Title: Miura operators as R-matrices from M-brane intersections

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Abstract:

In this talk, I will discuss how M2-M5 intersections in a twisted M-theory background yield the R-matrices of the quantum toroidal algebra of gl(1). These R-matrices are identified with the Miura operators for the q-deformed W- and Y-algebras. Additionally, I will show how the M2-M5 intersection (or equivalently, the Miura operator) generates the qq-characters of the 5d N=1 gauge theory, offering new insight into the algebraic meaning of the latter.

Miura Operators as R-matrices from M-brane Intersections

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Mathematical Physics Seminar, Perimeter Institute

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Based on 2407.15990 w/ N. Haouzi,
2408.15712 w/ N. Ishtiaque and Y. Zhou

Miura transformation for W-algebra

- The W-algebras of type gl(N) admit free-field realization, by Miura transformation.
- Basic building block: Miura operator $R_i = \varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_i(z)$ $\quad (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0)$

  $J_i(z)$: i-th $\widehat{\mathfrak{gl}}(1)$ current $\quad J_i(z) J_j(w) \sim \frac{-\frac{1}{\varepsilon_1 \varepsilon_2}}{(z-w)^2} \delta_{i,j}$

- The Miura transformation is obtained by taking the product of Miura operators. The coefficients of the differentials generate the W-algebra.

$$\underbrace{(\varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_N(z))}_{R_N} \cdots \underbrace{(\varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_1(z))}_{R_1} = (\varepsilon_3 \partial_z)^N + U_1(z)(\varepsilon_3 \partial_z)^{N-1} + \cdots + U_N(z)$$

Spin chain monodromy matrix?

(Product of R-matrices under coproduct operations)

Maulik-Okounkov R-matrix

[Maulik, Okounkov 12]

- The Miura transformation does not depend on the ordering; a change of ordering results in isomorphic W-algebra.
- This is because of the Maulik-Okounkov R-matrix $R_{\mathrm{MO}} \in \widehat{\mathfrak{gl}}(1) \widehat{\otimes} \widehat{\mathfrak{gl}}(1)$, satisfying

$$R_{\mathrm{MO}} \left(\varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_2(z)\right) \left(\varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_1(z)\right) = \left(\varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_1(z)\right) \left(\varepsilon_3 \partial_z - \varepsilon_1 \varepsilon_2 J_2(z)\right) R_{\mathrm{MO}}$$

$$R_{\mathrm{MO}} R_2 R_1 = R_1 R_2 R_{\mathrm{MO}}$$

Yang-Baxter equation?

- There is a good amount of understanding on the case just introduced [Gaiotto, Rapčák 20] [Ishtiaque, SJ, Zhou 24], but let me focus on a "q-deformation" of the problem [Haouzi, SJ 24] cf. [Harada, Matsuo, Noshita, Watanabe 21]

- In the twisted M-theory setup [Costello 16], the change amounts to:

$$\begin{cases} \mathbb{C} \times \mathbb{C} & \qquad Y_1(\widehat{\mathfrak{gl}}(1)) \\ \mathbb{C} \times \mathbb{C}^\times & \qquad Y(\widehat{\mathfrak{gl}}(1)) \\ \mathbb{C}^\times \times \mathbb{C}^\times & \qquad U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \end{cases}$$

cf. [Costello 17], [Gaiotto, Oh 19], [Gaiotto, Abajian 20], [Gaiotto, Rapčák 20]

Twisted M-theory [Costello 16]

Consider the M-theory on the 11-dimensional worldvolume $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}^2_{\varepsilon_3} \times \mathbb{R}_t \times \mathbb{C}^\times_X \times \mathbb{C}^\times_Z$ with

$(\varepsilon_3 = -\varepsilon_1 - \varepsilon_2)$

- Flat metric
- Ω-background $\mathcal{Q}^2_{\varepsilon_1,\varepsilon_2} = \varepsilon_1 \mathcal{L}_{V_1} + \varepsilon_2 \mathcal{L}_{V_2}$

$$C^{(3)} = (\varepsilon_1 V_1^\flat + \varepsilon_2 V_2^\flat) \wedge \frac{d\bar{X}}{\bar{X}} \wedge \frac{d\bar{Z}}{\bar{Z}}$$

$$V_1 = \partial_{\varphi_1} - \partial_{\varphi_3}$$
$$V_2 = \partial_{\varphi_2} - \partial_{\varphi_3}$$

- When $\varepsilon_1 = \varepsilon_2 = 0$, $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ is topological and $\mathbb{C}^\times_X \times \mathbb{C}^\times_Z$ is holomorphic in the cohomology of $\mathcal{Q}_{\varepsilon_1=\varepsilon_2=0}$

- Turning on $\varepsilon_1, \varepsilon_2 \neq 0$, the 11-dimensional supergravity localizes to the fixed point of the isometry, yielding the 5d $\mathfrak{gl}(1)$ Chern-Simons theory on $\mathbb{R}_t \times \mathbb{C}^\times_X \times \mathbb{C}^\times_Z$ as the effective theory. $(XZ = q_2 ZX)$

$$S = \frac{1}{\varepsilon_1} \int \frac{dX}{X} \wedge \frac{dZ}{Z} \wedge \left[\frac{1}{2} A \star_{\varepsilon_2} dA + \frac{1}{3} A \star_{\varepsilon_2} A \star_{\varepsilon_2} A\right]$$

$$f \star_{\varepsilon_2} g = \sum_{l=0}^{\infty} \frac{\varepsilon_2^l}{2^l l!} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \cdots \epsilon_{i_l j_l} \left(\partial_{z_{i_1}} \cdots \partial_{z_{i_l}} f\right) \wedge \left(\partial_{z_{j_1}} \cdots \partial_{z_{j_l}} g\right)$$

$$z_1 = \log X, \ z_2 = \log Z$$

- Triality of exchanging topological planes, seemingly broken in the 5d CS

M-branes and line/surface defects

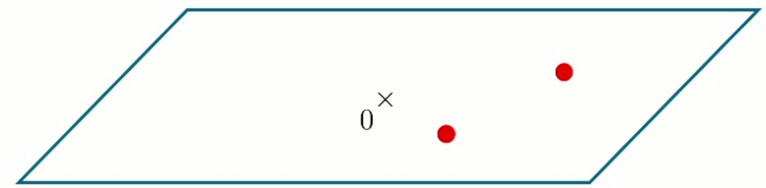
- M2-branes can be supported on $\mathbb{R}^2_{\varepsilon_a} \times \mathbb{R}_t$
- M5-branes can be supported on $\mathbb{R}^2_{\varepsilon_{c+1}} \times \mathbb{R}^2_{\varepsilon_{c-1}} \times C^{(\mathbf{p},\mathbf{q})}$ $\qquad C^{(\mathbf{p},\mathbf{q})} = \{X^{\mathbf{q}} Z^{-\mathbf{p}} = \text{const}\} \subset \mathbb{C}^\times_X \times \mathbb{C}^\times_Z$

	$\mathbb{R}^2_{\varepsilon_1}$		$\mathbb{R}^2_{\varepsilon_2}$		$\mathbb{R}^2_{\varepsilon_3}$		$\mathbb{R}_t$ (5d CS)	$\mathbb{C}^\times_X$ (5d CS)		$\mathbb{C}^\times_Z$ (5d CS)	
M-branes	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{M2}_{1,0,0}$	x	x					x				
$\mathrm{M2}_{0,1,0}$			x	x			x				
$\mathrm{M2}_{0,0,1}$					x	x	x				
$\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,1}$	x	x	x	x				x	x		
$\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,1,0}$	x	x			x	x		x	x		
$\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{1,0,0}$			x	x	x	x		x	x		

line defect $\mathcal{L}_{l,m,n}$

surface defect $\mathcal{S}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$

Topological line defect $\mathcal{L}_{l,m,n}$



Holomorphic surface defect $\mathcal{S}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$

- The local operators on the line defect form a non-commutative associative algebra by their OPEs. We denote this algebra by $\mathrm{M2}_{l,m,n}$

- The local operators on the surface defect form a chiral algebra by their OPEs. We denote its mode algebra by $\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$

M-branes and line/surface defects

- M2-branes can be supported on $\mathbb{R}^2_{\varepsilon_a} \times \mathbb{R}_t$
- M5-branes can be supported on $\mathbb{R}^2_{\varepsilon_{c+1}} \times \mathbb{R}^2_{\varepsilon_{c-1}} \times C^{(\mathbf{p},\mathbf{q})}$ $\qquad C^{(\mathbf{p},\mathbf{q})} = \{X^{\mathbf{q}} Z^{-\mathbf{p}} = \text{const}\} \subset \mathbb{C}^\times_X \times \mathbb{C}^\times_Z$

5d CS ($\mathbb{R}_t$, $\mathbb{C}^\times_X$, $\mathbb{C}^\times_Z$)

	$\mathbb{R}^2_{\varepsilon_1}$		$\mathbb{R}^2_{\varepsilon_2}$		$\mathbb{R}^2_{\varepsilon_3}$		$\mathbb{R}_t$	$\mathbb{C}^\times_X$		$\mathbb{C}^\times_Z$	
M-branes	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{M2}_{1,0,0}$	x	x					x				
$\mathrm{M2}_{0,1,0}$			x	x			x				
$\mathrm{M2}_{0,0,1}$					x	x	x				
$\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,1}$	x	x	x	x				x	x		
$\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,1,0}$	x	x			x	x		x	x		
$\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{1,0,0}$			x	x	x	x		x	x		

line defect $\mathcal{L}_{l,m,n}$

surface defect $\mathcal{S}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$

- The gauge-invariance of the coupling requires the algebra of local operators on the line/surface defects to be a representation of some universal associative algebra (Koszul duality). [Costello 13]

  (for the surface defects, we only consider the mode algebra, not the (quantum) vertex algebra)

- The suggestion is that, in our case, this universal associative algebra is the quantum toroidal algebra of gl(1):

Quantum torus algebra: $XZ = q_2 ZX$

$$U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) = \text{“}U_{q_1}(\mathscr{O}_{q_2}(\mathbb{C}^\times \times \mathbb{C}^\times))\text{”}$$

$(q_1 q_2 q_3 = 1)$

$q_a = e^{\varepsilon_a}$

- For the line defects, $\rho_{\mathrm{M2}_{l,m,n}} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M2}_{l,m,n}$
- For the surface defects, $\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$

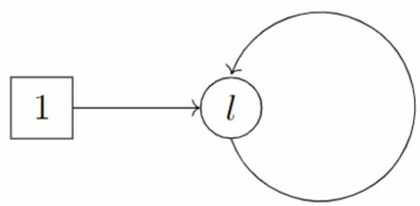
- In principle, the defining relations should be verified through the perturbative analysis of the 5d CS theory.

cf. [Costello, Witten, Yamazaki 17],
[Costello 17], [Costello, Paquette 20], [Oh, Zhou 20],
[Paquette, Williams 24] ,
[Ishtiaque, SJ, Zhou 24]

Line defects

- l parallel M2-brane supported on $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}_t$ $\xrightarrow[\mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}^2_{\varepsilon_3} \simeq \mathrm{TN}]{\text{IIA reduction}}$ l D2 on $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}_t$
  1 D6 on $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}_t \times \mathbb{C}^\times_X \times \mathbb{C}^\times_Z$

- At $\varepsilon_1 = 0$, the effective 3d N=4 theory on the D2-branes flows to the topological sigma model with the target space given by the Higgs branch.



$$A, B \in \mathrm{End}(\mathbb{C}^l)$$
$$I \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^l)$$
$$J \in \mathrm{Hom}(\mathbb{C}^l, \mathbb{C})$$

[Gorsky, Nekrasov, Rubtsov 99],
[Kapustin, Kuznetsov, Orlov 00],
[Jordan 14]

- The Higgs branch is the multiplicative quiver variety associated to the Jordan quiver:

$$\mathcal{M}^{(l)}_{q_2} = \{AB - q_2 BA = IJ\}/GL(l)$$

$$(A, B, I, J) \mapsto (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}), \qquad g \in GL(l).$$

(This is the phase space of the trigonometric Ruijsenaars-Schneider model of type gl(l))

- Thus, the algebra $\mathrm{M2}^{q_1=1}_{l,0,0}$ of local operators is the commutative algebra of holomorphic functions on $\mathcal{M}^{(l)}_{q_2}$

- By turning on $q_1 \neq 1$, the 3d theory localizes to the fixed locus $\{0\} \times \mathbb{R}_t$. The algebra of local operators gets non-commutative deformation. [Yagi 14]

- This quantized algebra $\mathrm{M2}_{l,0,0}$ is called spherical DAHA of type gl(l) $\mathbf{S\ddot{H}}^{(l)}_{q_1,q_2}$, generated by the difference operators (Macdonald representation), [Jordan 14], [Wen 23]

$$P^{(m)}_{0,r} = q_1^r \sum_{i=1}^{m} X_i^r, \qquad P^{(m)}_{0,-r} = \sum_{i=1}^{m} X_i^{-r}, \qquad r \in \mathbb{Z}_{>0}$$

$$P^{(m)}_{1,k} = q_1 \sum_{i=1}^{m} \left(\prod_{j \neq i} \frac{q_2 X_i - X_j}{X_i - X_j} \right) X_i^k q_1^{-D_{X_i}}$$

$$P^{(m)}_{-1,k} = \sum_{i=1}^{m} \left(\prod_{j \neq i} \frac{q_2^{-1} X_i - X_j}{X_i - X_j} \right) q_1^{D_{X_i}} X_i^k, \qquad k \in \mathbb{Z}$$

$$D_{X_1} = X_1 \partial_{X_1}$$

$$q_1^{D_{X_1}} f(X_1) = f(q_1 X_1) q_1^{D_{X_1}}$$

- $l = 1$: Quantum torus algebra, generated by $X_1^{\pm 1},\ q_1^{\pm D_{X_1}}$

Quantum toroidal algebra of gl(1)

- At each $l \in \mathbb{Z}_{\geq 0}$, there is a surjective algebra homomorphism [Schiffmann, Vasserot 11]

$$\rho_{\mathrm{M2}_{l,0,0}} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M2}_{l,0,0}$$

- In this sense, the quantum toroidal algebra of gl(1) is the "large-N" limit of the M2-brane algebra (i.e., spherical DAHA).

- Moreover, it is explicitly triality-invariant. Therefore, a natural guess is that it is the sought-after universal associative algebra.

Quantum toroidal algebra of gl(1) $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$

$(q_1 q_2 q_3 = 1)$

- Generating currents:

$$E(X) = \sum_{k\in\mathbb{Z}} E_k X^{-k}, \quad F(X) = \sum_{k\in\mathbb{Z}} F_k X^{-k}, \quad K^{\pm}(X) = (C^{\perp})^{\mp 1} \exp\left(\pm \sum_{r=1}^{\infty} \frac{\kappa_r}{r} H_{\pm r} X^{\mp r} \right).$$

$C, C^{\perp}$: central

- Relations:

$$K^{\pm}(X)K^{\pm}(X') = K^{\pm}(X')K^{\pm}(X)$$

$$\frac{g(C^{-1}X, X')}{g(CX, X')} K^{-}(X)K^{+}(X') = \frac{g(X', C^{-1}X)}{g(X', CX)} K^{+}(X')K^{-}(X)$$

$$g(X,X')E(X)E(X') + g(X',X)E(X')E(X) = 0$$

$$g(X',X)F(X)F(X') + g(X,X')F(X')F(X) = 0$$

$$g(X,X')K^{\pm}(C^{(1\mp1)/2}X)E(X') + g(X',X)E(X')K^{\pm}(C^{(1\mp1)/2}X) = 0$$

$$g(X',X)K^{\pm}(C^{(1\pm1)/2}X)F(X') + g(X,X')F(X')K^{\pm}(C^{(1\pm1)/2}X) = 0$$

$$[E(X), F(X')] = \tilde{g}\left\{ \delta\left(C\frac{X'}{X}\right) K^{+}(X) - \delta\left(C\frac{X}{X'}\right) K^{-}(X') \right\},$$

$$0 = \mathrm{Sym}_{i_1,i_2,i_3}[E_{i_1}, [E_{i_2+1}, E_{i_3-1}]]$$

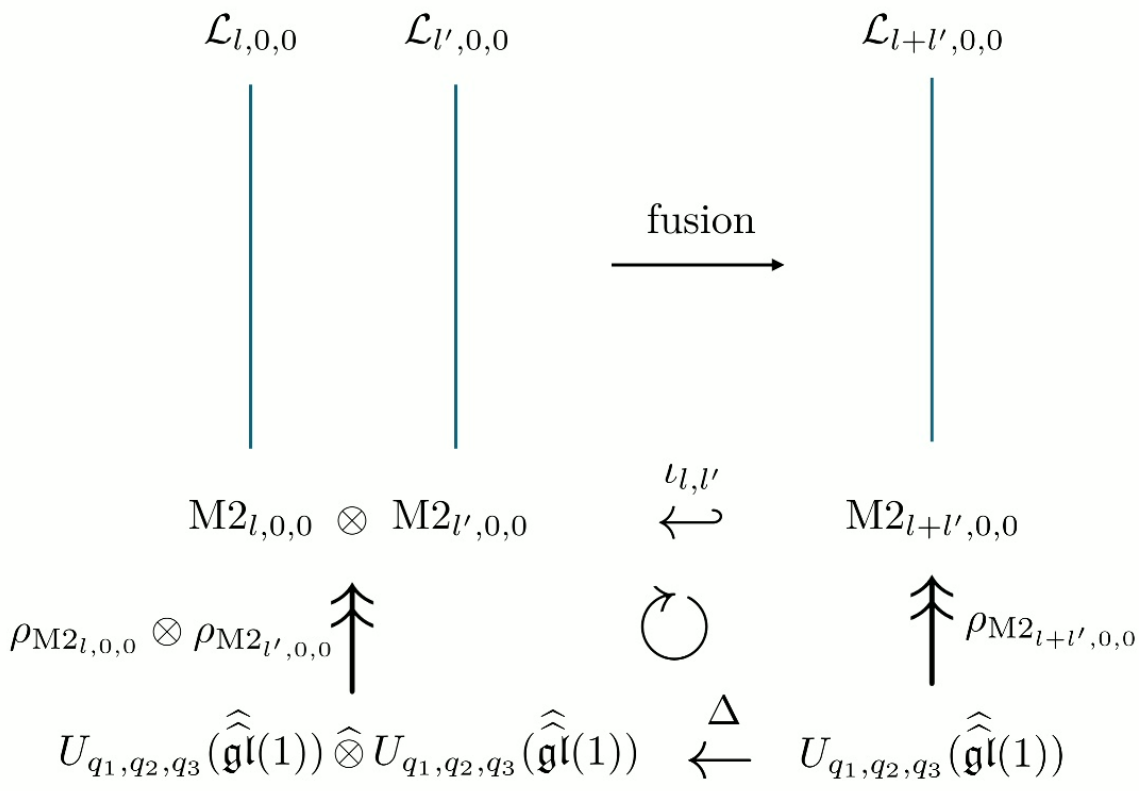
$$0 = \mathrm{Sym}_{i_1,i_2,i_3}[F_{i_1}, [F_{i_2+1}, F_{i_3-1}]].$$

$$g(X, X') = \prod_{a=1}^{3}(X - q_a X')$$

Triality of exchanging (q_1, q_2, q_3)

Defect fusion and coproduct

- Two line defects can be fused together to become single line defect:



- In the most generic case, we have non-parallel M2-branes:

  l M2-branes supported on $\mathbb{R}^2_{\varepsilon_1} \times \mathbb{R}_t$

  m M2-branes supported on $\mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}_t$ producing the line defect $\mathcal{L}_{l,m,n}$

  n M2-branes supported on $\mathbb{R}^2_{\varepsilon_3} \times \mathbb{R}_t$

- We can reconstruct the algebra $\mathrm{M2}_{l,m,n}$ by fusing three line defects:

$$\rho_{\mathrm{M2}_{l,m,n}} = (\rho_{\mathrm{M2}_{l,0,0}} \otimes \rho_{\mathrm{M2}_{0,m,0}} \otimes \rho_{\mathrm{M2}_{0,0,n}})\Delta^2 : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M2}_{l,m,n}$$

generalized Macdonald representation

$$
\begin{aligned}
E_k \mapsto\ & \frac{1}{1-q_1}\sum_{i=1}^{l}\prod_{\substack{j=1\\ j\neq i}}^{l}\frac{q_2X_i-X_j}{X_i-X_j}\prod_{j=1}^{m}\frac{q_1X_i-X'_j}{X_i-X'_j}\prod_{j=1}^{n}\frac{q_1X_i-X''_j}{X_i-X''_j}X_i^k q_1^{-D_{X_i}}\\
&+\frac{1}{1-q_2}\sum_{i=1}^{m}\prod_{j=1}^{l}\frac{q_2X'_i-X_j}{X'_i-X_j}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{q_3X'_i-X'_j}{X'_i-X'_j}\prod_{j=1}^{n}\frac{q_2X'_i-X''_j}{X'_i-X''_j}(X'_i)^k q_2^{-D_{X'_i}}\\
&+\frac{1}{1-q_3}\sum_{i=1}^{n}\prod_{j=1}^{l}\frac{q_3X''_i-X_j}{X''_i-X_j}\prod_{j=1}^{m}\frac{q_3X''_i-X'_j}{X''_i-X'_j}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{q_1X''_i-X''_j}{X''_i-X''_j}(X''_i)^k q_3^{-D_{X''_i}}
\qquad\qquad k\in\mathbb{Z},\\
F_k \mapsto\ & \frac{1}{1-q_1^{-1}}\sum_{i=1}^{l}\prod_{\substack{j=1\\ j\neq i}}^{l}\frac{q_2^{-1}X_i-X_j}{X_i-X_j}\prod_{j=1}^{m}\frac{q_2^{-1}X_i-X'_j}{q_1q_2^{-1}X_i-X'_j}\prod_{j=1}^{n}\frac{q_3^{-1}X_i-X''_j}{q_1q_3^{-1}X_i-X''_j}q_1^{D_{X_i}}X_i^k\\
&+\frac{1}{1-q_2^{-1}}\sum_{i=1}^{m}\prod_{j=1}^{l}\frac{q_1^{-1}X'_i-X_j}{q_1^{-1}q_2X'_i-X_j}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{q_3^{-1}X'_i-X'_j}{X'_i-X'_j}\prod_{j=1}^{n}\frac{q_3^{-1}X'_i-X''_j}{q_2q_3^{-1}X'_i-X''_j}q_2^{D_{X'_i}}(X'_i)^k\\
&+\frac{1}{1-q_3^{-1}}\sum_{i=1}^{n}\prod_{j=1}^{l}\frac{q_1^{-1}X''_i-X_j}{q_1^{-1}q_3X''_i-X_j}\prod_{j=1}^{m}\frac{q_2^{-1}X''_i-X'_j}{q_2^{-1}q_3X''_i-X'_j}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{q_1^{-1}X''_i-X''_j}{X''_i-X''_j}q_3^{D_{X''_i}}(X''_i)^k,\\
H_{\pm r}\mapsto\ & \frac{1}{1-q_1^{\mp r}}\sum_{i=1}^{l}X_i^{\pm r}+\frac{1}{1-q_2^{\mp r}}\sum_{i=1}^{m}(X'_i)^{\pm r}+\frac{1}{1-q_3^{\mp r}}\sum_{i=1}^{n}(X''_i)^{\pm r},\qquad r\in\mathbb{Z}_{>0},\\
C\mapsto\ & 1,\quad C^{\perp}\mapsto 1.
\end{aligned}
$$

Surface defects

- Single M5-brane supported on $\mathbb{R}^2_{\varepsilon_{c+1}} \times \mathbb{R}^2_{\varepsilon_{c-1}} \times \mathbb{C}^\times_X \xrightarrow{\text{IIA reduction}}$ 1 D4 on $\mathbb{R}^2_{\varepsilon_{c+1}} \times \mathbb{R}^2_{\varepsilon_{c-1}} \times \mathbb{R}_x$

- D4-D4 open string: 5d N=1 U(1) gauge theory compactified on a circle

- The Hilbert space is the equivariant K-theory of the moduli space of U(1) instantons, on which the algebra $\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}$ acts on.

- It is known that this space admits an action of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$ through the free q-boson algebra.

[Schiffmann, Vasserot 09]
[Feigin, Tsymbaliuk 09]

$$\left(\kappa_r = \prod_{a=1}^{3}(q_a^{r/2} - q_a^{-r/2})\right)$$

- Single M5-brane supported on $\mathbb{R}^2_{\varepsilon_{c+1}} \times \mathbb{R}^2_{\varepsilon_{c-1}} \times \mathbb{C}^\times_X$: a free q-boson $[a_r^{(c)}, a_s^{(c)}] = -\delta_{r+s,0}\frac{r}{\kappa_r}(q_c^{r/2} - q_c^{-r/2})^3.$

$$\rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}$$

$$E(X) \mapsto -e^{a_0^{(c)}\frac{\log q_{c+1}\log q_{c-1}}{\log q_c}}\frac{1-q_c}{\kappa_1}\eta_c(X)$$

$$F(X) \mapsto e^{-a_0^{(c)}\frac{\log q_{c+1}\log q_{c-1}}{\log q_c}}\frac{1-q_c^{-1}}{\kappa_1}\xi_c(X)$$

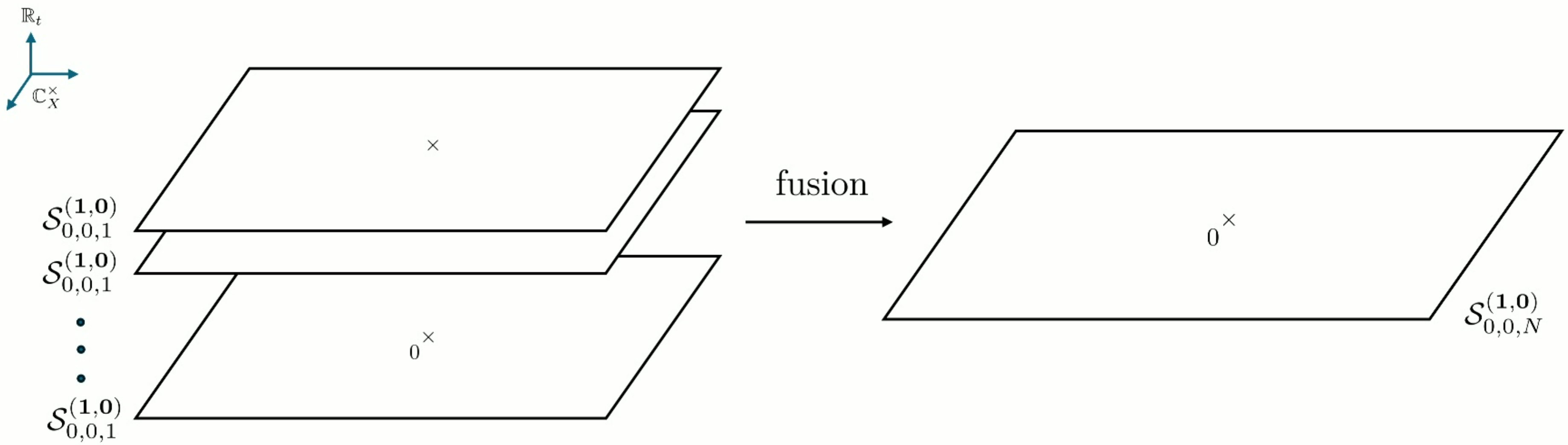
$$K^\pm(X) \mapsto \varphi_c^\pm(X)$$

$$(C, C^\perp) \mapsto (q_c^{1/2}, 1)$$

$$\eta_c(X) = \exp\left(-\sum_{r=1}^{\infty}\frac{\kappa_r}{r}\frac{1}{(q_c^{r/2}-q_c^{-r/2})^2}a_{-r}^{(c)}X^r\right)\exp\left(-\sum_{r=1}^{\infty}\frac{\kappa_r}{r}\frac{q_c^{-r/2}}{(q_c^{r/2}-q_c^{-r/2})^2}a_r^{(c)}X^{-r}\right) \qquad \varphi_c^\pm(X) = \exp\left(\sum_{r=1}^{\infty}\frac{\kappa_r}{r}\frac{a_{\pm r}^{(c)}}{q_c^{r/2}-q_c^{-r/2}}X^{\mp r}\right)$$

$$\xi_c(X) = \exp\left(\sum_{r=1}^{\infty}\frac{\kappa_r}{r}\frac{q_c^{r/2}}{(q_c^{r/2}-q_c^{-r/2})^2}a_{-r}^{(c)}X^r\right)\exp\left(\sum_{r=1}^{\infty}\frac{\kappa_r}{r}\frac{1}{(q_c^{r/2}-q_c^{-r/2})^2}a_r^{(c)}X^{-r}\right).$$

- Multiple parallel surface defects can be fused together into single surface defect.

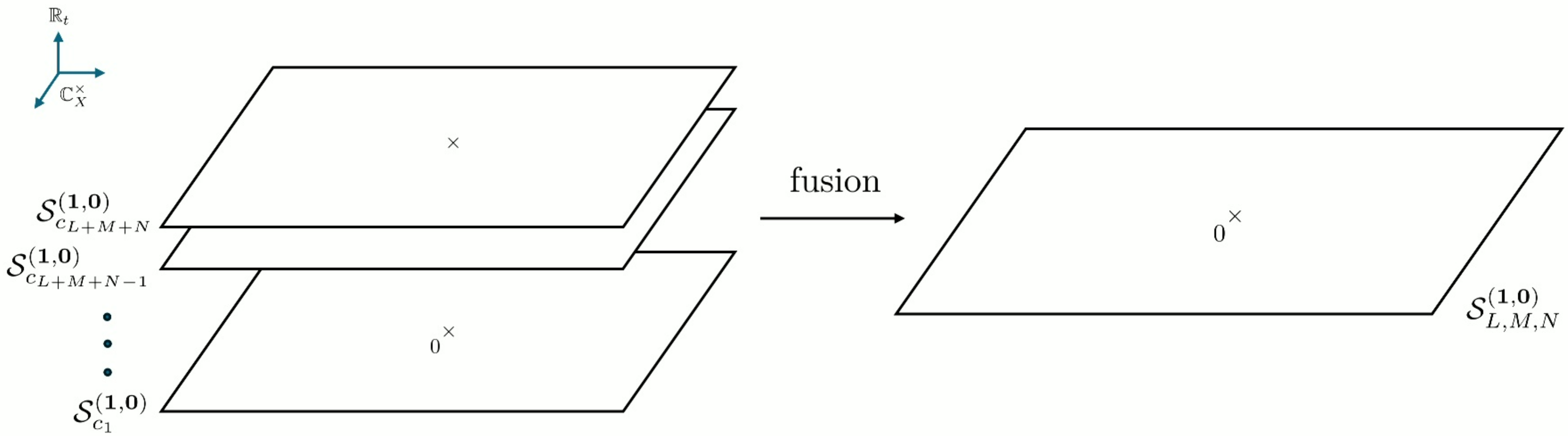


$$\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,N}} = \left(\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,1}} \otimes \cdots \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,1}}\right) \Delta^{N-1} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,N}$$

q-deformed W-algebra

cf. [Frenkel, Reshetikhin 95]
[Shiraishi, Kubo, Awata, Odake 95]

- Multiple non-parallel surface defects can be fused together into single surface defect.

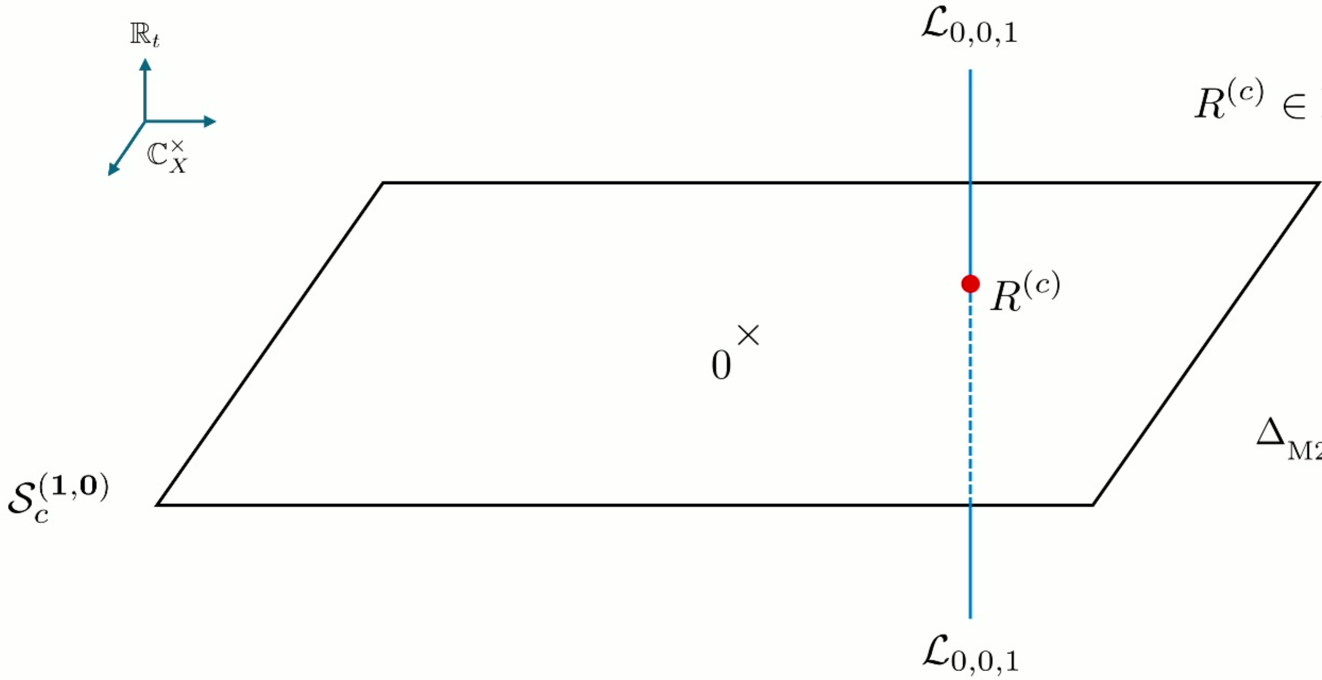


$$\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}} = \left(\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{L+M+N}}} \otimes \cdots \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_1}}\right) \Delta^{L+M+N-1} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$$

q-deformed Y-algebra

q-deformation of [Gaiotto, Rapčák 18]

M2-M5 intersections and R-matrices



$$R^{(c)} \in \mathrm{M2}_{0,0,1} \widehat{\otimes} \mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}$$

$$\Delta_{\mathrm{M2}_{0,0,1},\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}} := \left(\rho_{\mathrm{M2}_{0,0,1}} \otimes \rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}\right) \Delta$$

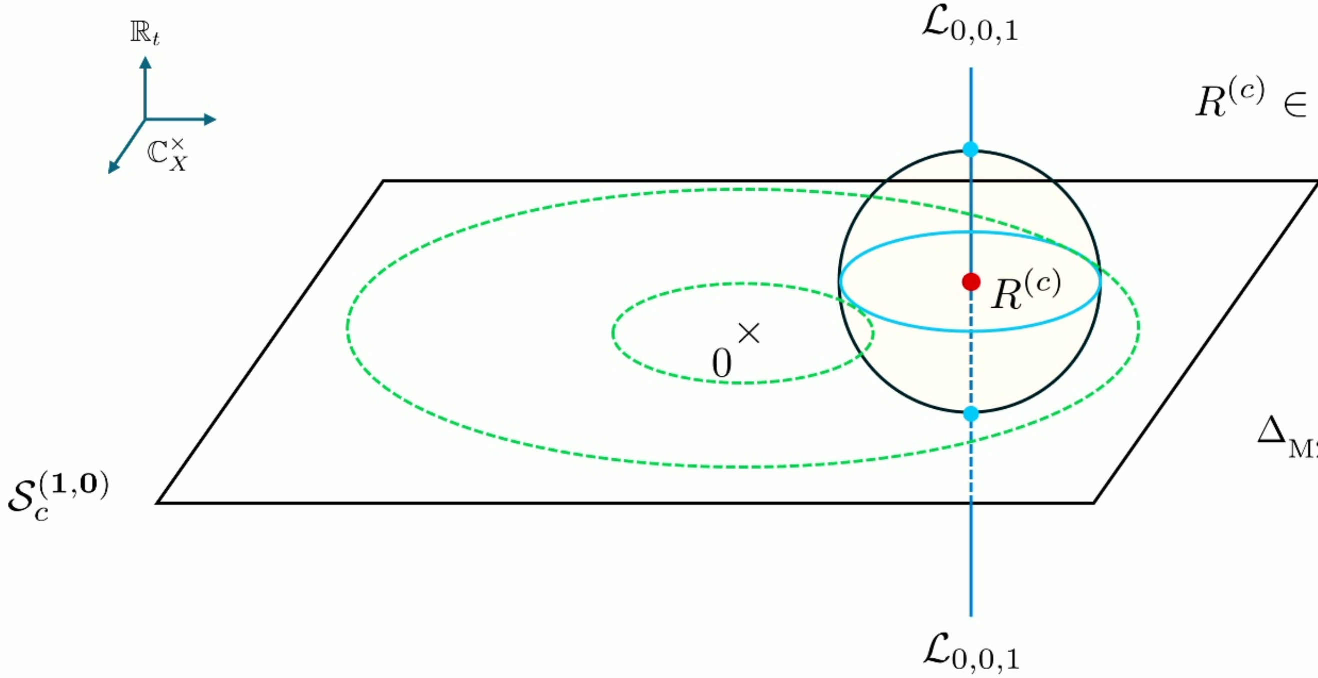
$$\Delta^{\mathrm{op}} = \sigma \circ \Delta$$

$$\sigma(g_1 \otimes g_2) = g_2 \otimes g_1$$

Classically, $R^{(c)} \left(\rho_{\mathrm{M2}_{0,0,1}}(g) \otimes 1 + 1 \otimes \rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}(g)\right) = \left(\rho_{\mathrm{M2}_{0,0,1}}(g) \otimes 1 + 1 \otimes \rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}(g)\right) R^{(c)}$

Quantum mechanically, $R^{(c)} \Delta_{\mathrm{M2}_{0,0,1},\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}(g) = \Delta^{\mathrm{op}}_{\mathrm{M2}_{0,0,1},\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}(g) R^{(c)}$

M2-M5 intersections and R-matrices



$$R^{(c)} \in \mathrm{M2}_{0,0,1} \widehat{\otimes} \mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}$$

$$\Delta_{\mathrm{M2}_{0,0,1},\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}} := \left(\rho_{\mathrm{M2}_{0,0,1}} \otimes \rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}\right) \Delta$$

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Quantum mechanically, $R^{(c)} \Delta_{\mathrm{M2}_{0,0,1},\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}(g) = \Delta^{\mathrm{op}}_{\mathrm{M2}_{0,0,1},\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}(g) R^{(c)}$

From universal R-matrix to Miura operators

- The quantum toroidal algebra of gl(1) has the universal R-matrix: $\mathcal{R} \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \,\widehat{\otimes}\, U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$

$$\mathcal{R}\Delta(g) = \Delta^{\mathrm{op}}(g)\mathcal{R}, \qquad \text{for any } g \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)),$$
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$$
$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}.$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

- Using the Drinfeld double construction, it can be expressed as a product [Garbali, Neguț 21]

$$\mathcal{R} = \bar{\mathcal{R}}\mathcal{K},$$
$$\mathcal{K} = e^{\log C \otimes d + d \otimes \log C + \log C^{\perp} \otimes d^{\perp} + d^{\perp} \otimes \log C^{\perp}} \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r}{r} H_{-r} \otimes H_r\right]$$
$$\bar{\mathcal{R}} = 1 \otimes 1 + \kappa_1 \sum_{k\in\mathbb{Z}} E_k \otimes F_{-k} + \cdots$$

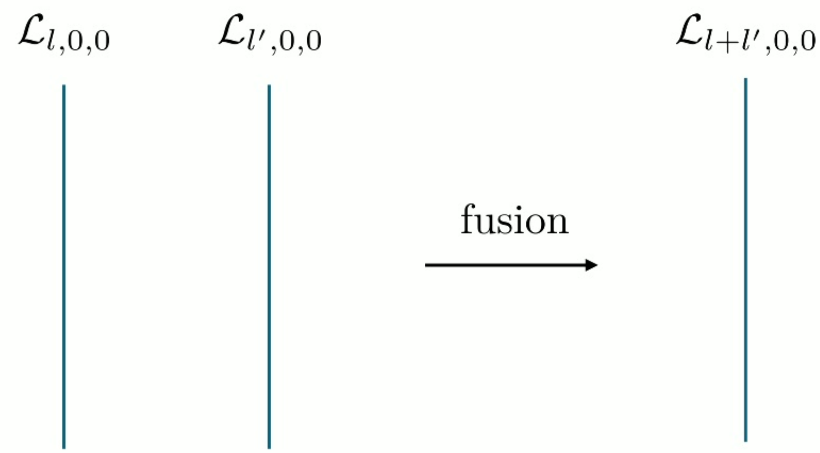
- The generic expressions for the higher terms are not available as of now, due to the lack of understanding of the Poincaré-Birkhoff-Witt (PBW) basis of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$

- Coproduct $\Delta : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \to U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \widehat{\otimes} U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$

$$
\begin{aligned}
&\Delta(E(X)) = E(X) \otimes 1 + K^-(C_1 X) \otimes E(C_1 X) \\
&\Delta(F(X)) = 1 \otimes F(X) + F(C_2 X) \otimes K^+(C_2 X) \\
&\Delta(K^+(X)) = K^+(X) \otimes K^+(C_1^{-1} X) \\
&\Delta(K^-(X)) = K^-(C_2^{-1} X) \otimes K^-(X) \\
&\Delta(C) = C \otimes C,
\end{aligned}
$$

Defect fusion and coproduct

- Two line defects can be fused together to become single line defect:



$$
\begin{array}{ccc}
\mathrm{M2}_{l,0,0} \otimes \mathrm{M2}_{l',0,0} & \xhookleftarrow{\iota_{l,l'}} & \mathrm{M2}_{l+l',0,0} \\
\rho_{\mathrm{M2}_{l,0,0}} \otimes \rho_{\mathrm{M2}_{l',0,0}} \Big\uparrow & \circlearrowright & \Big\uparrow \rho_{\mathrm{M2}_{l+l',0,0}} \\
U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \mathbin{\widehat{\otimes}} U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) & \xleftarrow{\Delta} & U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))
\end{array}
$$

From universal R-matrix to Miura operators

- The quantum toroidal algebra of gl(1) has the universal R-matrix: $\mathcal{R} \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \,\widehat{\otimes}\, U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$

$$\begin{aligned}
&\mathcal{R}\Delta(g) = \Delta^{\mathrm{op}}(g)\mathcal{R}, \qquad \text{for any } g \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)),\\
&(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\\
&(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}.
\end{aligned}$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

- Using the Drinfeld double construction, it can be expressed as a product [Garbali, Neguţ 21]

$$\mathcal{R} = \bar{\mathcal{R}}\mathcal{K},$$

$$\mathcal{K} = e^{\log C \otimes d + d \otimes \log C + \log C^{\perp} \otimes d^{\perp} + d^{\perp} \otimes \log C^{\perp}} \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r}{r} H_{-r} \otimes H_r\right]$$

$$\bar{\mathcal{R}} = 1 \otimes 1 + \kappa_1 \sum_{k \in \mathbb{Z}} E_k \otimes F_{-k} + \cdots$$

- The generic expressions for the higher terms are not available as of now, due to the lack of understanding of the Poincaré-Birkhoff-Witt (PBW) basis of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$

- However, we do not need the generic expression of the universal R-matrix for our purpose. We only need the ones mapped to the representations associated to M2- and M5-branes.

- Namely, we consider $R^{(c)} = (\rho_{\mathrm{M2}_{0,0,1}} \otimes \rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}})\mathcal{R} \in \mathrm{M2}_{0,0,1} \widehat{\otimes}\, \mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}$

- The constraints can be solved recursively, yielding

$$R^{(c)} = \bar{R}^{(c)} K^{(c)}$$

$$K^{(c)} = (q_c^{\frac{d}{2}} \otimes 1) \prod_{j=0}^{\infty} \varphi_c^{+}(q_c^{-j} X_1) = \left(q_c^{\frac{d}{2}} \otimes 1\right) \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r}{r} \frac{X_1^{-r}}{1-q_3^r} \otimes \frac{a_r^{(c)}}{q_c^{r/2} - q_c^{-r/2}}\right]$$

$$\begin{aligned}\bar{R}^{(c)} &= \sum_{m=0}^{\infty} \kappa_1^m (\rho_{\mathrm{M2}_{0,0,1}} \otimes \rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}}) \sum_{k_1,\cdots,k_m \in \mathbb{Z}} E_{k_1} \cdots E_{k_m} \otimes F_{-k_1} \cdots F_{-k_m} \\ &= \sum_{m=0}^{\infty} q_{\bar{c}}^m e^{-m a_0^{(c)} \frac{\log q_{c-1} \log q_{c+1}}{\log q_c}} \left(\prod_{j=0}^{m-1} \frac{1-q_c q_3^{-j}}{1-q_3^{-j-1}}\right) : \prod_{j=0}^{m-1} \xi_c(q_3^{-j} X_1) : q_3^{-m D_{X_1}}\end{aligned}$$

- Importantly, in the transverse case $c = 3$,

$$R^{(3)} = \bar{R}^{(3)} K^{(3)}$$

$$K^{(3)} = (q_3^{\frac{d}{2}} \otimes 1) \prod_{j=0}^{\infty} \varphi_3^+(q_3^{-j} X_1) = \left(q_3^{\frac{d}{2}} \otimes 1\right) \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r}{r} \frac{X_1^{-r}}{1 - q_3^r} \otimes \frac{a_r^{(c)}}{q_3^{r/2} - q_3^{-r/2}}\right]$$

$$\bar{R}^{(3)} = 1 - e^{-a_0^{(3)} \frac{\log q_1 \log q_2}{\log q_3}} q_3^{-1} \xi_3(X_1) q_3^{-D_{X_1}}$$

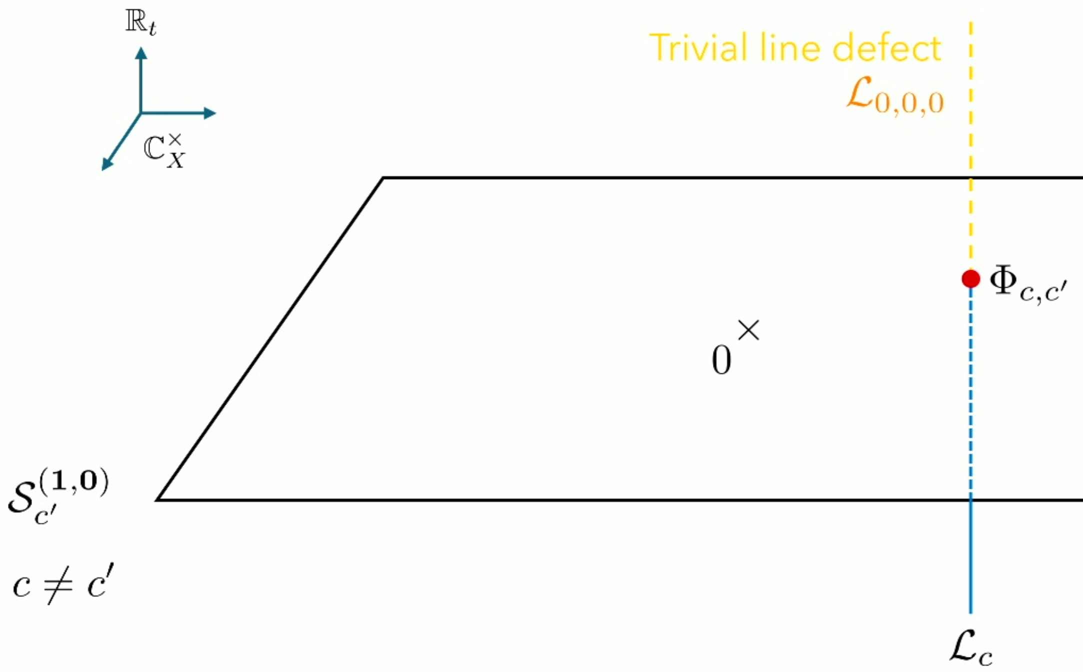
$$R_{\mathrm{M2}_{0,0,1},\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}} = R^{(c_{L+M+N})}_{L+M+N} R^{(c_{L+M+N-1})}_{N-1} \cdots R^{(c_1)}_{1}$$

$$= \sum_{m=0}^{\infty} (-1)^m T_m(X_1) q_3^{-mD_{X_1}}$$

- Expanding it in the generators of $\mathrm{M2}_{0,0,1}$ gives the generators of $\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$, by construction.

- To justify this is a Miura transformation, we need to show: do they generate ALL?

M2-M5 intersections and Intertwiners



$$\Phi_{c,c'} \in \mathrm{Hom}(\mathcal{V}_c, \mathbb{C}) \,\widehat{\otimes}\, \mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}$$

$$\epsilon : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \to \mathbb{C}$$

counit

$$\Phi_{c,c'}\Delta_{\mathrm{M2}_c,\mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}}(g) = \rho_{\mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}}(g)\Phi_{c,c'} = (\rho_{\mathrm{M2}_{0,0,0}} \otimes \rho_{\mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}})\Delta^{\mathrm{op}}(g)\Phi_{c,c'}$$

$$= (\epsilon \otimes \rho_{\mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}})\Delta^{\mathrm{op}}(g)\Phi_{c,c'}$$

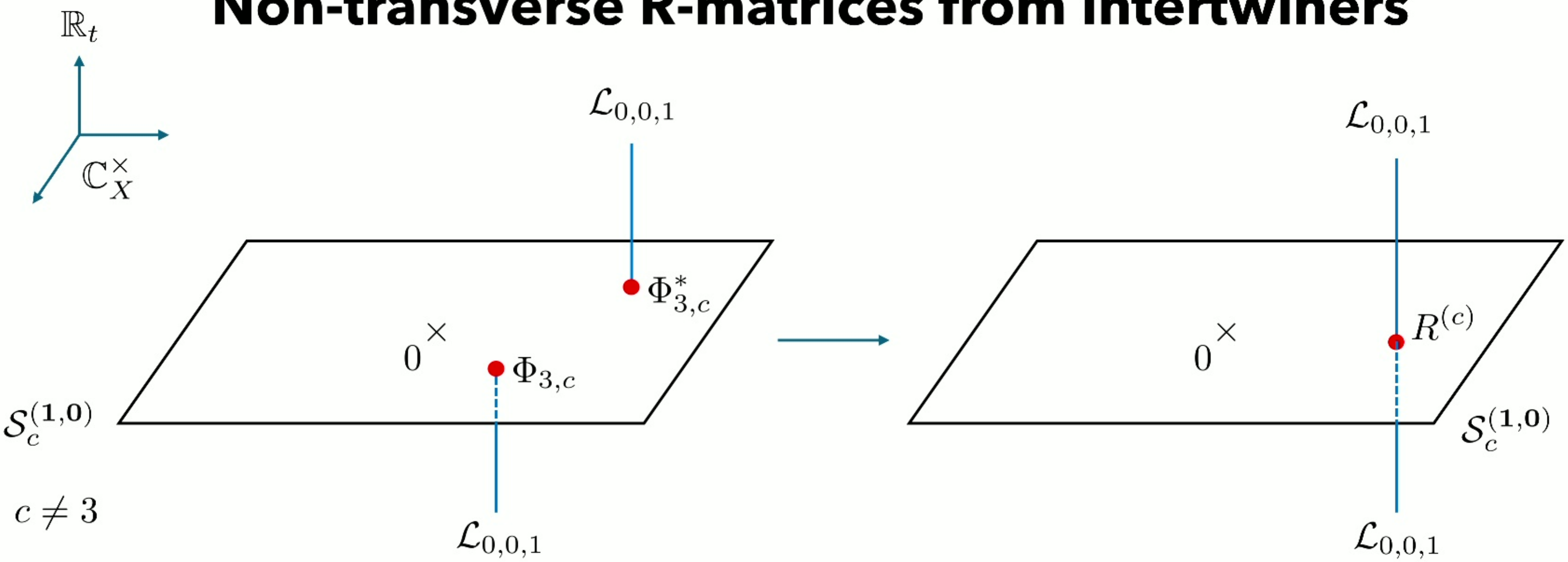
- This constraint can be solved: [Feigin, Hashizume, Hoshino, Shiraishi, Yanagida 09]

$\{\bar{c}\} = \{1,2\} \setminus \{c\}$

$$\Phi_{3,c} = e^{\frac{\log q_{\bar{c}}}{\log q_c}\left(\log X_1 \otimes a_0^{(c)}\right)} \left(1 \otimes q_{\bar{c}}^{d^\perp}\right) \exp\left[-\sum_{r=1}^{\infty} \frac{\kappa_r}{r} \frac{q_3^{\frac{r}{2}}}{\left(q_c^{\frac{r}{2}} - q_c^{-\frac{r}{2}}\right)^2 \left(q_3^{\frac{r}{2}} - q_3^{-\frac{r}{2}}\right)} X_1^r \otimes a_{-r}^{(c)}\right]$$
$$\times \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r}{r} \frac{q_3^{-\frac{r}{2}} q_c^{-\frac{r}{2}}}{\left(q_c^{\frac{r}{2}} - q_c^{-\frac{r}{2}}\right)^2 \left(q_3^{\frac{r}{2}} - q_3^{-\frac{r}{2}}\right)} X_1^{-r} \otimes a_r^{(c)}\right]$$

$$\Phi_{3,c}^* = \left(1 \otimes q_{\bar{c}}^{-d^\perp}\right) e^{-\frac{\log q_{\bar{c}}}{\log q_c}\left(\log X_1 \otimes a_0^{(c)}\right)} \exp\left[\sum_{r=1}^{\infty} \frac{\kappa_r}{r} \frac{q_3^{\frac{r}{2}} q_c^{\frac{r}{2}}}{\left(q_c^{\frac{r}{2}} - q_c^{-\frac{r}{2}}\right)^2 \left(q_3^{\frac{r}{2}} - q_3^{-\frac{r}{2}}\right)} X_1^r \otimes a_{-r}^{(c)}\right]$$
$$\times \exp\left[-\sum_{r=1}^{\infty} \frac{\kappa_r}{r} \frac{q_3^{-\frac{r}{2}}}{\left(q_c^{\frac{r}{2}} - q_c^{-\frac{r}{2}}\right)^2 \left(q_3^{\frac{r}{2}} - q_3^{-\frac{r}{2}}\right)} X_1^{-r} \otimes a_r^{(c)}\right]$$

Non-transverse R-matrices from intertwiners

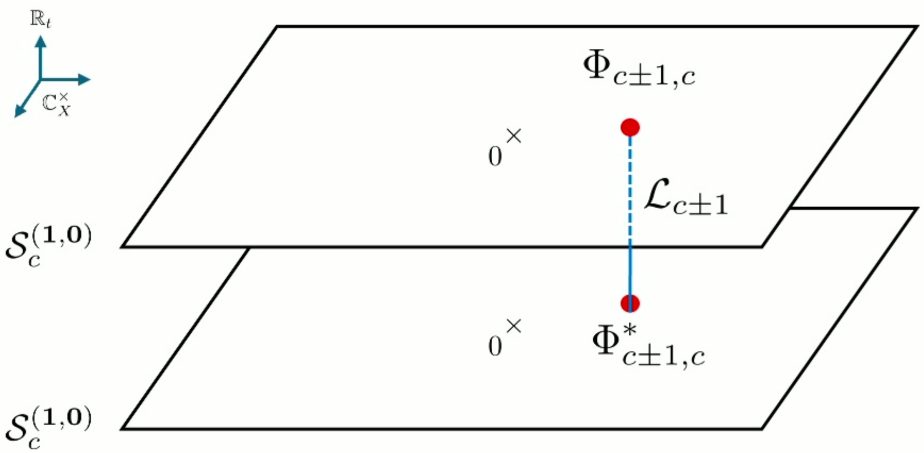


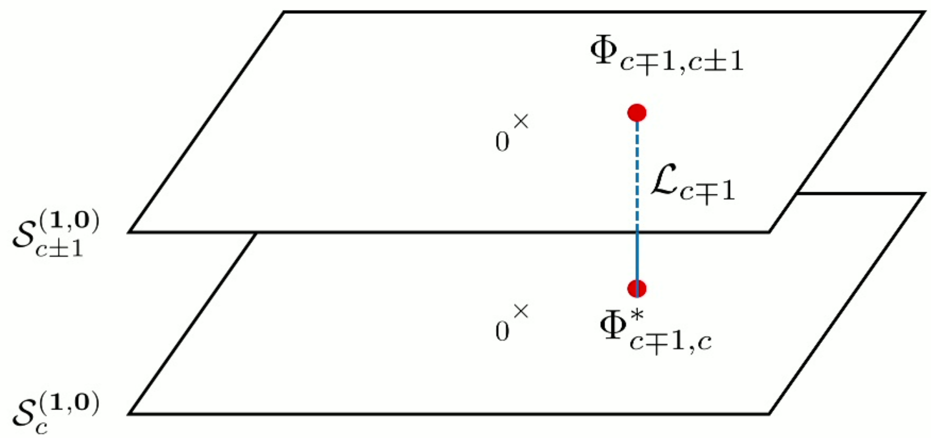
$$R^{(c)} = -(q_3;q_3)_{\nu_c} q_{\bar{c}}^{-\frac{1}{2}(1\otimes a_0^{(c)})} \oint \frac{dX}{2\pi i X_1} \frac{\left(q_{\bar{c}}^{-1}\frac{X}{X_1};q_3\right)_\infty}{\left(\frac{X}{X_1};q_3\right)_\infty} e^{\frac{\log q_c}{\log q_3}\log\frac{X}{X_1}} :\Phi^*_{3,c}(X_1)\Phi_{3,c}(q_c^{\frac{1}{2}}X): Z_1^{-\frac{\log\frac{X}{X_1}}{\log q_3}} \left(q_c^{\frac{d}{2}}\otimes 1\right)$$

$$= -(q_3;q_3)_{\nu_c} q_{\bar{c}}^{\frac{1}{2}\left(\frac{\log q_c}{\log q_3}-1\otimes a_0^{(c)}\right)} \left(\oint \frac{dX}{2\pi i X}\, \Phi^*_{3,c}(X_1)\Phi_{3,c}(q_c^{\frac{1}{2}}X)\, Z_1^{-\frac{\log\frac{X}{X_1}}{\log q_3}}\right)\left(q_c^{\frac{d}{2}}\otimes 1\right)$$

Screening charges from finite M2-branes

- There are only two cases for any two adjacent M5-branes:





$$S^{c\pm1}_{c,c} := \sigma \circ \left(\Phi_{c\pm1,c} \otimes \Phi^*_{c\pm1,c}\right) \in \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i=c} \otimes \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}=c}$$

$$S^{c\mp1}_{c,c\pm1} := \sigma \circ \left(\Phi_{c\mp1,c\pm1} \otimes \Phi^*_{c\mp1,c}\right) \in \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i=c} \otimes \mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}=c\pm1}.$$

$$S^{c\pm1}_{c,c} = q_{c\mp1}^{-d^\perp\otimes1+1\otimes d^\perp} \sum_{i\in\mathbb{Z}} (uq^i_{c\pm1})^{-\frac{\log q_{c\mp1}}{\log q_c}(a_0^{(c)}\otimes1-1\otimes a_0^{(c)})-\frac{\log q_{c\mp1}}{\log q_{c\pm1}}}$$

$$\times \exp\left[\sum_{r=1}^{\infty}\frac{1}{r}\frac{q_{c\mp1}^{r/2}-q_{c\mp1}^{-r/2}}{q_c^{r/2}-q_c^{-r/2}}q_{c\mp1}^{-r/2}(a_{-r}^{(c)}\otimes1-1\otimes q_c^{-r/2}a_{-r}^{(c)})(uq^i_{c\pm1})^r\right]$$

$$\times \exp\left[-\sum_{r=1}^{\infty}\frac{1}{r}\frac{q_{c\mp1}^{r/2}-q_{c\mp1}^{-r/2}}{q_c^{r/2}-q_c^{-r/2}}q_{c\mp1}^{r/2}(q_c^{r/2}a_r^{(c)}\otimes1-1\otimes a_r^{(c)})(uq^i_{c\pm1})^{-r}\right]$$

$$S^{c\mp1}_{c,c\pm1} = q_{c\pm1}^{-d^\perp\otimes1}q_c^{1\otimes d^\perp} \sum_{i\in\mathbb{Z}} (uq^i_{c\mp1})^{-\frac{\log q_{c\pm1}}{\log q_c}(a_0^{(c)}\otimes1)+\frac{\log q_c}{\log q_{c\pm1}}(1\otimes a_0^{(c\pm1)})-\frac{\log q_c}{\log q_{c\mp1}}}$$

$$\times \exp\left[\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{q_{c\pm1}^{r/2}-q_{c\pm1}^{-r/2}}{q_c^{r/2}-q_c^{-r/2}}q_{c\pm1}^{-r/2}(a_{-r}^{(c)}\otimes1)-\frac{q_c^{r/2}-q_c^{-r/2}}{q_{c\pm1}^{r/2}-q_{c\pm1}^{-r/2}}q_{c\mp1}^{r/2}(1\otimes a_{-r}^{(c\pm1)})\right)(uq^i_{c\mp1})^r\right]$$

$$\times \exp\left[-\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{q_{c\pm1}^{r/2}-q_{c\pm1}^{-r/2}}{q_c^{r/2}-q_c^{-r/2}}q_{c\mp1}^{-r/2}(a_r^{(c)}\otimes1)-\frac{q_c^{r/2}-q_c^{-r/2}}{q_{c\pm1}^{r/2}-q_{c\pm1}^{-r/2}}q_c^{r/2}(1\otimes a_r^{(c\pm1)})\right)(uq^i_{c\mp1})^{-r}\right]$$

- The gauge-invariance conditions for the intersections imply that all the generators of $\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$ commute with the screening charges:

$$\left[S^{c_{\mathrm{M2}}}_{c_i,c_{i+1}}, \Delta_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i},\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}}(g)\right] = 0, \qquad \text{for any } g \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$$

$$\begin{aligned}
&\Delta_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i},\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}}(g) S^{c_{\mathrm{M2}}}_{c_i,c_{i+1}} \\
&= \sigma\left(\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}}(g_{(2)}) \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}}(g_{(1)})\right)\left(\Phi_{c_{\mathrm{M2}},c_{i+1}} \otimes \Phi^*_{c_{\mathrm{M2}},c_i}\right) \\
&= \sigma\left[\Phi_{c_{\mathrm{M2}},c_{i+1}}\left(\rho_{\mathrm{M2}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i}}\right)(g_{(2),(1)} \otimes g_{(2),(2)} \otimes g_{(1)})\Phi^*_{c_{\mathrm{M2}},c_i}\right] \\
&= \sigma\left[\Phi_{c_{\mathrm{M2}},c_{i+1}}\left(\rho_{\mathrm{M2}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i}}\right)(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(g_{(1)} \otimes g_{(2),(1)} \otimes g_{(2),(2)})\Phi^*_{c_{\mathrm{M2}},c_i}\right] \\
&= \sigma\left[\Phi_{c_{\mathrm{M2}},c_{i+1}}\left(\rho_{\mathrm{M2}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i}}\right)(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(g_{(1),(1)} \otimes g_{(1),(2)} \otimes g_{(2)})\Phi^*_{c_{\mathrm{M2}},c_i}\right] \\
&= \sigma\left[\Phi_{c_{\mathrm{M2}},c_{i+1}}\left(\rho_{\mathrm{M2}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}} \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i}}\right)(g_{(1),(2)} \otimes g_{(2)} \otimes g_{(1),(1)})\Phi^*_{c_{\mathrm{M2}},c_i}\right] \\
&= \sigma\left[\left(\Phi_{c_{\mathrm{M2}},c_{i+1}} \otimes \Phi^*_{c_{\mathrm{M2}},c_i}\right)\left(\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}}(g_{(2)}) \otimes \rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i}}(g_{(1)})\right)\right] \\
&= S^{c_{\mathrm{M2}}}_{c_i,c_{i+1}}\Delta_{\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_i},\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{c_{i+1}}}(g)
\end{aligned}$$

- Finally, it can be shown that there is no more operator that commutes with all the generators obtained from the product of R-matrices.

Thus,

1. The Miura transformation constructed by the R-matrices generates the whole q-deformed Y-algebra $\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{L,M,N}$

2. The q-deformed Y-algebra is completely characterized as commutants of the screening charges in the tensor product of free q-bosons.

Ordering-independence by Yang-Baxter equation

- Consider the R-matrices between q-boson representations:

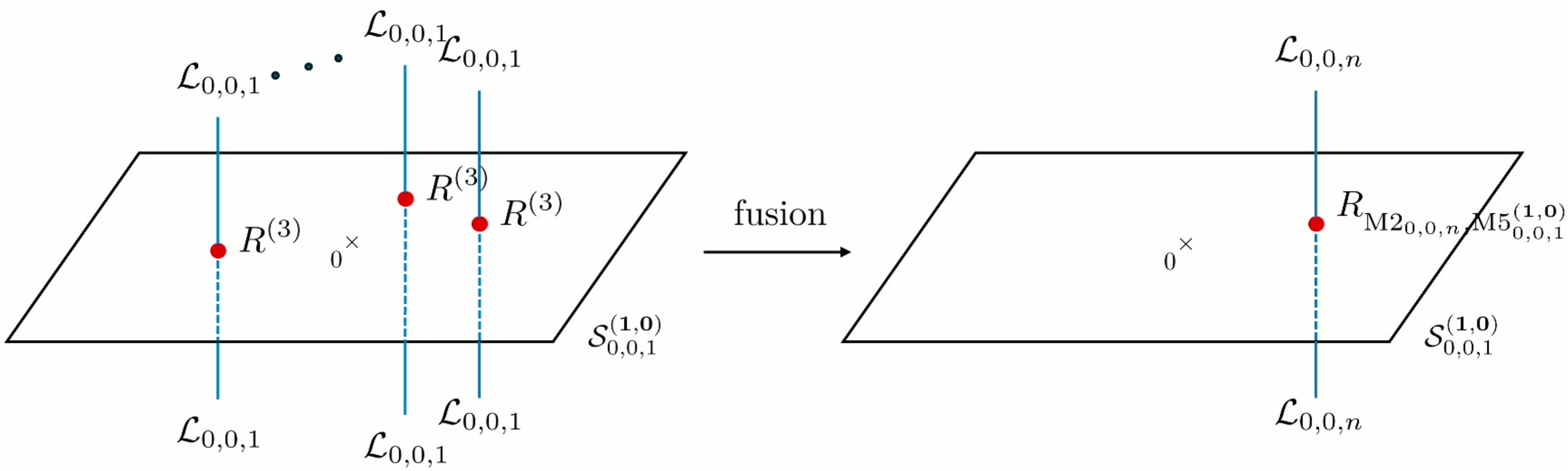
$$R_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})},\mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}} = (\rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{0})}} \otimes \rho_{\mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}})\mathcal{R} \in \mathrm{M5}_c^{(\mathbf{1},\mathbf{0})} \otimes \mathrm{M5}_{c'}^{(\mathbf{1},\mathbf{0})}, \qquad c, c' \in \{1,2,3\}$$

We may call them q-deformed Maulik-Okounkov R-matrices. cf. [Fukuda, Harada, Matsuo, Zhu 17], [Garbali, de Gier 20], [Negut 20], [Garbali, Negut 21]

- The universal Yang-Baxter directly implies the change of ordering of Miura operators yields equivalent Miura transformation up to the conjugation of this R-matrix:

$$R_{\mathrm{M5}_{c_i}^{(\mathbf{1},\mathbf{0})},\mathrm{M5}_{c_{i+1}}^{(\mathbf{1},\mathbf{0})}} R_{i+1}^{(c_{i+1})} R_i^{(c_i)} = R_i^{(c_i)} R_{i+1}^{(c_{i+1})} R_{\mathrm{M5}_{c_i}^{(\mathbf{1},\mathbf{0})},\mathrm{M5}_{c_{i+1}}^{(\mathbf{1},\mathbf{0})}}, \qquad \text{for any } i$$

Multiple M2s and single M5

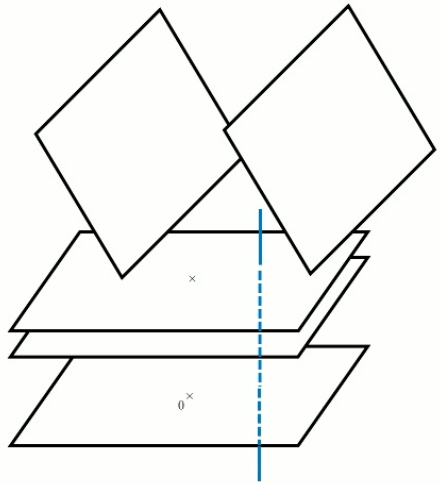


- The relation $(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ implies:

$$R_{\mathrm{M2}_{0,0,n},\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,1}} = R^{(3)}_1 R^{(3)}_2 \cdots R^{(3)}_n$$

M-brane Network

- Consider more general configurations of parallel and/or intersecting M2- and M5-branes, e.g.,



- In 5d CS theory, they descend to configurations of line and surface defects which overlap and intersect with one another.

- For this, we need M5-branes supported on generic holomorphic curve $C^{(\mathbf{p},\mathbf{q})}$
- The reparametrization of $\mathbb{C}^{\times}_X \times \mathbb{C}^{\times}_Z$ by $X' = X^a Z^b,\ Z' = X^c Z^d$ preserves the background.

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$$

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- Indeed, there is an $SL(2,\mathbb{Z})$ automorphism group of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1))$ [Miki 07]

- By repetitively applying T to $C^{(\mathbf{1},\mathbf{0})}$, we get all the curves of the form $C^{(\mathbf{1},\mathbf{n})}$

$$\rho_{\mathrm{M5}_c^{(\mathbf{1},\mathbf{n})}} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M5}_c^{(\mathbf{1},\mathbf{n})} = \mathrm{End}(\mathcal{F}_c^{(\mathbf{1},\mathbf{n})}(v)).$$

$$E(X) \mapsto -e^{a_0^{(c)} \frac{\log q_{c+1} \log q_{c-1}}{\log q_c}} \frac{1-q_c}{\kappa_1} \left(\frac{q_c^{1/2}}{X} \right)^{\mathbf{n}} \eta_c(X),$$
$$F(X) \mapsto e^{-a_0^{(c)} \frac{\log q_{c+1} \log q_{c-1}}{\log q_c}} \frac{1-q_c^{-1}}{\kappa_1} \left(\frac{X}{q_c^{1/2}} \right)^{\mathbf{n}} \xi_c(X),$$
$$K^{\pm}(X) \mapsto q_c^{\mp \mathbf{n}/2} \varphi_c^{\pm}(X),$$

- We can also get $C^{(\mathbf{0},\mathbf{1})} = \mathbb{C}_Z^\times$ by applying S

$$\rho_{\mathrm{M5}_c^{(\mathbf{0},\mathbf{1})}} : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}(1)) \twoheadrightarrow \mathrm{M5}_c^{(\mathbf{0},\mathbf{1})} = \mathrm{End}(\mathcal{F}_c^{(\mathbf{0},\mathbf{1})}(a)).$$

[Feigin, Tsymbaliuk 09], [Schiffmann, Vasserot 09]

$$E(X)|a,\lambda\rangle^{(c)} = \frac{1-q_c}{\kappa_1} \sum_{\Box\in\partial_+\lambda} \delta(\chi_\Box/X) \operatorname*{Res}_{X=\chi_\Box} X^{-1} \mathcal{Y}_\lambda^{(c)}(X,a)^{-1} |a,\lambda+\Box\rangle^{(c)},$$

$$F(X)|a,\lambda\rangle^{(c)} = -\frac{1-q_c^{-1}}{\kappa_1} q_c^{-1/2} \sum_{\Box\in\partial_-\lambda} \delta(\chi_\Box/X) \operatorname*{Res}_{X=\chi_\Box} X^{-1} \mathcal{Y}_\lambda^{(c)}(Xq_c^{-1},a) |a,\lambda-\Box\rangle^{(c)},$$

$$K^\pm(X)|a,\lambda\rangle^{(c)} = q_c^{-1/2} \left(\frac{\mathcal{Y}_\lambda^{(c)}(Xq_c^{-1},a)}{\mathcal{Y}_\lambda^{(c)}(X,a)} \right)_\pm |a,\lambda\rangle^{(c)}, \qquad C|a,\lambda\rangle^{(c)} = |a,\lambda\rangle^{(c)},$$

$$\mathcal{Y}_\lambda^{(c)}(X,a) = \left(1-\frac{a}{X}\right) \prod_{\Box\in\lambda} \frac{\left(1-\frac{q_{c+1}\chi_\Box}{X}\right)\left(1-\frac{q_{c-1}\chi_\Box}{X}\right)}{\left(1-\frac{\chi_\Box}{X}\right)\left(1-\frac{q_c^{-1}\chi_\Box}{X}\right)}$$

$$\chi_{\Box_{(i,j)}} = a q_{c+1}^{i-1} q_{c-1}^{j-1}$$

- How do the M5-branes intersect? It is more convenient to work in the IIB dual frame.

In a dual IIB frame,

- M2-branes become D3-branes
- M5-branes on $C^{(\mathbf{p},\mathbf{q})}$ become $(\mathbf{p},\mathbf{q})$-fivebranes

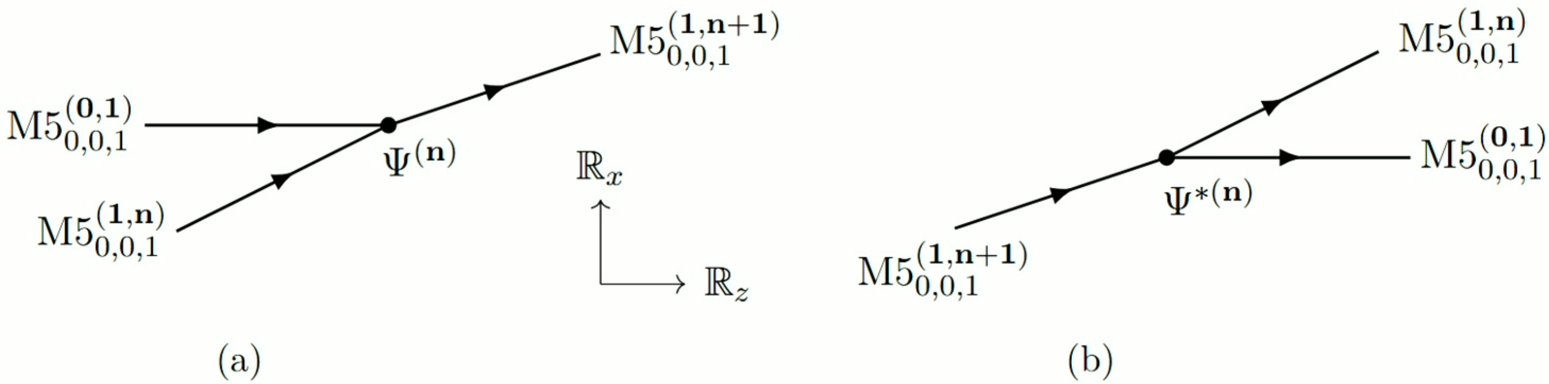
$(\mathbf{1},\mathbf{0}) = \mathrm{NS5}$
$(\mathbf{0},\mathbf{1}) = \mathrm{D5}$

cf. [Awata, Feigin, Shiraishi 11] and many others

	$\mathbb{R}^2_{\varepsilon_1}$		$\mathbb{R}^2_{\varepsilon_2}$		$\mathbb{R}^2_{\varepsilon_3}$		$\mathbb{R}_t$	$\mathbb{R}_x$	$\mathbb{R}_z$	S^1
IIB branes	0	1	2	3	4	5	6	7	8	9
$\mathrm{D3}_{1,0,0}$	x	x					x			x
$\mathrm{D3}_{0,1,0}$			x	x			x			x
$\mathrm{D3}_{0,0,1}$					x	x	x			x
$\mathrm{NS5}_{0,0,1}$	x	x	x	x				x		x
$\mathrm{NS5}_{0,1,0}$	x	x			x	x		x		x
$\mathrm{NS5}_{1,0,0}$			x	x	x	x		x		x
$(\mathbf{p},\mathbf{q})_{0,0,1}$	x	x	x	x				$l^{(\mathbf{p},\mathbf{q})}$		x

$$l^{(\mathbf{p},\mathbf{q})} = \{\mathbf{q}x - \mathbf{p}z = \mathrm{const}\} \subset \mathbb{R}_x \times \mathbb{R}_z$$

- How do the M5-branes intersect? It is more convenient to work in the IIB dual frame.



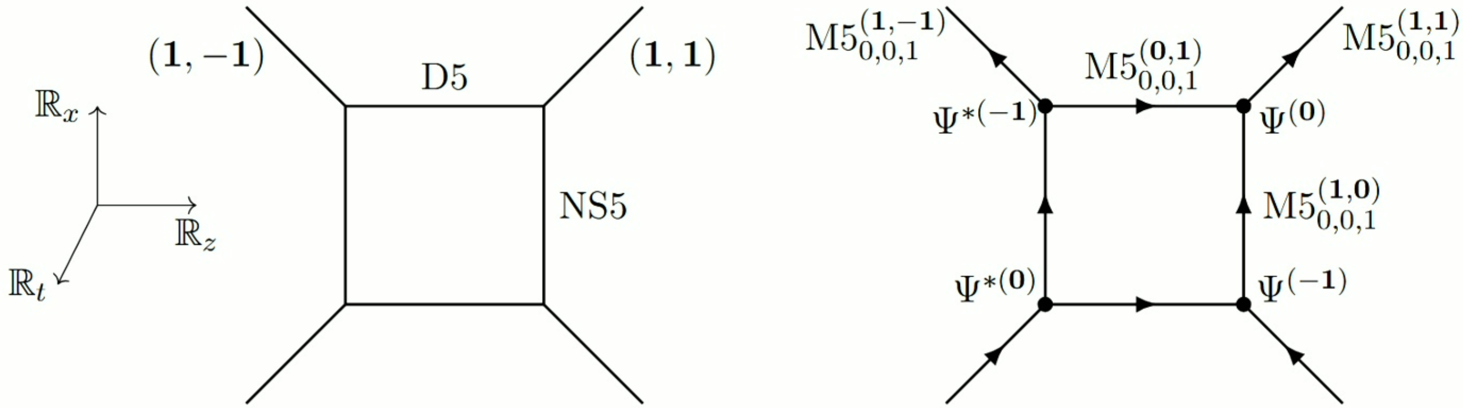
$$\Psi^{(\mathbf{n})}(v,v') : \mathcal{F}_3^{(\mathbf{0},\mathbf{1})}(v') \otimes \mathcal{F}_3^{(\mathbf{1},\mathbf{n})}(v) \to \mathcal{F}_3^{(\mathbf{1},\mathbf{n}+\mathbf{1})}(-vv')$$

$$\Psi^{*(\mathbf{n})}(v,v') : \mathcal{F}_3^{(\mathbf{1},\mathbf{n}+\mathbf{1})}(-vv') \to \mathcal{F}_3^{(\mathbf{0},\mathbf{1})}(v') \otimes \mathcal{F}_3^{(\mathbf{1},\mathbf{n})}(v)$$

$$\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{n}+\mathbf{1})}_{0,0,1}}(g)\Psi^{(\mathbf{n})} = \Psi^{(\mathbf{n})}\Delta_{\mathrm{M5}^{(\mathbf{0},\mathbf{1})}_{0,0,1},\mathrm{M5}^{(\mathbf{1},\mathbf{n})}_{0,0,1}}(g)$$

$$\Delta^{\mathrm{op}}_{\mathrm{M5}^{(\mathbf{0},\mathbf{1})}_{0,0,1},\mathrm{M5}^{(\mathbf{1},\mathbf{n})}_{0,0,1}}(g)\Psi^{*(\mathbf{n})} = \Psi^{*(\mathbf{n})}\rho_{\mathrm{M5}^{(\mathbf{1},\mathbf{n}+\mathbf{1})}_{0,0,1}}(g)$$

- The network of M5-branes engineers a 5d N=1 gauge theory as the effective theory. For example, pure U(2) gauge theory:



(a) (b)

$$\mathcal{T} := \sigma\left(\Psi^{(\mathbf{0})}(-v_2a_1, a_2)\otimes\Psi^{*(-\mathbf{1})}(-v_1,a_2)\right)\circ\left(\Psi^{(-\mathbf{1})}(v_2,a_1)\otimes\Psi^{*(\mathbf{0})}(v_1a_2,a_1)\right)\sigma$$
$$: \mathcal{F}_3^{(\mathbf{1},\mathbf{1})}(-v_1a_1a_2)\otimes\mathcal{F}_3^{(\mathbf{1},-\mathbf{1})}(v_2)\to\mathcal{F}_3^{(\mathbf{1},-\mathbf{1})}(-v_1)\otimes\mathcal{F}_3^{(\mathbf{1},\mathbf{1})}(v_2a_1a_2).$$

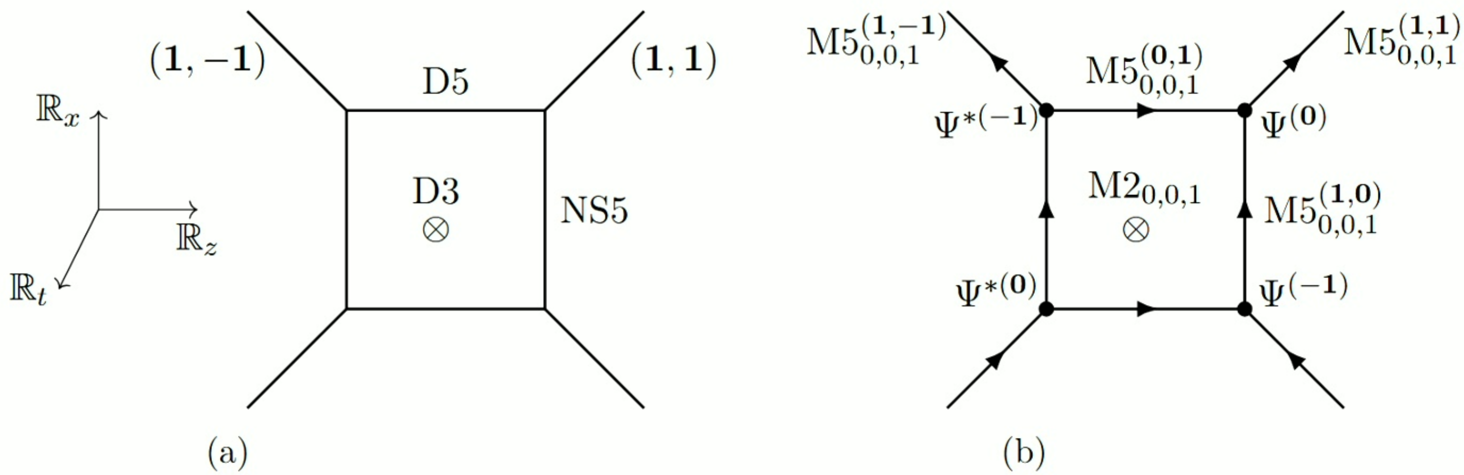
cf. [Alday, Gaiotto, Tachikawa 09] [Awata, Feigin, Shiraishi 11]

$$\mathcal{Z}(a;\mathfrak{q}) := (\langle\varnothing|\otimes\langle\varnothing|)\mathcal{T}(|\varnothing\rangle\otimes|\varnothing\rangle) = \mathcal{Z}^{\text{1-loop}}(a)\sum_{\{\lambda_1,\lambda_2\}}\mathfrak{q}^{|\lambda_1|+|\lambda_2|}\frac{1}{\prod_{\alpha,\beta=1,2}N_{\lambda_\alpha\lambda_\beta}(a_\alpha/a_\beta;q_1,q_2)}$$

Vev of intersecting surface defects in 5d CS theory

Partition function of 5d N=1 gauge theory

- Then, M2-branes can be added to the network, providing certain observables in the gauge theory. E.g., a single M2-brane:



$$\mathcal{X} := \left(\sigma(\Psi^{(\mathbf{0})} \otimes \Psi^{*(-\mathbf{1})})\sigma\right) R_{\mathrm{M2}_{0,0,1},\mathrm{M5}^{(\mathbf{1},\mathbf{0})}_{0,0,2}} \left(\sigma(\Psi^{(-\mathbf{1})} \otimes \Psi^{*(\mathbf{0})})\sigma\right)$$
$$\in \mathrm{M2}_{0,0,1} \otimes \mathrm{Hom}\left(\mathcal{F}_3^{(\mathbf{1},\mathbf{1})}(-v_1 a_1 a_2) \otimes \mathcal{F}_3^{(\mathbf{1},-\mathbf{1})}(v_2), \mathcal{F}_3^{(\mathbf{1},-\mathbf{1})}(-v_1) \otimes \mathcal{F}_3^{(\mathbf{1},\mathbf{1})}(v_2, a_1 a_2)\right)$$

$$\langle \mathcal{X}(u) \rangle_k := \left([uq_3^{-1}]^*_{i+k} \otimes \langle \varnothing| \otimes \langle \varnothing|\right) \mathcal{X}\left([u]_i \otimes |\varnothing\rangle \otimes |\varnothing\rangle\right)$$

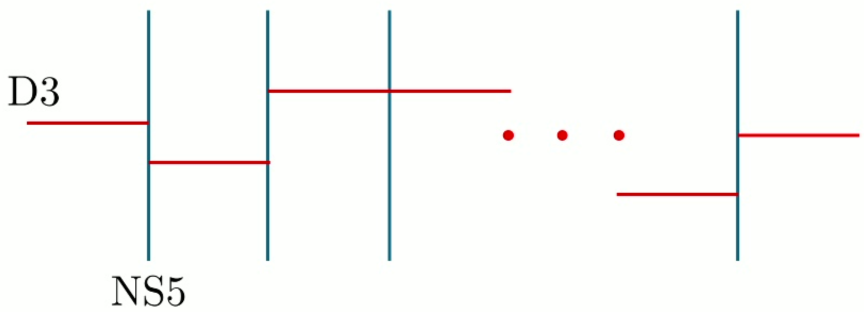
$$\langle \mathcal{X}(X_1)\rangle_1 = \mathcal{Z}^{\text{1-loop}}(a) \sum_{\{\lambda_1,\lambda_2\}} \mathfrak{q}^{|\lambda_1|+|\lambda_2|} \frac{1}{\prod_{\alpha,\beta=1,2} N_{\lambda_\alpha\lambda_\beta}(a_\alpha/a_\beta; q_1, q_2)} \times \frac{q_3^{-1/2}}{v_2} \left(\frac{q_3^{-1}}{a_1 a_2} X_1 \mathcal{Y}(X_1 q_3^{-1/2}) + \mathfrak{q} \frac{1}{X_1 \mathcal{Y}(X_1 q_3^{1/2})} \right)$$

- This exactly recovers the qq-character of the 5d N=1 pure U(2) gauge theory [Nekrasov 15]

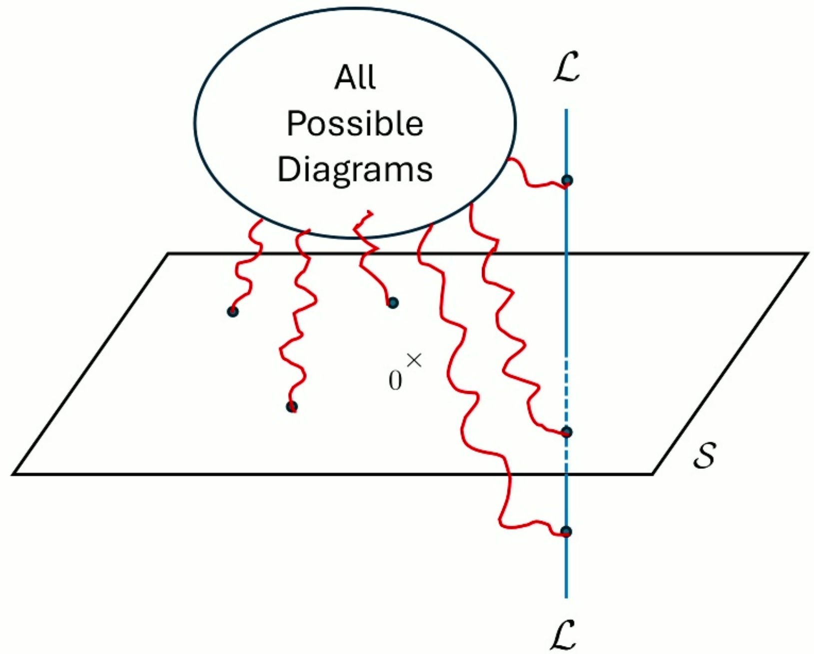
- Namely, the vev of the network of line/surface defects in 5d CS theory is shown to match with the vev of the qq-character.

Outlook

- Half-index of 3d quiver abelian gauge theory with some 1d defects = Miura transformation



- Direct perturbative analysis of 5d CS theory on $\mathbb{R}_t \times \mathbb{C}^\times_X \times \mathbb{C}^\times_Z$



$\mathbb{C}_x \times \mathbb{C}_Z^\times$

- Twisted M-theory on $\mathbb{R}^2_{\varepsilon_1} \times (\mathbb{R}^2_{\varepsilon_2} \times \mathbb{R}^2_{\varepsilon_3})/\mathbb{Z}_K \times \mathbb{R}_t \times \mathbb{C}_X^\times \times \mathbb{C}_Z^\times$, 5d $\mathfrak{gl}(K)$ CS theory

- The M5-brane network engineers a 4d N=2 (or 5d N=1) theory with a codimension-two monodromy defect. Its vev was shown to give common eigenfunctions of the quantum integrable model of Hitchin type (or its multiplicative uplift). The transverse M2-branes was shown to give rise to Lax matrices. The non-transverse M2-branes provides Hecke operators (or Q-operators).

cf. [Alday, Tachikawa 10], [Kanno, Tachikawa 11]
[SJ, Nekrasov 18]
[SJ, Lee, Nekrasov 23], [SJ, Lee, Nekrasov 24]

Thank you!