Title: The dynamics of dRGT massive gravity

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Collection/Series: Strong Gravity

Subject: Strong Gravity

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Abstract:

After reviewing the motivation and challenges connected with the dRGT theory of ghost-free massive gravity, we discuss our recent progress in understanding non-linear dynamics of this model. In spherical symmetry, numerical studies suggest the formation of naked singularities during gravitational collapse of matter. Analytically, the same can be seen in the limit where the graviton mass is much smaller than the scales of the matter present. Both of these results underline the need to move beyond spherical symmetry to try and obtain realistic predictions. To that end, we present a new 'harmonic-inspired' formulation of the minimal model and argue that it is well-posed, opening the door to full 3+1 numerical simulations.

Dynamics of dRGT massive gravity

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Based on work with Emma Albertini, Toby Wiseman, Claudia de Rham and Andrew Tolley

Outline

1. Motivation
2. Massive gravity crash-course
3. Dynamical formulation and symmetric dynamics
4. Overview of well-posedness
5. The new strongly hyperbolic formulation
6. Corollaries, conclusions, future directions

Motivation

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Modifying gravity

- GR has been good to us.
- A lot of interest in modifications.
- Mostly UV!
- Adding a mass is the most natural IR modification.
- Despite considerable efforts, poorly understood.

Simulating gravity

- Gravity = difficult
- Strong gravity = more difficult
- Strong modified gravity = even more difficult!
- Need numerics for comparison with GR
- How can we be sure we can trust the numerics?

Overview of massive gravity

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Massive gravity 101

[de Rham, 2014]

$$\mathscr{L}_{GR} = -\frac{1}{4} h^T \mathscr{E}_{Lich} h$$

Unique stable mass term ↓ Breaks gauge invariance

$$\mathscr{L}_{MG} = \mathscr{L}_{GR} - \frac{1}{8} m^2 (h_{\mu\nu}^2 - [h]^2)$$

[Fierz & Pauli, 1939]

Problem I

[van Dam & Veltman 1970; Zakharov 1970]

$$-\frac{1}{2}\left(\nabla^2 - m^2\right) h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} = T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}[T]$$

vs

discontinuity!

$$-\frac{1}{2}\nabla^2 h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}[T]$$

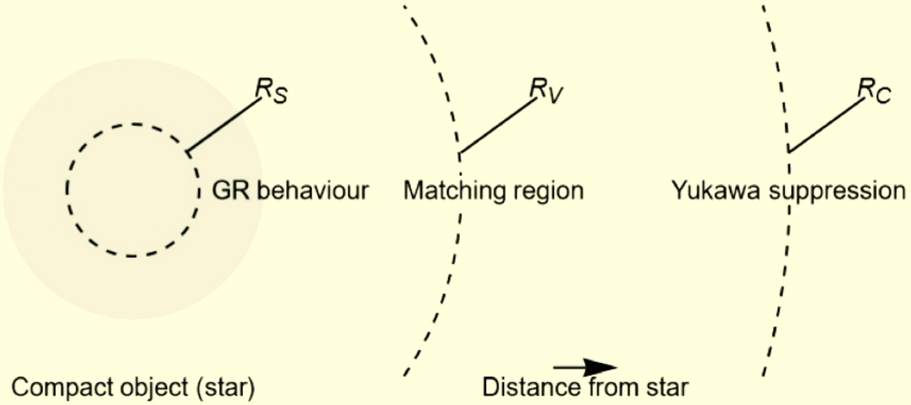
Where $\xi_\mu = \partial^\rho h_{\rho\mu} - \frac{1}{2}\partial_\mu[h]$

Solution I

[Vainshtein 1972]

$$[h] = -\frac{2}{3m^2}[T]$$

- Linear theory breaks down in the small-mass limit.
- Have to account for nonlinearities!



Problem II

[Boulware & Deser 1972]

Suppose we have some non-linear extension

$$G_{\mu\nu} + m^2 M_{\mu\nu} = T_{\mu\nu}$$

Then taking the divergence we get 4 constraints

$$\nabla^{\mu} M_{\mu\nu} = 0$$

For a total of 6 dofs. But we want 5! The 6th is a **ghost**.

Solution II

[de Rham, Gabadadze & Tolley 2011&2012; Hassan & Rosen 2011; …]

Must use the **dRGT mass term**

$$\mathscr{L}_{dRGT} = \frac{m^2}{4}\sqrt{-g}\sum_{n=0}^{4}\beta_n \mathscr{L}_n[E]$$

$$E^{\mu}{}_{\alpha}E^{\alpha}{}_{\nu} = f^{\mu}{}_{\nu} = g^{\mu\alpha}f_{\alpha\nu}$$

Where $f_{\mu\nu}$ is some fixed Lorentzian 'reference metric' and $\mathscr{L}_n$ are symmetric polynomials in the eigenvalues of E.

Equations of motion

With these restrictions, our equations of motion become

$$\mathscr{E}_{\mu\nu} \equiv G_{\mu\nu} + m_{(1)}^2 M_{\mu\nu}^{(1)} + m_{(2)}^2 M_{\mu\nu}^{(2)} - T_{\mu\nu} = 0$$

$$M_{\mu\nu}^{(1)} = -E_{\mu\nu} + [E]g_{\mu\nu} - 3g_{\mu\nu}$$

$$M_{\mu\nu}^{(2)} = \frac{1}{2}E_{\mu\alpha}E^{\alpha}{}_{\nu} - \frac{1}{2}[E]E_{\mu\nu} - \frac{1}{4}\left([E^2] - [E]^2\right)g_{\mu\nu} - \frac{3}{2}g_{\mu\nu}$$

Conventions for this talk

- We take a **flat reference metric**.
- The metric g is **asymptotically flat** (in particular there is no cosmological constant).
- Working **directly with the vierbein**.
- Eliminate the (unstable) ‘cubic’ mass term.

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$$M_{\mu\nu}^{(1)} = -E_{\mu\nu} + [E] g_{\mu\nu} - 3 g_{\mu\nu}$$ ⟵ Minimal model

$$M_{\mu\nu}^{(2)} = \frac{1}{2} E_{\mu\alpha} E^{\alpha}{}_{\nu} - \frac{1}{2} [E] E_{\mu\nu} - \frac{1}{4} \left([E^2] - [E]^2 \right) g_{\mu\nu} - \frac{3}{2} g_{\mu\nu}$$

Vector constraint

[de Rham, JK, Tolley & Wiseman 2023]

Taking the divergence of the EoM gives 4 constraints:

$$V_\mu = \nabla^\alpha \mathscr{E}_{\alpha\mu} = \nabla^\alpha M_{\alpha\mu}$$

With a slight redefinition, this can be written as

$$0 = \xi^\mu \equiv E^\mu{}_\alpha \eta^{\alpha\beta} V_\beta = V^{\mu\nu\alpha\beta} \partial_{[\nu} E_{\alpha]\beta}$$

For some known expression $V^{\mu\nu\alpha\beta}$.

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For some known expression $V^{\mu\nu\alpha\beta}$.

$$\nabla^{(f)}_{[\nu} E_{\alpha]\beta}$$

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Scalar constraint

[de Rham, JK, Tolley & Wiseman 2023]

There is one further constraint, given by

$$0 = \mathcal{S} \equiv \frac{1}{2}\left(m_{(1)}^2 g^{\mu\nu} + m_{(2)}^2 E^{\mu\nu}\right)\mathscr{E}_{\mu\nu} + \nabla \cdot \xi$$

Which can be written as

$$0 = A^{\alpha\beta\gamma\mu\nu\rho}\partial_{[\alpha}E_{\beta]\gamma}\partial_{[\mu}E_{\nu]\rho} + [\textit{trace terms}]$$

Where A is a little ugly, but known analytically.
We see that the above has no time derivatives of E_{tt} at all, so it is indeed a constraint. This is how the ghost is avoided.

First look at dynamics

Derivative structure

[de Rham, JK, Tolley & Wiseman 2023]

Set $m_{(2)} = 0$. Then terms containing derivatives in $\mathcal{S}$ are

$$0 = \mathcal{S} \sim R + 2\nabla \cdot \xi$$

So the action is equivalent to

$$S = \int d^4x \,|\det E|\left(-\frac{1}{2}A_{(1)}^{\alpha\beta\gamma\mu\nu\rho}K_{\alpha\beta\gamma}K_{\mu\nu\rho} - \ldots\right)$$

$$K_{\mu\nu\rho} = \partial_\mu E_{\nu\rho} - \partial_\nu E_{\mu\rho}$$

Derivative structure

[de Rham, JK, Tolley & Wiseman 2023]

We use $K_{\mu\nu\rho}$ as our ‘momenta’.

Then the EoM have to be expressible in the form

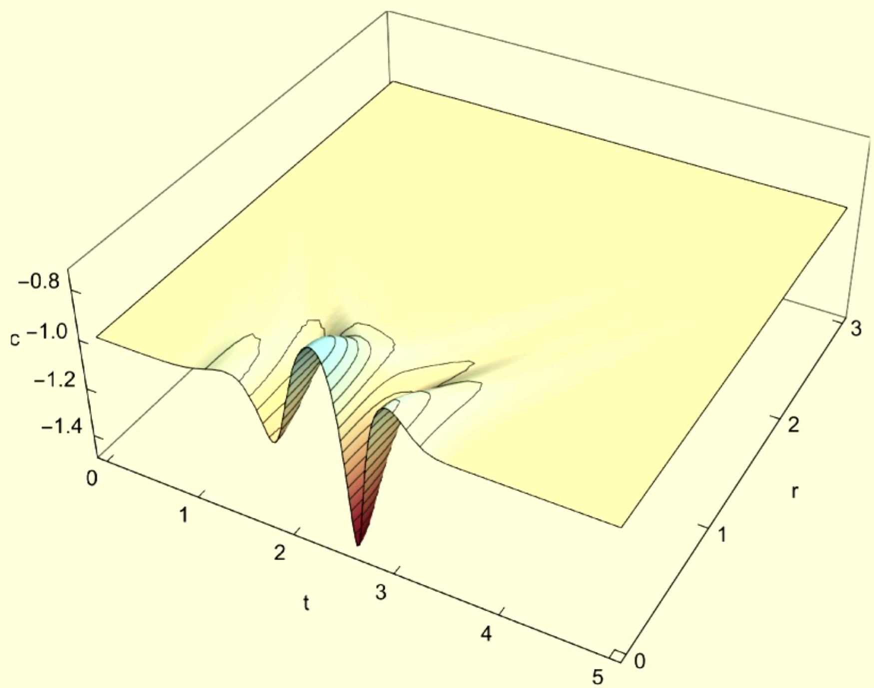
$$\mathscr{E} = \#\partial K + \#K\partial E + K^T \#K + \dots$$

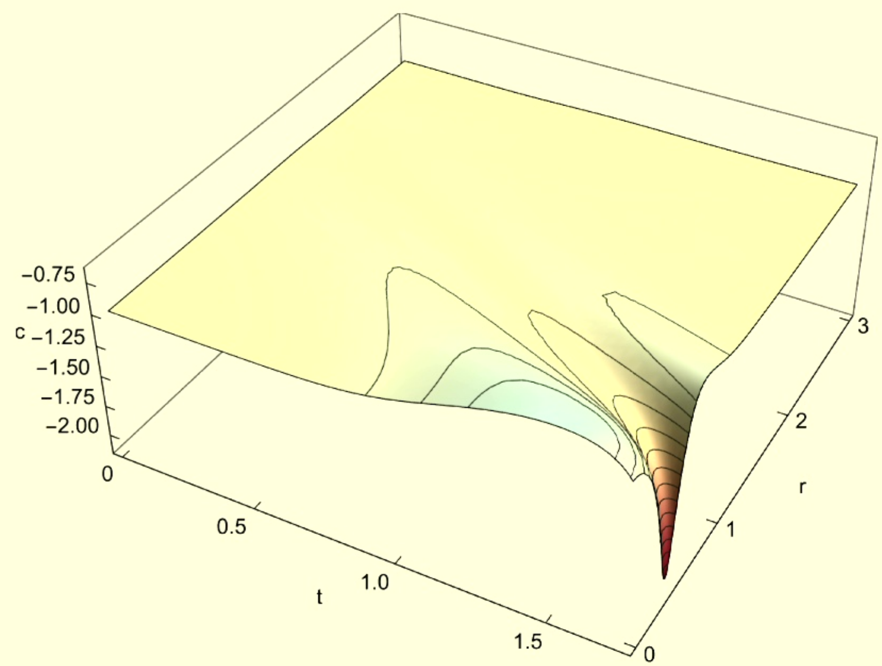
And now all the derivatives appear linearly!

Spherical collapse

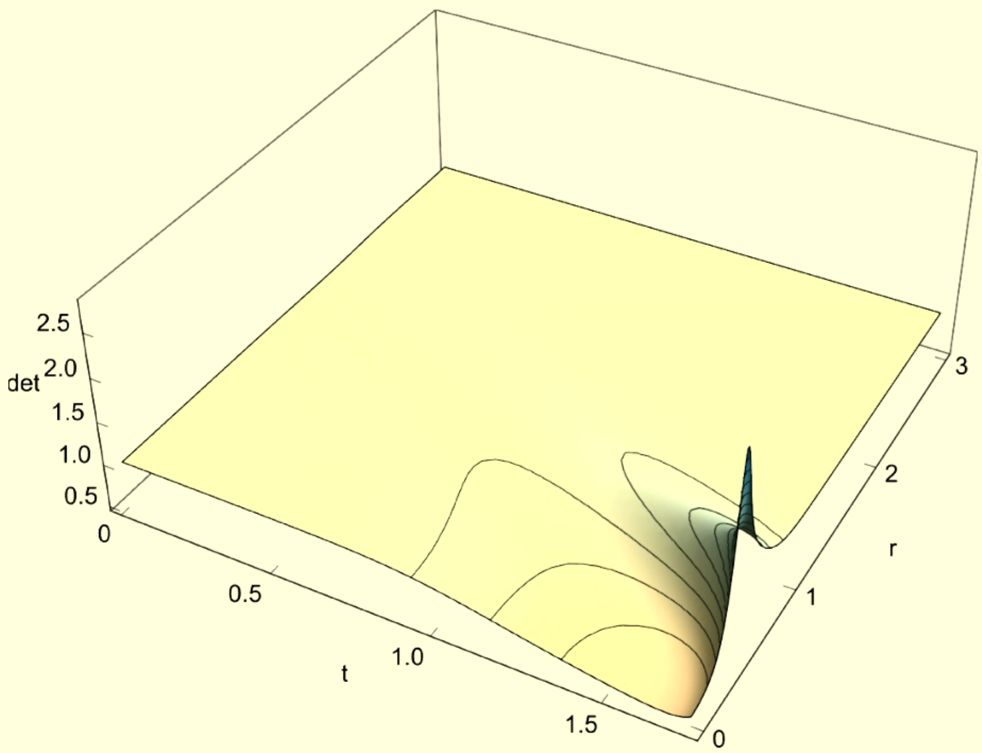
- Couple to massless scalar field.
- Linear data behaves fine.
- Singularities form with when more matter included.
- Vacuum gravity also dynamical
- No horizons!
- Some analytic understanding of the issues.

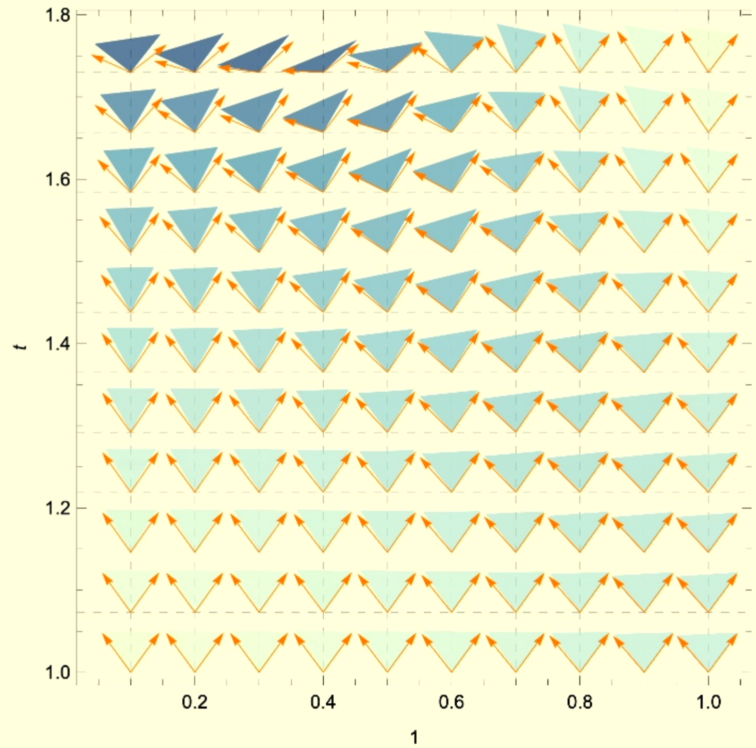
Slide with pictures 1





Slide with pictures 2





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Well-posedness

Initial value problems

something $\xrightarrow[\textit{time}]{}$ something else

OR

$$u_t = \mathscr{L}(u, \partial u, \ldots)$$

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Initial value problems

1. Does the solution exist?

2. Is it unique?

3. Does it depend continuously on initial data?

(1) + (2) + (3) = Well-posedness

Hyperbolicity intro

[Papallo & Reall, 2017]

Consider a first-order pde, linearised about a background

$$Au_t + P^i \partial_i u + Cu = 0$$

FT gives the formal solution

$$u = \frac{1}{(2\pi)^{d-1}} \int d^{d-1}k \exp(-ik_i x^i) \times \exp(i\mathcal{M}(k_i)t) \times \hat{u}(0,k_i)$$

$$\mathcal{M}(k_i) \equiv A^{-1}(-P^i k_i + iC)$$

Hyperbolicity intro

Integral might not converge as $|k| \to \infty$

To ensure it does we need a bound

$$\exp(i\mathscr{M}(k_i)t) \leq f(t)$$

Or, in the high-frequency limit

$$\forall t, \hat{k}_i \quad \exp\left(iM\left(\hat{k}_i\right)t\right) \leq const$$

$$M\left(\hat{k}_i\right) \equiv -A^{-1}P^i\hat{k}_i$$

Hyperbolicity and eigendecomposition

How to ensure a bound exists?

$$Mv = (\lambda_1 - i\lambda_2)v \implies \exp(iMt)v = e^{i\lambda_1 t}e^{\lambda_2 t}v$$

So the eigenvalues must be real (weak hyperbolicity).

$$M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \implies \exp(iMt) = e^{i\lambda t}\begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}$$

So the matrix must be diagonalisable.

Hyperbolicity

The system which at a point linearises to

$$Au_t + P^i \partial_i u + Cu = 0$$

is **strongly hyperbolic** (at this point) if

$$M\left(\hat{k}_i\right) \equiv -A^{-1} P^i \hat{k}_i$$

is diagonalisable with real eigenvalues for all unit vectors $\hat{k}$.

Hyperbolicity

This is sufficient for us in the sense that

(Strong) hyperbolicity $\Longrightarrow$ well-posedness $\Longrightarrow$ sensible numerics

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Hyperbolicity is Not Everything

- Famously shown for GR in [Choquet-Bruhat 1952]
- But full numerics had to wait until [Pretorius, 2005]
- Strongly formulation-dependent (not physical)
- 'Irrelevant' from EFT point of view
- Still it is foundational - lot of research into modified gravities:

[Cayuso, Ortiz & Lehner 2017; Allwright & Lehner 2018; Gerhardinger, Giblin, Tolley & Trodden 2022; Saló, Clough & Figueras 2023; de Rham, JK, Tolley & Wiseman 2023; **Figueras, Held & Kovács 2024**; …]

Hyperbolicity example

Consider a system of two wave equations

$$\ddot{\phi} - \partial_x^2 \phi = 0, \quad \ddot{\psi} - \partial_x^2 \psi = 0$$

Define the variables vector

$$u = \left(\pi_t \equiv \dot{\phi}, \pi_x \equiv \phi', \phi, \Pi_t \equiv \dot{\psi}, \Pi_x \equiv \psi', \psi\right)$$

Then a simple computation gives

$$\text{Jordan}[M] = \text{diag}\,(+1, +1, -1, -1, 0, 0)$$

And the system is strongly hyperbolic.

Hyperbolicity example

But now instead define

$$u = \left(\pi_t \equiv \dot{\phi}, \pi_x \equiv \phi', \phi, \Pi_t \equiv \dot{\psi} - \phi', \Pi_x \equiv \psi', \psi\right)$$

Then we find

$$\text{Jordan}[M] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hyperbolicity example moral

- The hyperbolicity of the system in the example depended on the way we extended it to first order.
- The ‘physical’ eigenvalues and corresponding Jordan blocks remained unchanged in both examples.
- The two systems are equivalent provided constraints on initial data are obeyed.

A hyperbolic first-order formulation

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2410.19491 - out now!!

What can we hope for?

1. Establish hyperbolicity in flat space.
2. Hope things behave well as one deforms background.
3. Result in ‘some open neighbourhood’ of Minkowski.

Conventions for this section

- Take **Minkowski coordinates**,

$$f_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$$

- We're in the **vacuum**.
- Finally we restrict to the **minimal model**.

Harmonic formulation for GR

Modify the Einstein equations as

$$0 = R_{\mu\nu} - \nabla_{(\mu} v_{\nu)}$$

$$v^\alpha = g^{\mu\nu}\left(\Gamma^\alpha_{\mu\nu} - \bar{\Gamma}^\alpha_{\mu\nu}\right)$$

Easily shown to be hyperbolic, while v_α obeys

$$\nabla^2 v_\alpha + R_\alpha^{\ \beta} v_\beta = 0$$

4 'constraint-violating' dofs that remain zero if they vanish at first.

Harmonic formulation for dRGT

Try a similar trick:

$$\mathscr{E}^H_{\mu\nu} \equiv R_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)} + m^2\bar{M}_{\mu\nu} = 0$$

Then

$$\nabla^2\xi_\alpha + R_\alpha^{\ \beta}\xi_\beta = m^2\eta_{\alpha\beta}(E^{-1})^{\beta\gamma}\xi_\gamma$$

And the scalar constraint is given by

$$\mathscr{S} = -\,g^{\mu\nu}\mathscr{E}^H_{\mu\nu}$$

Harmonic formulation for dRGT

This preserves the nice derivative structure:

$$\mathscr{E}^H_{\mu\nu} = \mathscr{A}^{\sigma\alpha\rho\beta}_{\mu\nu}\partial_\sigma K_{\alpha\rho\beta} + \mathscr{B}^{\sigma\rho\delta\alpha\beta\gamma}_{\mu\nu} K_{\sigma\rho\delta} K_{\alpha\beta\gamma} + \mathscr{C}^{\sigma\rho\delta\alpha\beta\gamma}_{\mu\nu} K_{\sigma\rho\delta}\partial_{(\alpha} E_{\beta)\gamma} + \dots$$

Where the coefficients are known.

Note that the final term involves derivatives of E_{tt} which then couple back to ∂K via

$$\partial_t \mathcal{S} = \mathcal{S}_1 \partial_t E_{tt} + 2A^{\alpha\beta\gamma\mu\nu\rho} K_{\alpha\beta\gamma}\partial_t K_{\mu\nu\rho} + \mathcal{S}_2$$

This is what really makes the problem difficult.

Full dynamical system?

Notation:

$$E_{\mu\nu} = \begin{pmatrix} \phi & V_i \\ & e_{ij} \end{pmatrix}$$

Evolve:

- V_i, e_{ij} through the 'momenta' $P_i \equiv K_{tit}$ and $P_{ij} \equiv K_{tij}$.
- Spatial derivatives $Q_{ij\mu} \equiv K_{ij\mu}$ through consistency relations.
- Momenta through the harmonic Einstein equations.
- ϕ through $\partial_t \mathcal{S}$.

Full dynamical system?

Writing

$$u = \left(\phi, P_i, P_{ij}, V_i, e_{ij}, Q_{ij\mu}\right)$$

Where $ij \in \{xx, yy, zz, xy, yz, zx\}$ to preserve rotation symmetry, we get an initial value problem for 31 variables, of the form

$$0 = A[u]\partial_t u + P^i[u]\partial_i u + C[u]$$

Full dynamical system?

On a particular background $u = u_0 + \delta u$

$$0 = A[u_0]\partial_t \delta u + P^i[u_0]\partial_i \delta u + \ldots$$

Full dynamical system?

On a particular background $u = u_0 + \delta u$

$$0 = A[u_0]\partial_t \delta u + P^i[u_0]\partial_i \delta u + \dots$$

This is not hyperbolic even in flat space!

The problem is not 'too bad'

The problem is exactly analogous to our toy example, coming from

$$\begin{aligned} \partial_t e_{ij} &= P_{ij} + \partial_i V_j \\ \partial_t V_j &= P_j + \partial_j \phi \end{aligned} \implies \begin{aligned} \partial_t e_{xy} &= \partial_x V_y + \dots \\ \partial_t V_y &= \dots \end{aligned}$$

- It's **'not physical'**. Can be solved by just adding zero!
- Adds new dof's but they remain zero.
- Not diffusive.

Full dynamical system!

Extend the variables vector

$$u = \left(\phi, P_i, P_{ij}, V_i, e_{ij}, Q_{ij\mu}, \tilde{e}_{ij}\right)$$

Now have a system of 34 variables and 34 equations

$$0 = A[u]\partial_t u + P^i[u]\partial_i u + \dots$$

Must investigate the eigendecomposition of (wlog $\hat{k} = (1,0,0)$)

$$M[\hat{k}] = -A^{-1}[u] \cdot \hat{k}_i P^i[u]$$

Flat space dynamics

On a flat background, we set $\phi = -1$, $e_{ij} = \delta_{ij}$, other vars $= 0$.

Then M is diagonalisable with the following eigenvalues:

- 9 pairs of ± 1: 5 physical dofs + 4 'constraint-violating' modes.
- 3 pairs (2 degenerate) due to our fixing.
- 10 eigenvalues equal to zero.

Flat space dynamics

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- 3 pairs (2 degenerate) due to our fixing.
- 10 eigenvalues equal to zero.

How to move away from flat space?

What might go wrong?

1. A degenerate eigenvalue pair splits into complex conjugates.

$$\begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}$$

2. An eigenvector ceases to exist as an evalue becomes defective.

$$\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$$

Proof strategy

1. The Jordan block of the 0's remains unchanged.
2. The 3 'fixing' pairs are under control.
3. 6 dofs are governed by the inverse metric **on any background.**
4. The remaining 3 (± 1)-evalue pairs split already at linear order away from Minkowski.

Proof strategy

1. The Jordan block of the 0's remains unchanged.
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This works! Except possibly (4) in some non-generic directions.

Corollaries from the proof

- Spin-2 always stays on light cone.
- Spins 1 and 0 split away.
- Spin-1 modes are generically birefringent.
- The diffusion in old simulations was not necessary for WP.

What did we learn?

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Conclusions and future research

- Minimal model well-posed.
- Characteristics beyond perturbation theory?
- Extend to non-minimal.
- Actually write the simulation (available code?).
- Make a more definitive statement on viability of dRGT.
- The usual: extensions to different reference metrics, asymptotic.

Questions?