Title: Derived differential geometry and applications

Speakers: Pelle Steffens

Collection/Series: Mathematical Physics

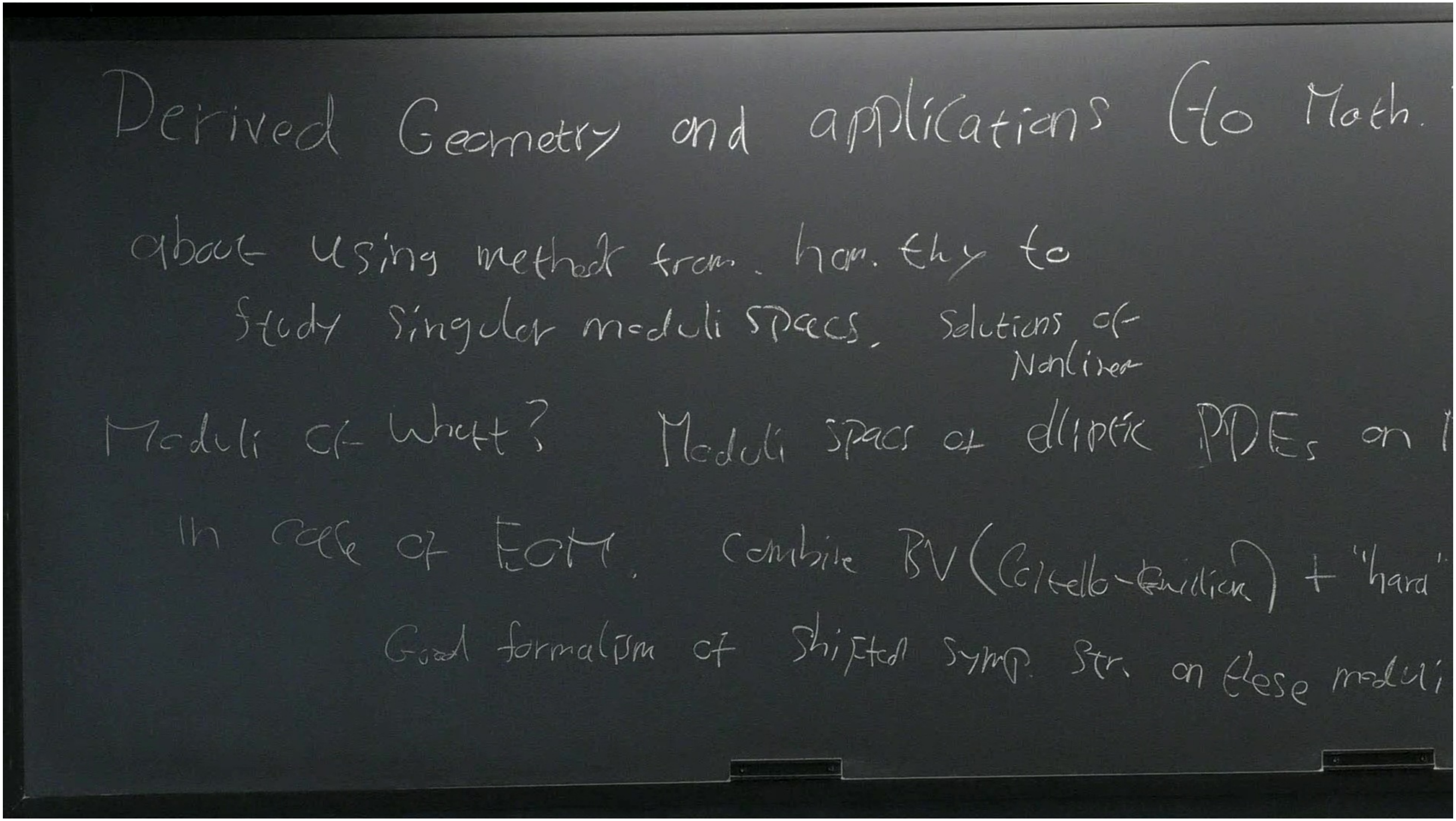
Subject: Mathematical physics

Date: November 07, 2024 - 11:00 AM

URL: https://pirsa.org/24110061

Abstract:

I will review some recent progress in derived differential geometry, in particular pertaining to moduli stacks of solutions of elliptic partial differential equations on manifolds (with boundaries, and also with `logarithmic' boundaries, which include, for instance, manifolds with asymptotically cylindrical ends). In particular, this framework allows one to work efficiently with the compactified moduli spaces of symplectic topology and gauge theory. In another direction, I will explain some work in progress on the derived geometry of jet spaces, which can be used to endow moduli stacks of solutions of EOMs of a classical field theory with shifted symplectic structures.


Derived Geometry and applications (to Math.
about using methods from hom. thy to
study singular moduli spaces. Solutions of
Nonlinear
Moduli of what? Moduli spaces of elliptic PDEs on
In case of EOM. Combine BV (Costello-Gwilliam) + "hard"
Good formalism of shifted symp. str. on these moduli

nd applications (to Math. Physics, Maybe).

m. hom. thy to

li spaces. Solutions of
Nonlinear

oduli spaces of elliptic PDEs on Mfd's (bdy/corners).

combine BV (Costello-Gwilliam) + "hard" geometric analysis + HTT/DAG

of shifted symp. str. on these moduli spaces

Fact (Kuranishi '65) ⇝ "Hard" analysis.

M cpt mfd $E \searrow M \swarrow F$ VB$/M$.

$P : \Gamma(E) \to \Gamma(F)$ Zeroes(P) locally homeo

nonl. elliptic PDE   to Zeroes(f), $f : \mathbb{R}^n \to \mathbb{R}^m$

$T_x f = T_x P$

analysis.

Corollary: If all f's are ..., ⟹ Zeroes(P) is a C-mfd.

Could be equipped w a f.c.

...s(P) locally homeo.

...roes(f), $f: \mathbb{R}^n \to \mathbb{R}^m$

P

In good cases,

Moduli of Pseudohol. curves in a Symp. Mfd.

Q: What if $-S_{\not{A}} 0$.

$\underline{A}$: [DAE ; $f: \mathbb{R}^n \to \mathbb{R}^m$, Polynomial]

$Z(f)$ underlying space of

Spec $A \subset$ derived alg. $\leftrightarrow$ cdga $dCAlg^{\geq 0}_{\mathbb{R}}$,

$\mathbb{R}[x_1, \ldots x_n] \otimes^{\mathbb{L}}_{\mathbb{R}[x_1, \ldots, x_n]} \mathbb{R} = A$ Pushout / derived $\otimes$ product

$\exists$ a functor

$$dCAlg^{\geq 0}_{\mathbb{R}//\mathbb{R}} \xrightarrow{\mathbb{L}} Ch(Vect_{\mathbb{R}})[w^{-1}]$$

$\chi(\mathbb{L}_{A,p})$

∃ a functor

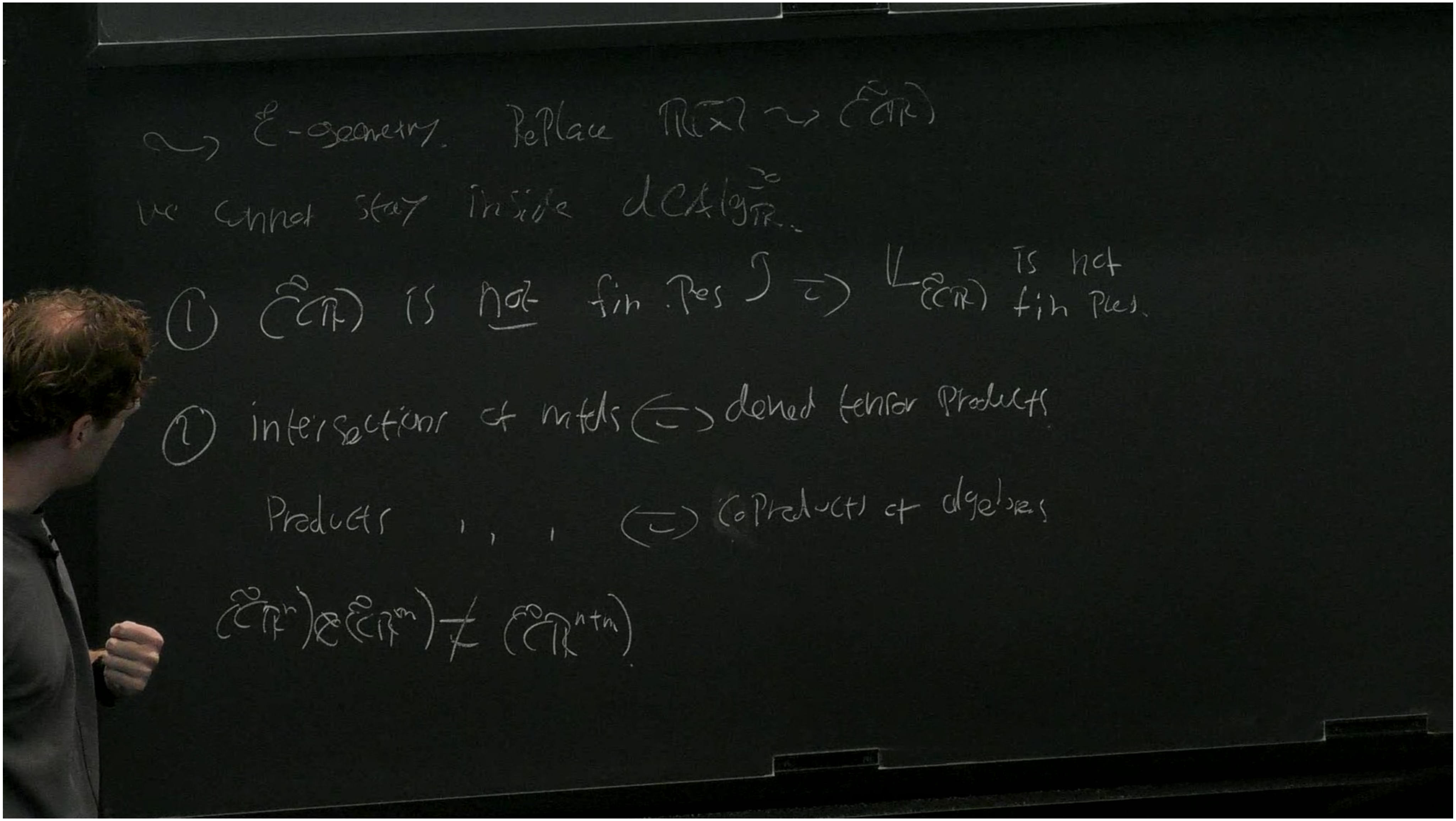
$$\mathrm{dCAlg}^{\geq 0}_{\mathbb{R}//\mathbb{R}} \xrightarrow{\mathbb{L}} \mathrm{Ch}(\mathrm{Vect}_{\mathbb{R}})[w^{-1}]$$

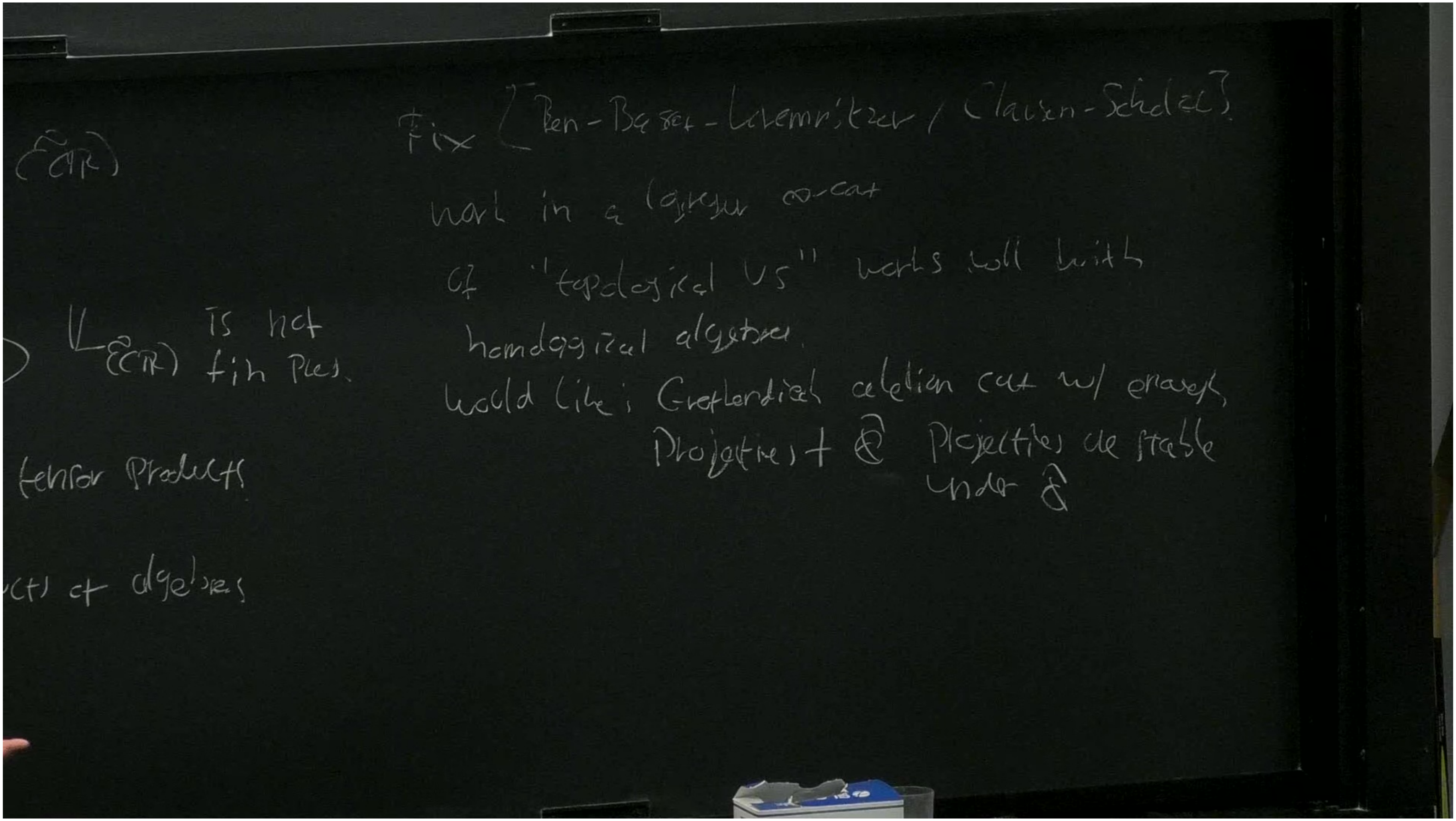
$$\chi(\mathbb{L}_{A,p})$$

$\mathrm{dCAlg}^{\geq 0}_{\mathbb{R}}$, Glue these Spec A's together.

$$\infty\text{-cat} \quad \mathrm{Top}^{\mathrm{loc}}_{\mathbb{R}}(\mathrm{dCAlg}) \underset{\mathrm{Spec}}{\overset{\Gamma}{\underset{\perp}{\rightleftarrows}}} \mathrm{dCAlg}^{op}$$

$$\mathrm{dSch} \subset \mathrm{Top}^{\mathrm{loc}}_{\mathbb{R}}(\mathrm{dCAlg}) \ni (X, \mathcal{O}_X)$$


⇝ C^∞-geometry. Replace ℝ[x] ⇝ C^∞(ℝ)
we cannot stay inside dCAlg_ℝ^{C^∞}...
① C^∞(ℝ) is not fin. Pres ∫ ⇒ L_{C^∞(ℝ)} is not fin Pres.
② intersections of mfds ⟺ derived tensor products
Products ⟺ (co)Products of algebras
C^∞(ℝ^n) ⊗ C^∞(ℝ^m) ≠ C^∞(ℝ^{n+m})


Fix [Ben-Bassat-Kremnitzer / Clausen-Scholze]
work in a ∞-cat
of "topological VS" works well with
homological algebra.
would like: Grothendieck abelian cat w/ enough
projectives + ⊗ projectives are stable
under ⊗
is not
fin pres.
tensor products
algebras

Idea: formalize bounded sequences

Note: $\mathrm{Vect}_{\mathbb{R}} = \mathcal{P}^{\mathrm{fct}}_{\Sigma}(\mathrm{Vect}^{fd}_{\mathbb{R}})$ ← sifted cocompletion in Set!

$\mathrm{Mod}^{\geq 0}_{\mathbb{R}} = \mathcal{P}_{\Sigma}(\mathrm{Vect}^{fd}_{\mathbb{R}}) = \mathrm{Fun}^{PP}(\mathrm{Vect}^{fd}, \mathcal{S})$ ← ∞-cat of spaces

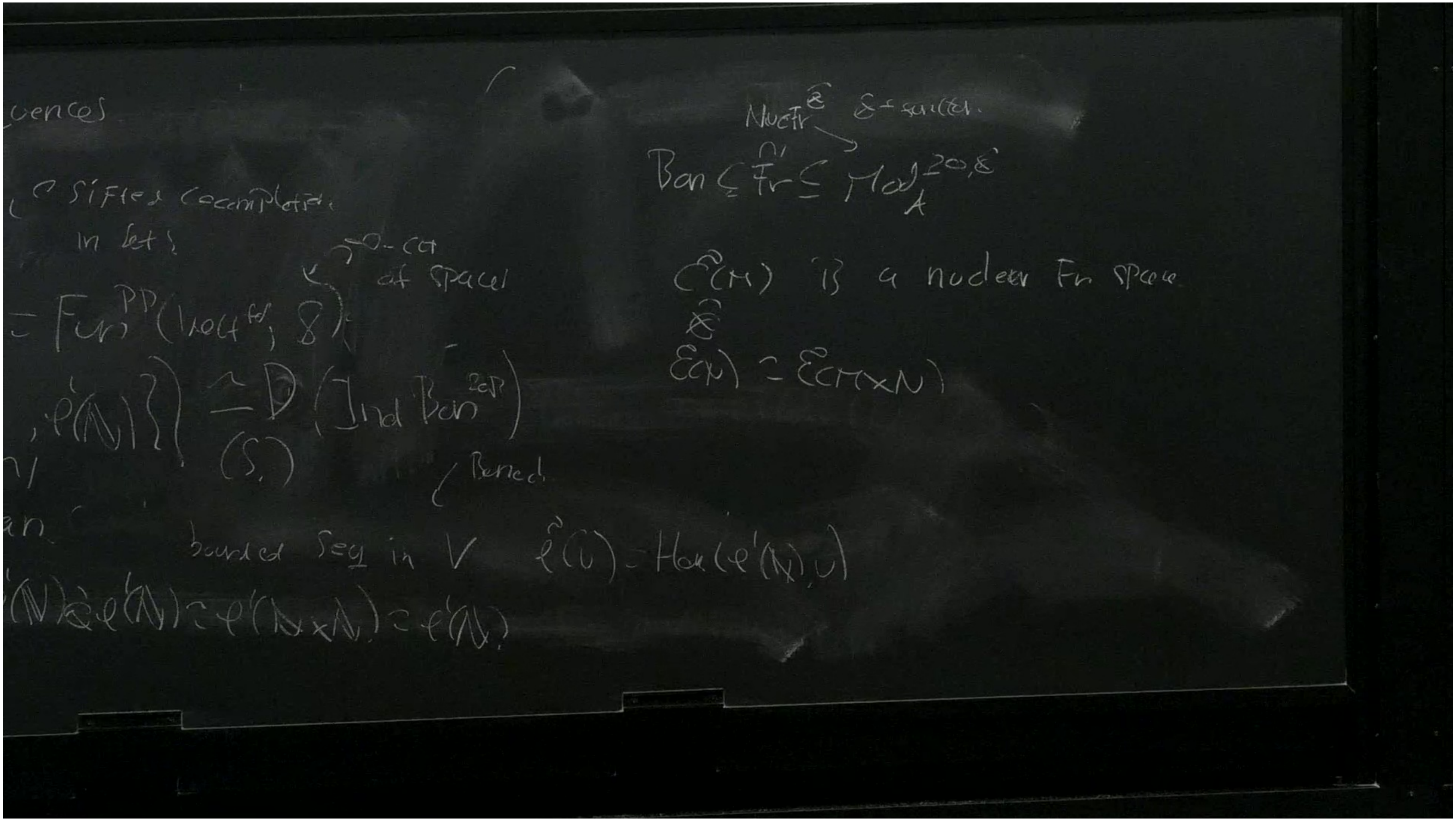
$\mathrm{Mod}^{\geq 0}_{A} = \mathcal{P}_{\Sigma}(\{\mathbb{R}^0, \mathbb{R}^1, \ldots, \ell^1(\mathbb{N})\}) \simeq \mathcal{D}_{(\geq 0)}(\mathrm{Ind}\,\mathrm{Ban}^{\mathrm{conv}})$ ← Bornol.

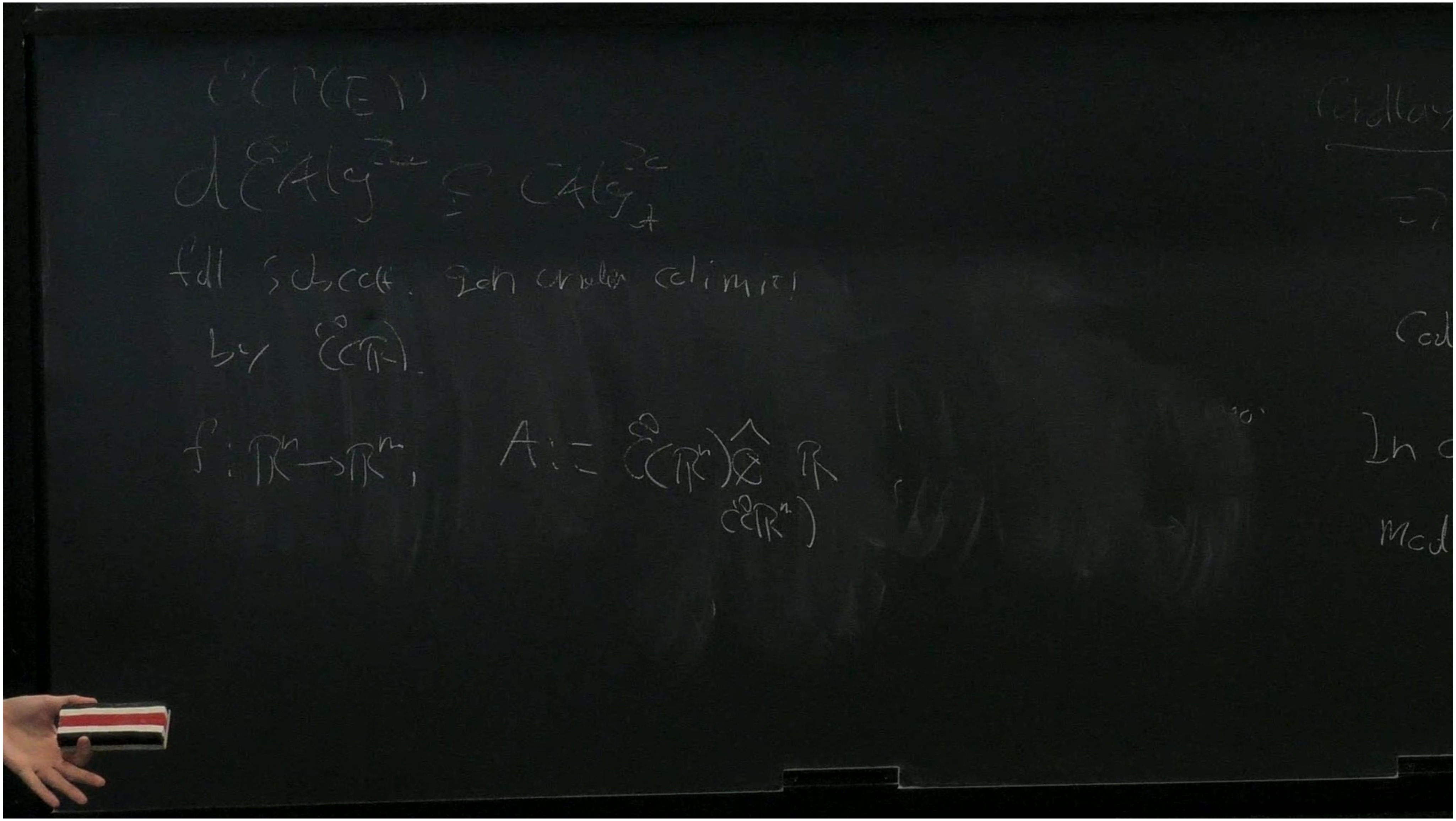
↑ ∞-cat of aggregated vector spaces

$\cap$ Ban

bounded seq in V: $\hat{\ell}(V) = \mathrm{Hom}(\ell^1(\mathbb{N}), V)$

$\ell^1(\mathbb{N}) \hat{\otimes} \ell^1(\mathbb{N}) = \ell^1(\mathbb{N} \times \mathbb{N}) = \ell^1(\mathbb{N})$


uences
sifted cocompletion
in set
Ban ⊆ Fr ⊆ Mod
nuclear Fr space
bounded seq in V
Bornol.


C^∞(P(E))
dC^∞Alg ⊆ CAlg
full subcat. gen under colimits
by C^∞(R^n).
f: R^n → R^m,
A := C^∞(R^n) ⊗ R
C^∞(R^m)

$d\mathcal{C}^\infty Alg^{\geq -} \subseteq CAlg^{\geq c}_{st}$

full subcat. gen under colimits by $C^\infty(\mathbb{R}^n)$.

$f : \mathbb{R}^n \to \mathbb{R}^m$, $A := C^\infty(\mathbb{R}^n) \hat{\otimes}_{C^\infty(\mathbb{R}^m)} \mathbb{R}$

Glue Spec A in an ∞-cat.

$Top^{loc}(d C^\infty Alg^{\geq -})$

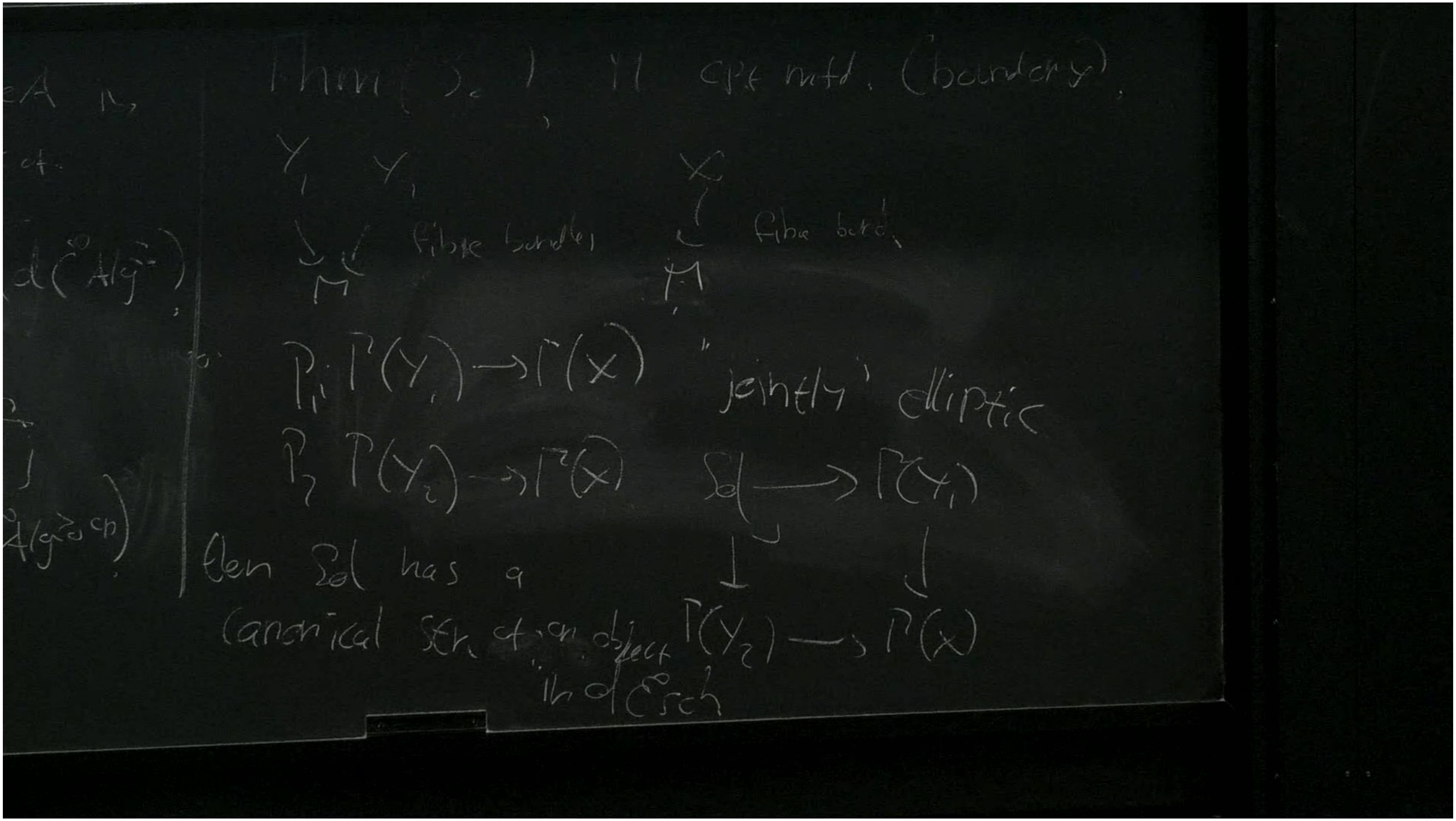
$\cup$

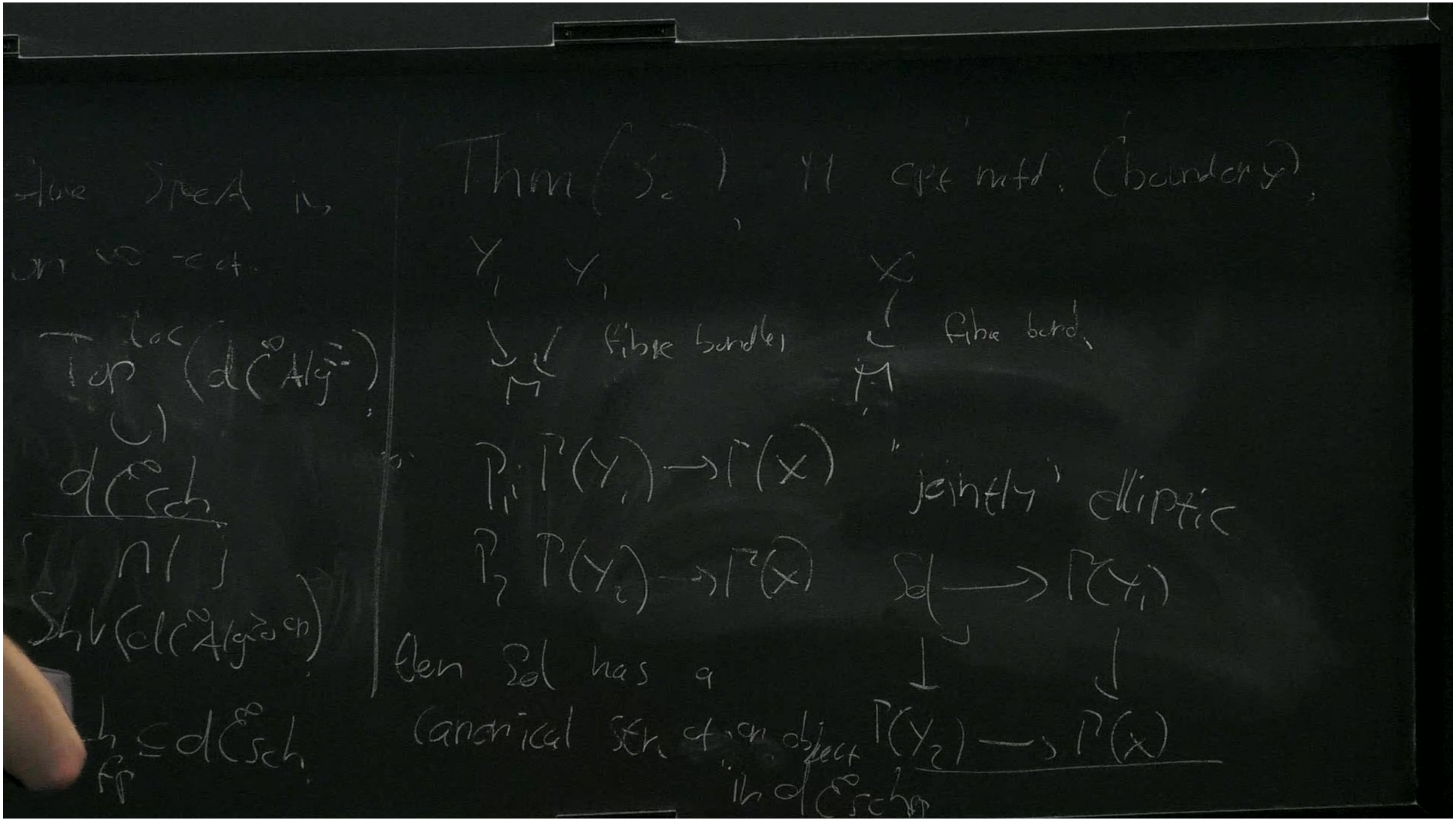
$d C^\infty Sch$

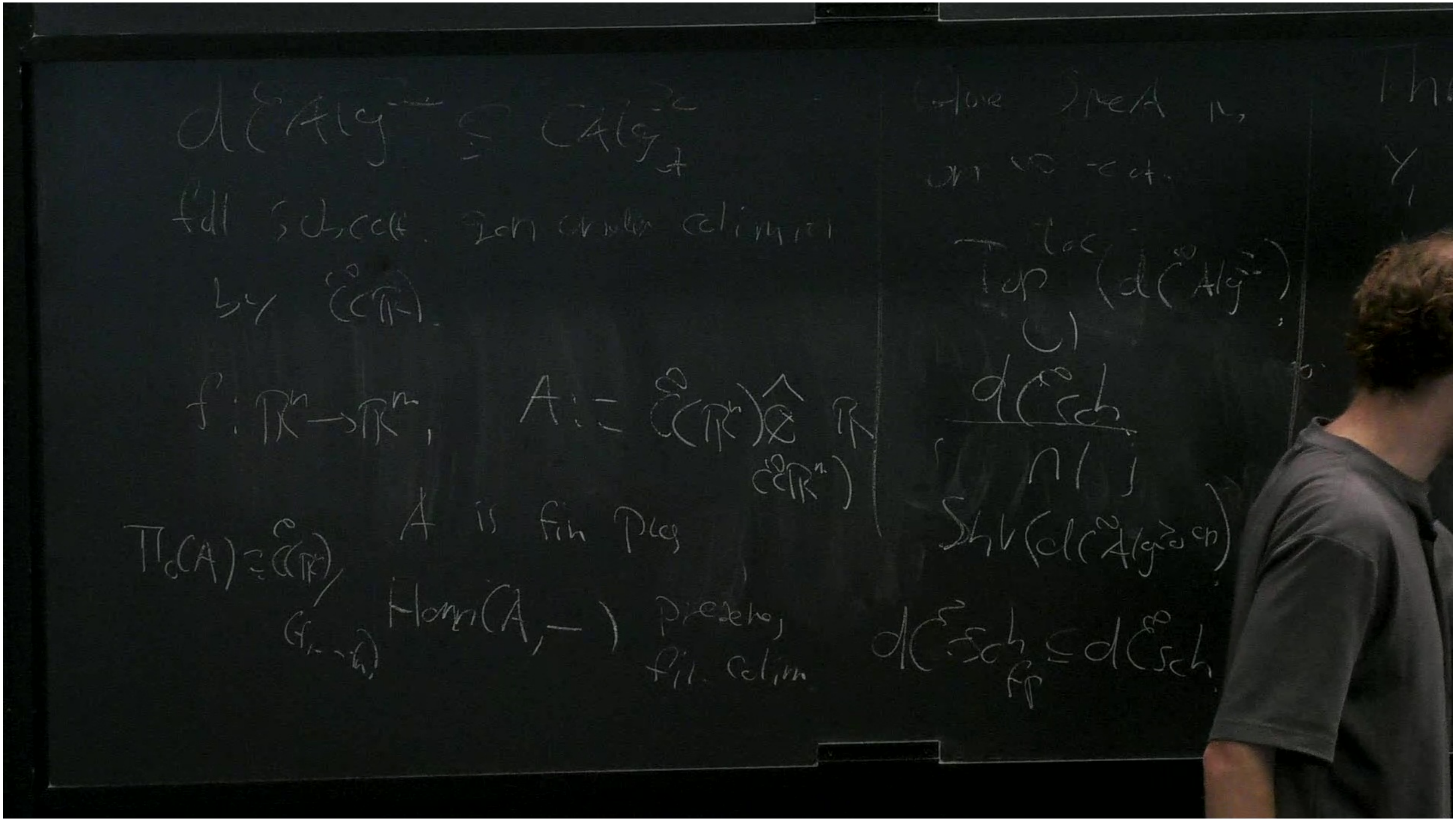
$\cap$

$Shv(dC^\infty Alg^{\geq 0\ op})$

Th


Thm (S.) M cpt mfd. (boundary)
Y1 Y2 X
fibre bundles
fibre bund.
M M
P1: Γ(Y1) → Γ(X)
P2: Γ(Y2) → Γ(X)
"jointly" elliptic
Sol → Γ(Y1)
Γ(Y2) → Γ(X)
then Sol has a
canonical str. of an object
in dEsch


Thm
cpt mfd. (boundary)
fibre bundles
fibre bund.
"jointly" elliptic
then Sol has a
canonical str. of an object
in dEsch


full subcat. gen under colimits
by C∞(R^n)
A is fin pres
Hom(A, –) preserves fil. colim
d C∞Sch_fp ⊆ d C∞Sch

Rmk [WIP w/ J. Pardon]

there is a "derived geometry" of "log-smooth" mfd's w boundary (corners) for which similar representability thm holds ⇒ X symp. mfd, J compatible almost cplx

$\overline{\mathcal{M}}(X, J)$ has a canonical structure of dlog Esd.

∃ a functor

$dCAlg^{\geq 0}_{\mathbb{R}/\!/\mathbb{R}} \xrightarrow{\mathbb{L}} Ch(Ve$

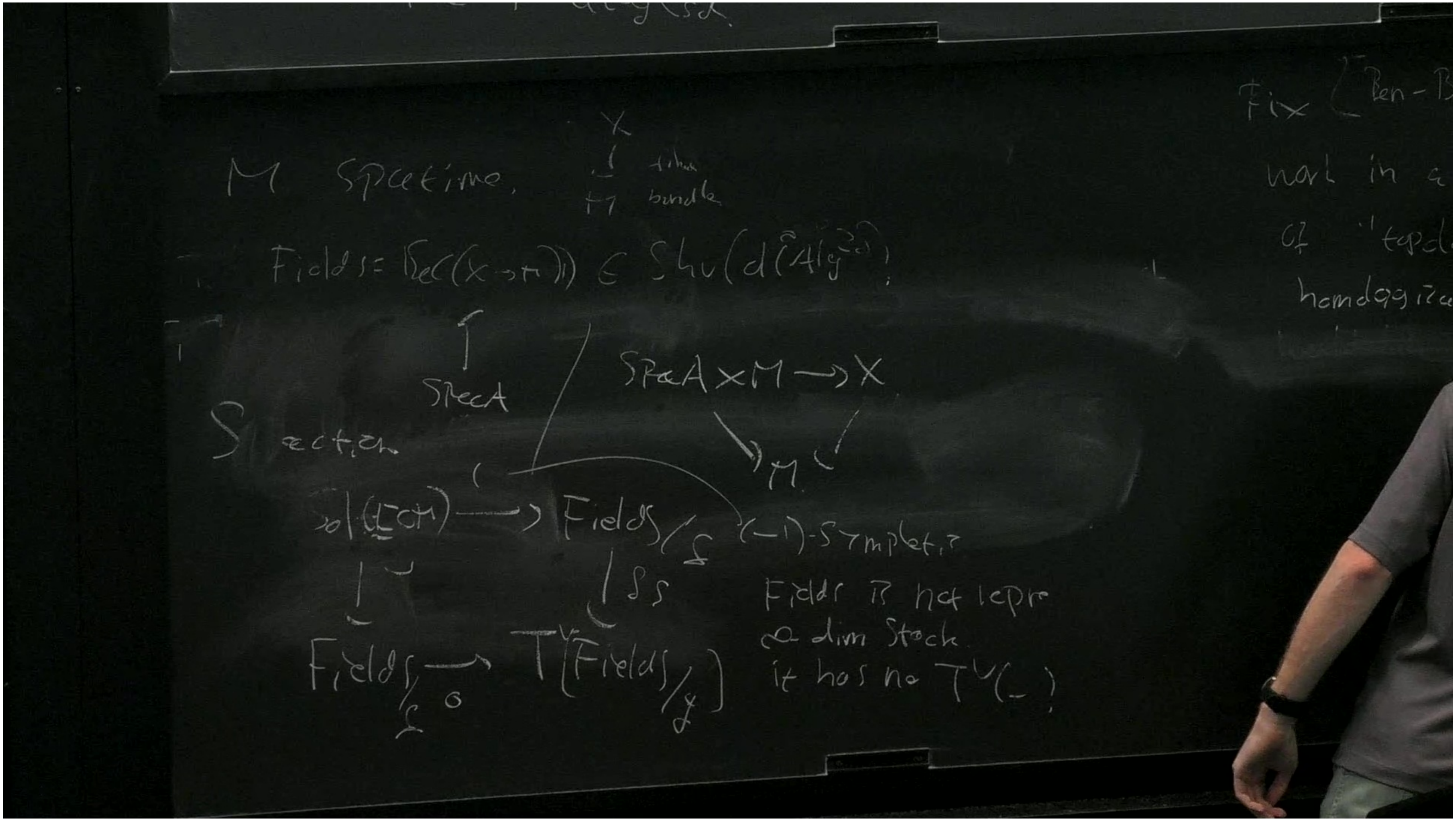
$\chi(\mathbb{L}_{A,P})$

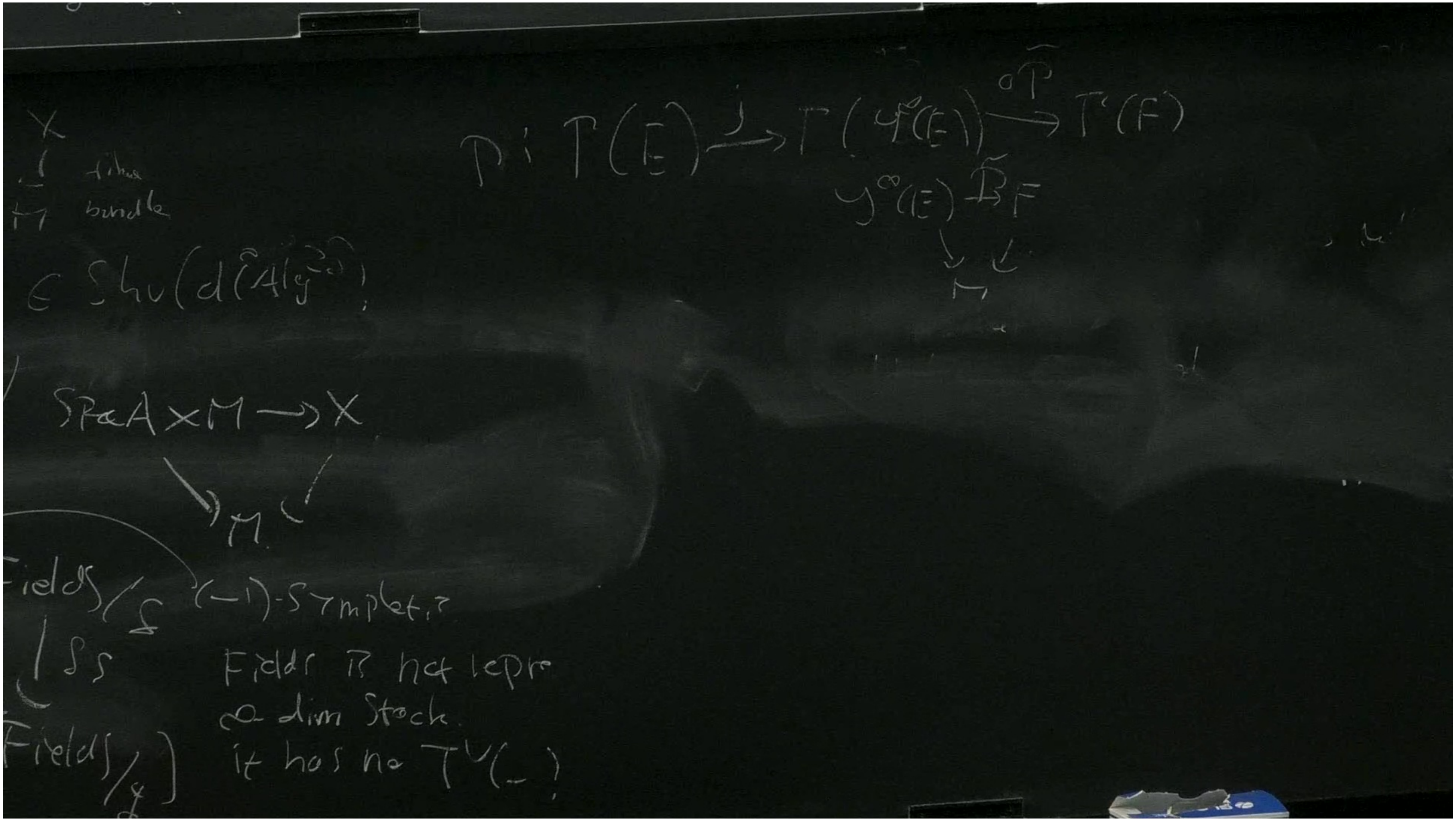
Glue these Spec A

∞-cat $Top^{loc}(dC$

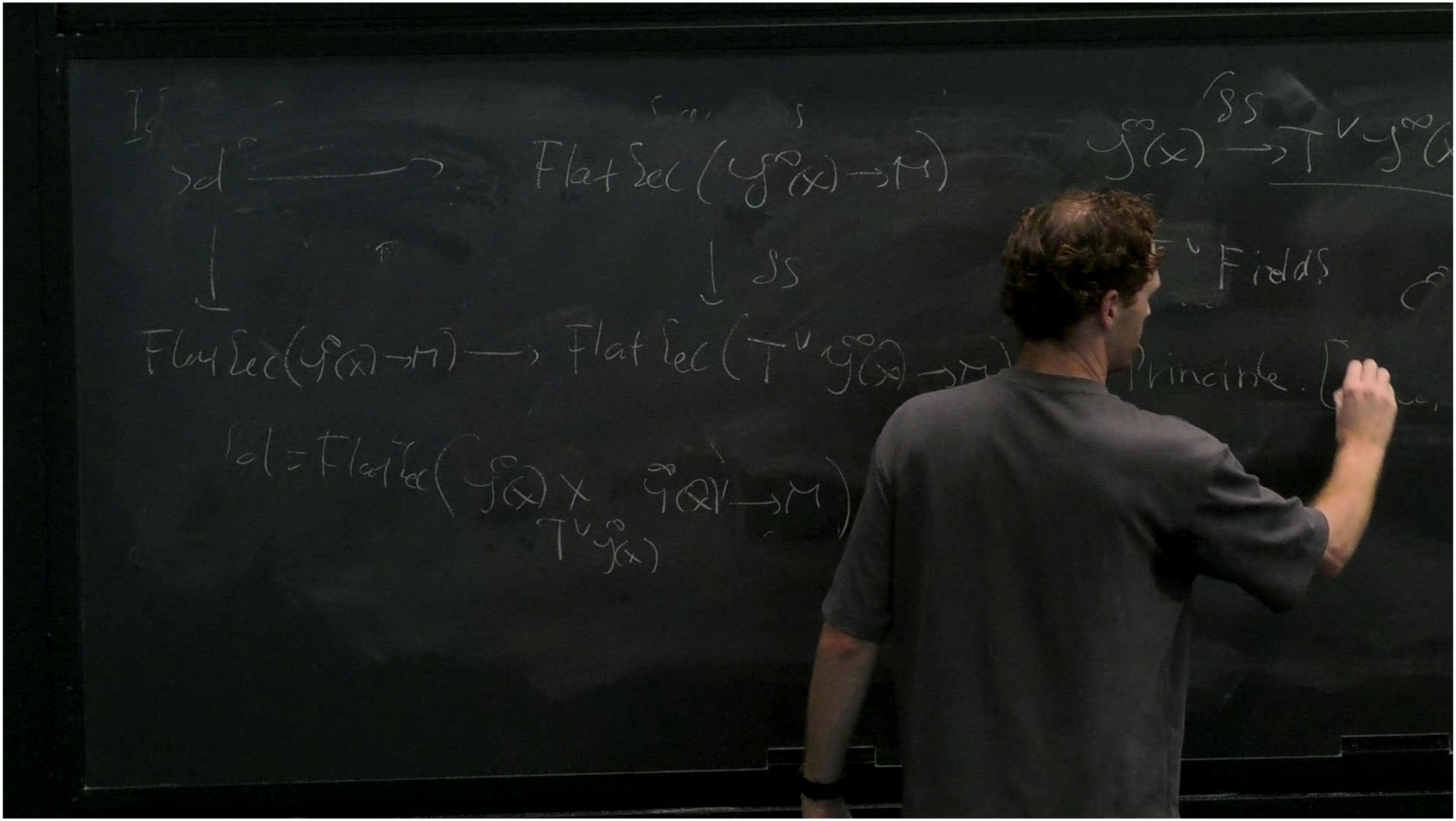
∪| ∋

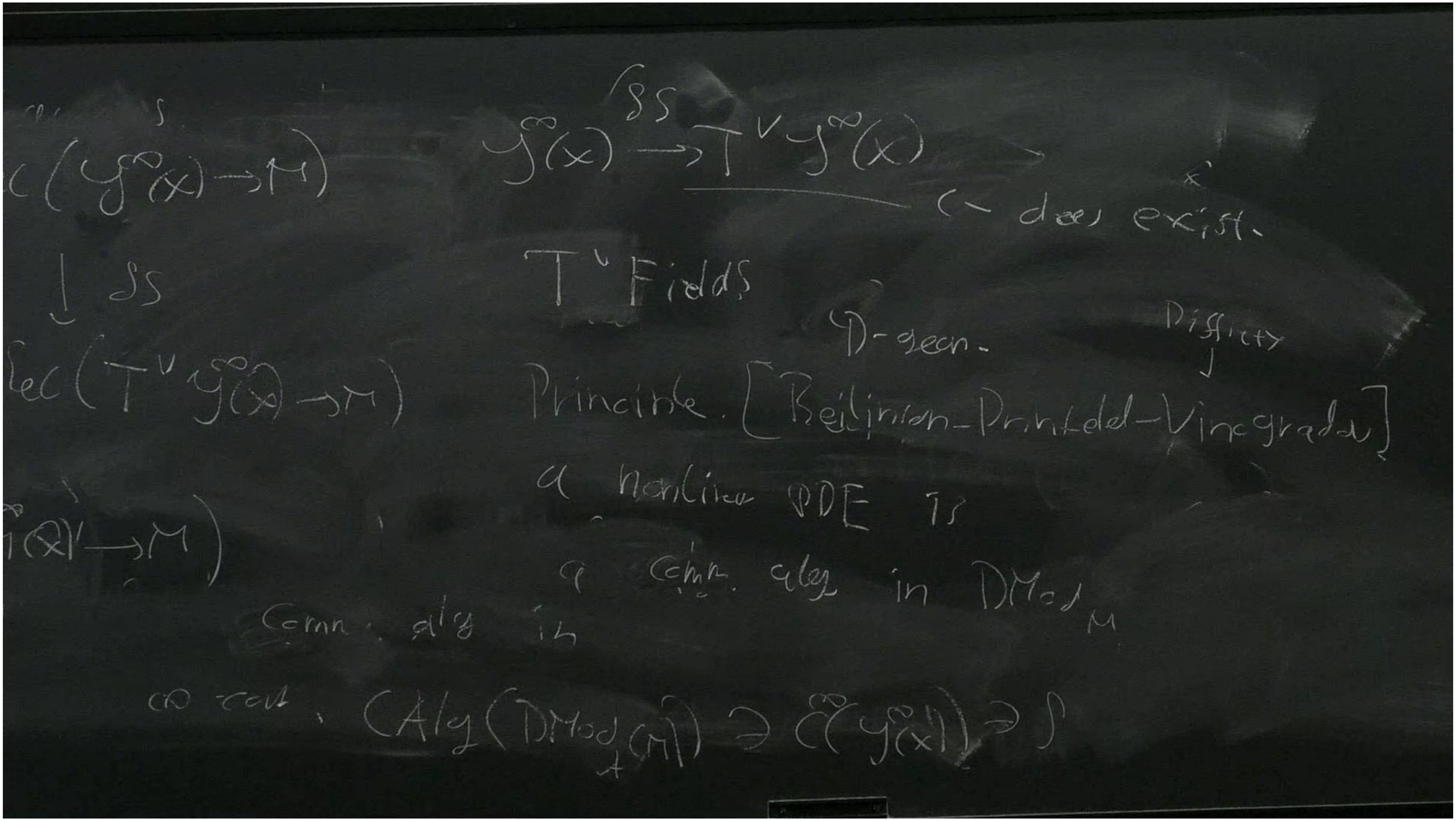
dSch (X

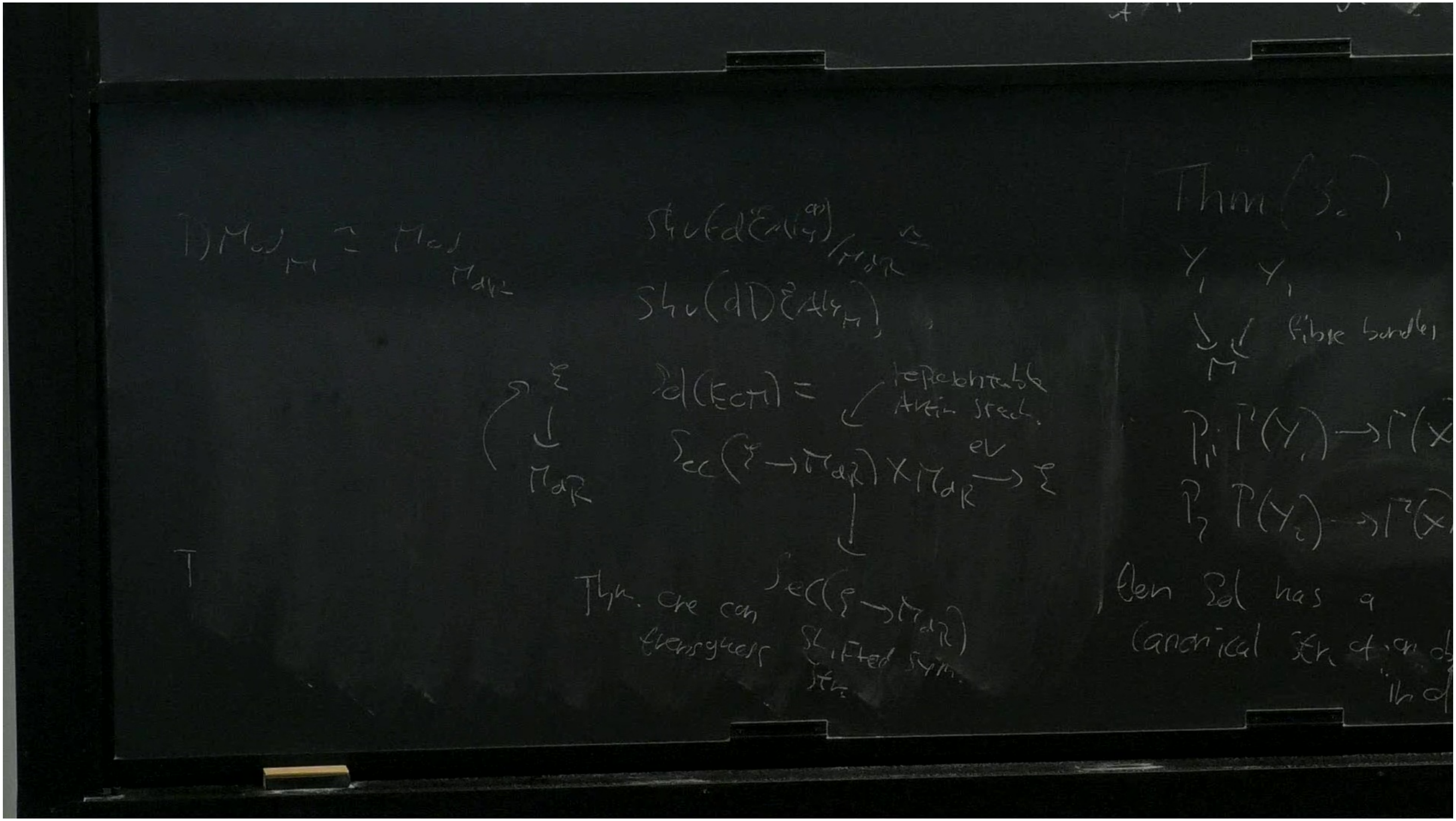

M spacetime,
bundle
SpecA × M → X
SpecA
action
Fields
(−1)-symplectic
Fields is not repr
it has no



X
fiber
bundle
M
Spec A × M → X
M
Fields/S
(−1)-symplectic
Fields is not repr
∞ dim stack
it has no T^∨(−)
Fields/S
P : Γ(E) → Γ(J^∞(E)) → Γ(F)
J^∞(E)
F
M


Flat Sec
Fields
Principle


$\mathcal{Y}^\infty(X) \to T^\vee \mathcal{Y}^\infty(X)$
← does exist.
$T^\vee$ Fields
D-geom.
Difficulty
Principle. [Beilinson–Drinfeld–Vinogradov]
a nonlinear PDE is
a comm. alg in $DMod_M$
comm. alg in
$CAlg(DMod(M)) \ni \mathcal{E}(\mathcal{Y}^\infty(X)) \ni S$


Thm
fibre bundles
representable
ev
has a
canonical
Shifted