langle g|_R \otimes I_S$

STRUCTURE OF PN FRAMEWORK TO QRFs

(a) Kinematical space $\mathcal{H}_{\mathrm{kin}}$

space of externally distinguishable states

(b) Perspective-neutral (gauge-invariant) space $\mathcal{H}_{\mathrm{pn}}$

encodes external frame-index., internally distinguishable physics prior to having chosen an internal QRF

(c) Internal QRF perspective space(s) $\mathcal{H}_{|R} = \mathcal{H}_S$

redundancy-free description of S relative to R

$$\Pi_{\mathrm{pn}} = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} U^g$$

Encoding and decodings?

- **Page-Wootters reduction**

$$\mathcal{R}_R^g = \sqrt{|\mathcal{G}|}(\langle g|_R \otimes I_S)\Pi_{\mathrm{pn}}$$

$$\mathcal{R}_R^g|\psi_{\mathrm{pn}}\rangle = U_S^g|\psi(e)\rangle_S$$

- **Trivialization (QRF disentangler)**

$$T_R = \sum_g |g\rangle\langle g|_R \otimes (U_S^g)^\dagger$$

$$T_R|\psi_{\mathrm{pn}}\rangle = |\bar{0}\rangle_R \otimes |\psi(e)\rangle_S$$

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PW encodings & trivializations

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(d) "electric" charge sectors $\mathcal{H}_{i\neq 1} = E_i(\mathcal{H}_{\text{pn}})$

external frame info reintroduced (as in QI approaches to QRFs)

(e) "magnetic" charge sectors $\mathcal{H}_g = E_g(\mathcal{H}_{\text{pn}}) = |g\rangle_R \otimes \mathcal{H}_S$

QRF-aligned spaces (as in perspectival QRF approach)

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$\mathcal{H}_R$ ↙ redundant; $\mathcal{H}_S$ ↘ non-redundant

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QECC interpretation via Pontryagin duality

gauge group	$\mathcal{G}$	$\leftrightarrow$	$\hat{\mathcal{G}}$	Pontryagin dual
gauge transformation	U^g	$\leftrightarrow$	$\hat{U}^i$	dual group transformation
electric charge label	$i \in \hat{\mathcal{G}}$	$\leftrightarrow$	$g \in \mathcal{G}$	magnetic charge label
electric excitation	E_i	$\leftrightarrow$	$\tilde{U}_g$	magnetic excitation
	$P_i = \|i\rangle\langle i\|$	$\leftrightarrow$	$E_g = \|g\rangle\langle g\|_R \otimes I_S$	

CHOICE OF QRF = PARTITION BETWEEN REDUNDANT & NON-REDUNDANT INFO

HOW DO WE CHOOSE IT FROM THE QECC SIDE?

Consider a maximal set of correctable errors for the 3-qubit code: $\mathcal{E} = \{E_0, E_1, E_2, E_3\} = \{I, X_1, X_2, X_1 X_2\}$

Algebra $\mathcal{A}_R = \text{Span}_{\mathbb{C}}\{E_i \Pi_{\text{pn}} E_j, \forall\, i, j = 0, \ldots, 3\}$ isomorphic to $\mathcal{L}((\mathbb{C}^2)^{\otimes 2}) \otimes I_{2\times 2}$ and contains stabilizer elements

$\mathcal{A}_R$, $\mathcal{A}'_R$ generate a tensor factorization $\mathcal{H}_{\text{kin}} \simeq \mathcal{H}_R \otimes \mathcal{H}_S$ non-locally related to the original 3 qubit partition (trivialization map)

In this TPS, the code space restriction of the errors and the stabilizer elements act non-trivially only on the redundant R factor

Error-set/QRF correspondence

maximal set of correctable errors	$\mathcal{E} = \{E_i \Pi_{\text{code}}\}$		$\mathcal{A}_R = \mathcal{L}(\mathcal{H}_R)$	QRF algebra
choice of correctable error set	$\mathcal{E}$	$\longleftrightarrow$	$\mathcal{H}_R, \mathcal{A}_R$	choice of internal QRF
			s.t. $\mathcal{H}_{\text{kin}} \simeq \mathcal{H}_R \otimes \mathcal{H}_S$	subsystem (and complementary system S)

QRF-reinterpretation of KL condition

error set $\mathcal{E} = \{E_0, \ldots, E_m\}$ correctable
(not necessarily maximal)

$\longleftrightarrow$

$\exists$ subsystem partition $\mathcal{H}_{\text{kin}} \simeq \mathcal{H}_R \otimes \mathcal{H}_S$
s.t. both $\mathcal{G}$ and $E_i \Pi_{\text{pn}}$ act only on the frame

OUTLOOK

So far, simple math but a beautiful story is starting to unfold.

Plenty of room for generalizations and potential applications:

- More general groups & frames (encompassed by PN framework)
- Continuous groups, non-Abelian,… more studied on the gauge theory-QRF side than on the QECC side
- Study distribution of gauge-invariant info & algebras in spacetime regions (edge modes reference frames)
- QEC viewpoint on coarse-graining gauge-invariant information via gluing of corner data
- QEC may give a new angle on dealing with frame-system interactions
- Use of QEC in QG beyond holography (e.g. bulk-to-boundary maps in LQG as QECC? and vice versa? coarse-graining in QG,…)

THANK YOU FOR YOUR ATTENTION !