Title: Lecture - Relativity, PHYS 604

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Subject: Cosmology, Strong Gravity

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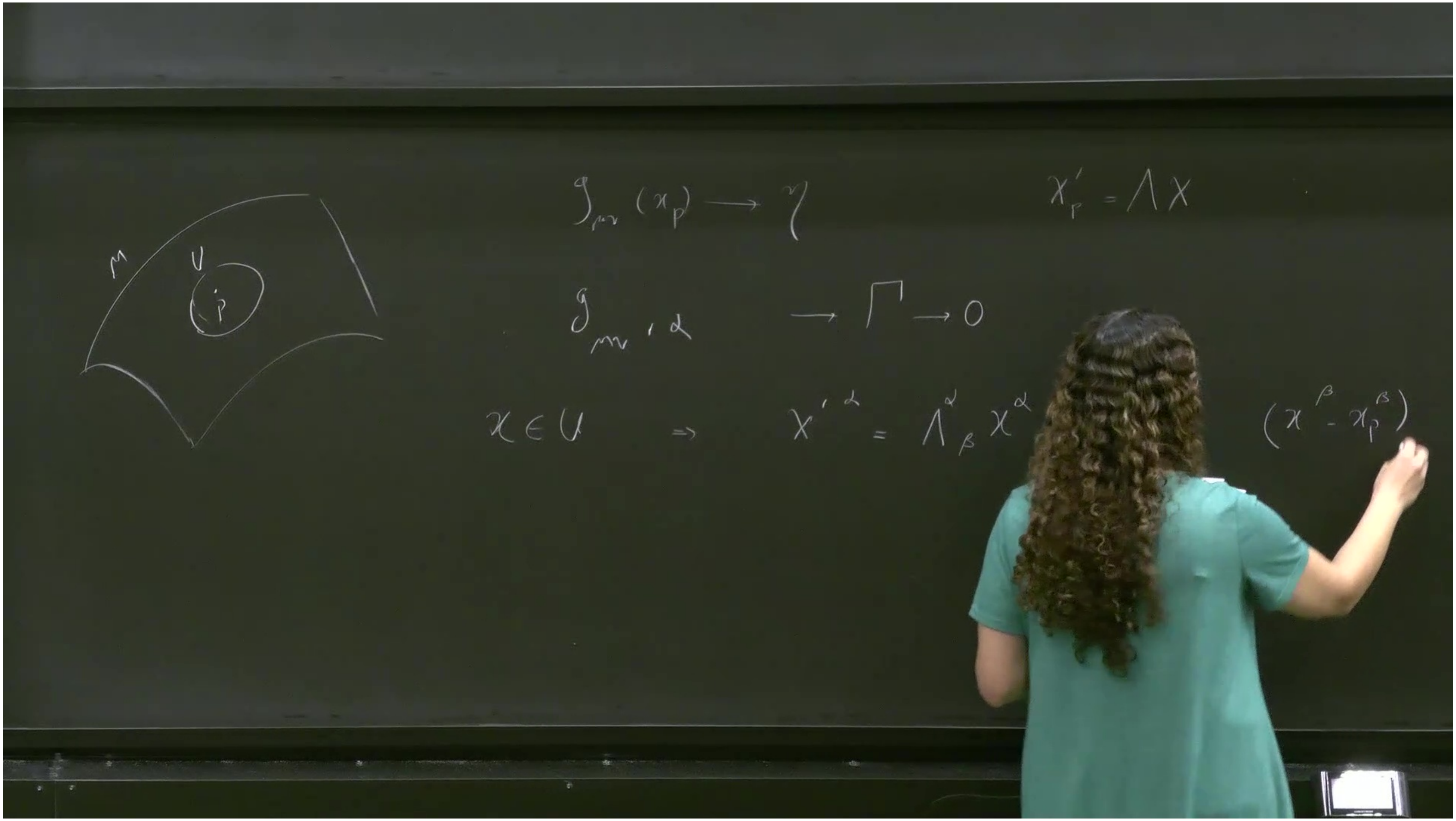
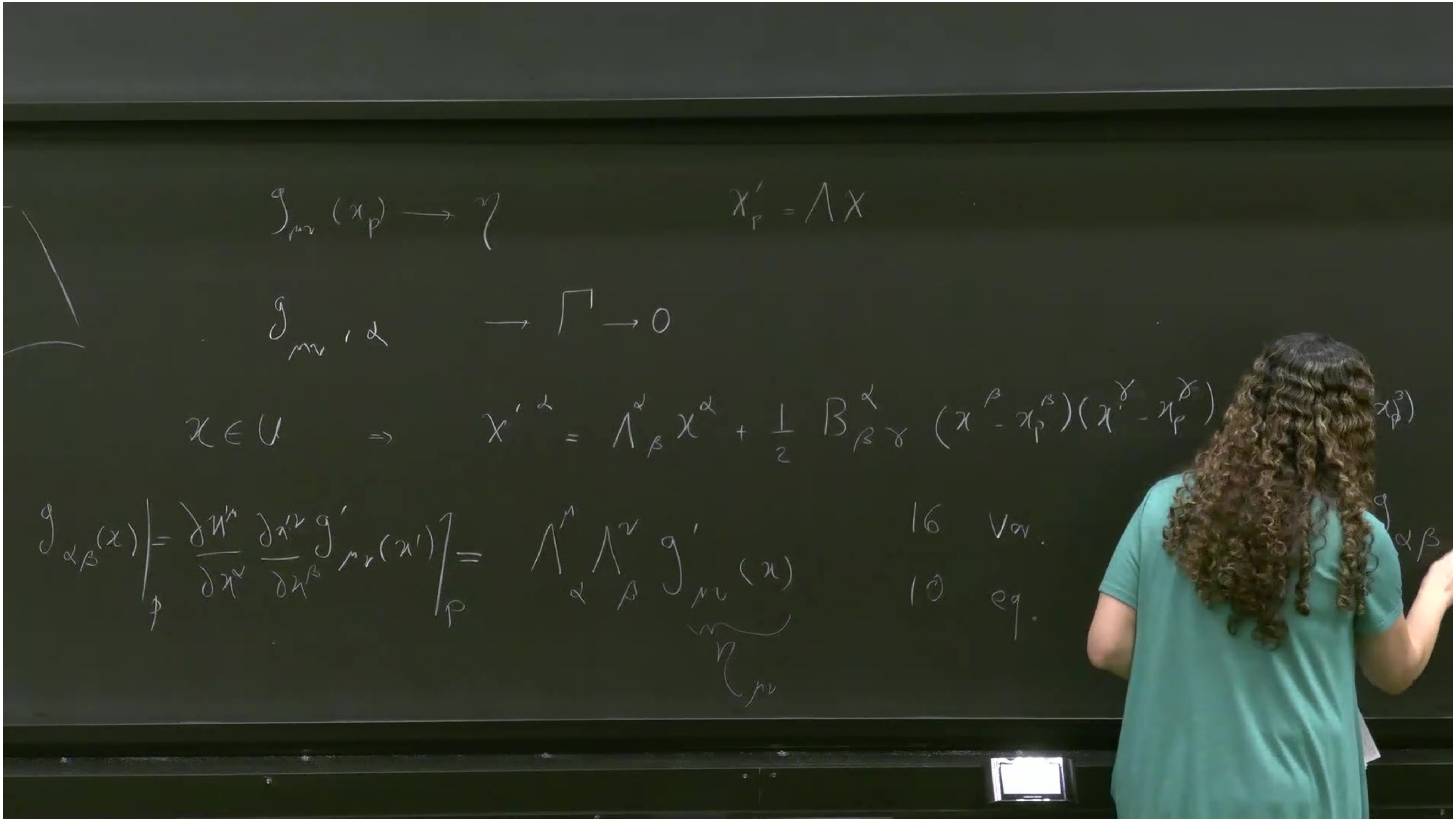
Abstract:


M
U
p
$g_{\mu\nu}(x_p) \longrightarrow \eta$
$x'_p = \Lambda X$
$g_{\mu\nu,\alpha} \longrightarrow \Gamma \longrightarrow 0$
$x \in U \Rightarrow x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, x^{\alpha}$
$(x^{\beta} - x_p^{\beta})$


$g_{\mu\nu}(x_p) \longrightarrow \eta$
$x'_p = \Lambda x$
$g_{\mu\nu,\alpha} \longrightarrow \Gamma \longrightarrow 0$
$x \in U \Rightarrow x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\alpha} + \frac{1}{2} B^{\alpha}_{\beta\gamma} (x^{\beta} - x_p^{\beta})(x^{\gamma} - x_p^{\gamma})$
$x_p^3)$
$g_{\alpha\beta}(x)\Big|_p = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g'_{\mu\nu}(x')\Big|_p = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \underbrace{g'_{\mu\nu}(x)}_{\eta_{\mu\nu}}$
16 var.
10 eq.
$g_{\alpha\beta}$

$$\ldots{}^{\alpha}_{\beta\gamma}\,(x^{\beta}-x_{p}^{\beta})(x^{\gamma}-x_{p}^{\gamma}) + C^{\alpha}_{\beta\gamma\sigma}\,(x^{\beta}-x_{p}^{\beta})(x^{\gamma}-x_{p}^{\gamma})(x^{\sigma}-x_{p}^{\sigma})$$

16 Var.

10 eq. ✓

$$g'_{\alpha\beta,\mu}=0 \longrightarrow \Gamma=0$$

$\widetilde{10}\times 4 \longrightarrow 40$ eq.

40 Var ✓

$$g'_{\alpha\beta,\mu\nu}=0$$

$10\times\widetilde{10}$ 100 eq

$$20_{\text{var.}} \to R^{\alpha}{}_{\beta\gamma\delta}(g, \partial g, \partial^2 g)$$

$$\alpha_{\beta\gamma}\,(x^{\beta} - x_p^{\beta})(x^{\gamma} - x_p^{\gamma}) + C^{\alpha}{}_{\beta\gamma\sigma}(x^{\beta} - x_p^{\beta})(x^{\gamma} - x_p^{\gamma})(x^{\sigma} - x_p^{\sigma})$$

$2 + \frac{4!}{2!2!} + 4 = 8 + 6 = 14$

16 Var.

10 eq. ✓

$$g'_{\alpha\beta,\mu} = 0 \longrightarrow \Gamma = 0$$

$10 \times 4 \longrightarrow 40$ eq.

40 Var ✓

$$g'_{\alpha\beta,\mu\nu} = 0$$

10×10 100 eq

$4 \times 20 = 80$ Var

$g_{\mu\nu,\alpha} \to \Gamma \to 0$

$20_{\text{var}} \to R^{\alpha}_{\beta\gamma\delta}(g_{\mu\nu}, g_{\mu\nu,\alpha})$

$$x \in U \Rightarrow x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\alpha} + \frac{1}{2} B^{\alpha}_{\beta\gamma}(x^{\beta} - x^{\beta}_{p})(x^{\gamma} - x^{\gamma}_{p}) + C^{\alpha}_{\beta\gamma\sigma}(x^{\beta} - x^{\beta}_{p})(x^{\gamma} - x^{\gamma}_{p})(x^{\sigma} - x^{\sigma}_{p})$$

$4 + \frac{9 \cdot 4}{2 \cdot 3 \cdot 1} + 4 = 8 + 6 = 14$

$$g_{\alpha\beta}(x)\Big|_{p} = \frac{\partial x^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\nu}}{\partial x^{\beta}} g'_{\mu\nu}(x')\Big|_{p} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \underbrace{g'_{\mu\nu}(x)}_{\eta_{\mu\nu}}$$

16 Var.

10 eq. ✓

$g'_{\alpha\beta,\mu} = 0 \longrightarrow \Gamma = 0$

$10 \times 4 \to 40$ eq.

40 Var ✓

$g'_{\alpha\beta,\mu\nu} = 0$

10×10 100 eq

$4 \times 20 = 80$ Var

Principle of equivalence:

$\forall\ p \in M$ $\exists$ small $U \subset M$ for which $\exists$ a locally inertial coordinate sys ("free falling frame") $\to$ laws of SR apply

$g_{\mu\nu,\alpha} \longrightarrow \Gamma \longrightarrow 0$

$$x \in U \Rightarrow x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\alpha} + \frac{1}{2} B^{\alpha}_{\beta\gamma} (x^{\beta} - x_p^{\beta})(x^{\gamma} - x_p^{\gamma}) + C^{\alpha}_{\beta\gamma\sigma} (x^{\beta} - x_p^{\beta})(x^{\gamma} - x_p^{\gamma})(x^{\sigma} - x_p^{\sigma})$$

$4 + \frac{3!}{2!\,1!} + 4 = 8 + 6 = 14$

$$g_{\alpha\beta}(x)\Big|_p = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g'_{\mu\nu}(x')\Big|_p = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \underbrace{g'_{\mu\nu}(x)}_{\eta_{\mu\nu}}$$

16 var.

10 eq. ✓

$g'_{\alpha\beta,\mu} = 0 \longrightarrow \Gamma = 0$

$10 \times 4 \longrightarrow 40$ eq.

40 var ✓

$g'_{\alpha\beta,\mu\nu} = 0$

10×10 100 eq

$4 \times 20 = 80$ var

Principle of equivalence.

$\forall\ p \in M$ $\exists$ small $U \subset M$ for which $\exists$ a locally inertial coordinate sys ("free falling frame") $\longrightarrow$ laws of SR. apply

Invariant Volume element

$g = \det(g_{\mu\nu})$

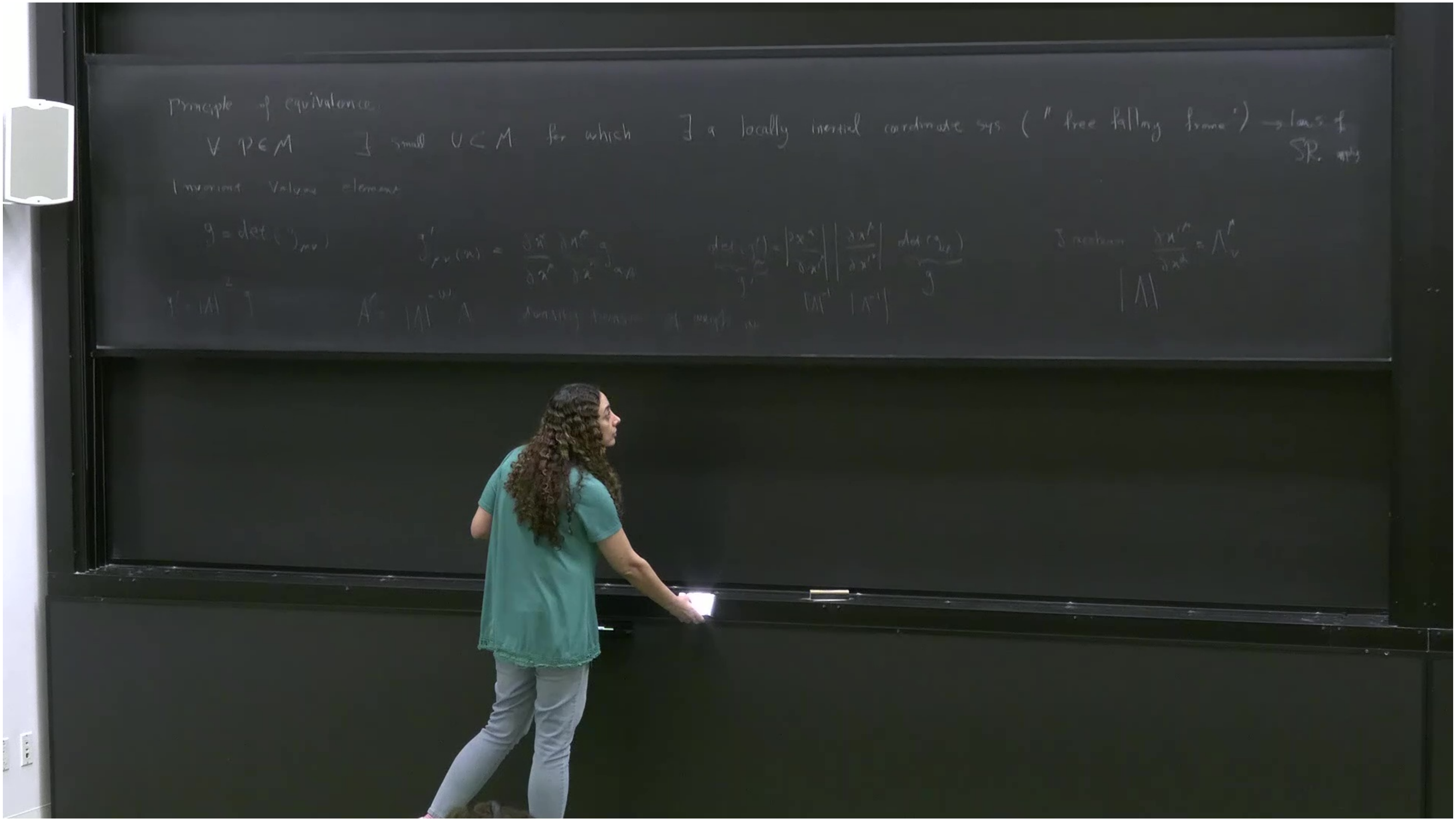
$$g'_{\mu\nu}(x) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

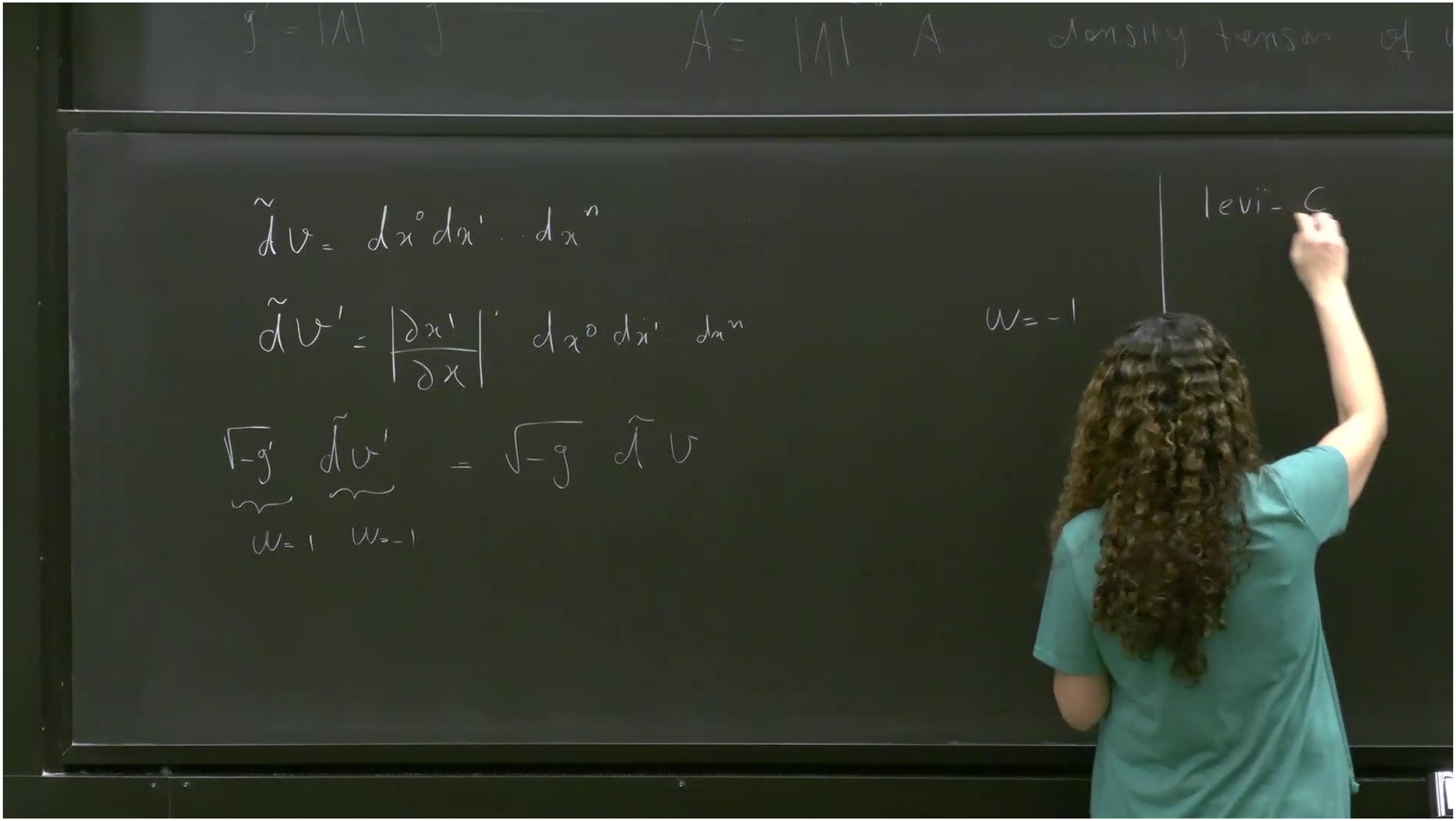
$$\underbrace{\det(g'_{\mu\nu})}_{g'} = \underbrace{\left|\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\right|}_{|\Lambda|^{-1}} \underbrace{\left|\frac{\partial x^{\beta}}{\partial x'^{\nu}}\right|}_{|\Lambda^{-1}|} \underbrace{\det(g_{\alpha\beta})}_{g}$$

Jacobian $\dfrac{\partial x'^{\mu}}{\partial x^{\alpha}} = \Lambda^{\mu}_{\nu}$

$|\Lambda|$

$g' = |\Lambda|^{-2} g$


Principle of equivalence
∀ p∈M ∃ small U⊂M for which ∃ a locally inertial coordinate sys ("free falling frame") → laws of SR apply
Invariant Volume element


$g' = |\Lambda| \; g$
$A' = |\Lambda| \; A$ density tensor of
$\tilde{d}V = dx^0 dx^1 \ldots dx^n$
$\tilde{d}V' = \left| \frac{\partial x'}{\partial x} \right| dx^0 dx^1 \ldots dx^n$
$\underbrace{\sqrt{-g'}}_{w=1} \; \underbrace{\tilde{d}V'}_{w=-1} = \sqrt{-g} \; \tilde{d}V$
$w = -1$
levi-C

levi-Civita tensor

$$[\mu\nu\rho\sigma] = \begin{cases} +1 & \text{for even perm. of } [0,1,2,3] \\ -1 & \text{" odd " "} \\ 0 & \text{otherwise} \end{cases}$$

$$\left|\frac{\partial x'}{\partial x}\right| = [\alpha'\beta'\gamma'\delta'] \frac{\partial x'^{\alpha'}}{\partial x^0} \frac{\partial x'^{\beta'}}{\partial x^1} \frac{\partial x'^{\gamma'}}{\partial x^2} \frac{\partial x'^{\delta'}}{\partial x^3}$$

$$g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$$

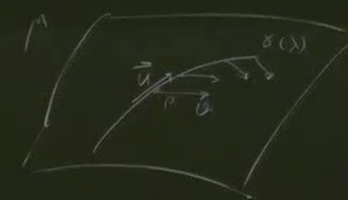
$$\boxed{\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,[\alpha\beta\gamma\delta]}$$

$$[\alpha'\beta'\gamma'\delta'] \frac{\partial x'^{\alpha'}}{\partial x^\mu} \frac{\partial x'^{\beta'}}{\partial x^\nu} \frac{\partial x'}{\partial x^\alpha} \frac{\partial x'}{\partial x^\beta} = \lambda\,[\mu\nu\alpha\beta]$$

$g' = \left| \frac{\partial x}{\partial x'} \right| g$

$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-}$

Parallel Transport $\longrightarrow$ Directional derivative



$$\frac{Df}{d\lambda} = \frac{df}{d\lambda} = u^\alpha f_{,\alpha} = f_{,\alpha} \frac{dx^\alpha}{d\lambda} = u(f)$$

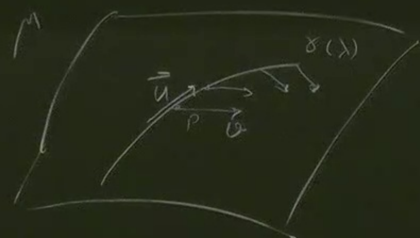
$$u = u^\alpha \partial_\alpha$$

$$\frac{Dv}{d\lambda} = u^\mu \nabla_\mu v^\alpha = \nabla_u v^\alpha$$

$$\frac{dv^\alpha}{d\lambda} = u^\beta v^\alpha_{\ ,\beta}$$

$$\frac{DT^{\alpha\beta}}{d\lambda} = u^\mu \nabla_\mu T^{\alpha\beta}$$

if $\nabla_u v = 0 \rightarrow v$



if $\nabla_u v = 0$ $\rightarrow$ v is parallel transported along γ (u)

$$\nabla_u v^\alpha = u^\mu \nabla_\mu v^\alpha = u^\mu\left(\partial_\mu v^\alpha + \Gamma^\alpha_{\mu\sigma} v^\sigma\right) = \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\sigma} v^\sigma u^\mu = 0$$

if u (tangent vector) is P.T. along γ $\Rightarrow$ $\gamma(\lambda)$ is a geodesic
$\Leftarrow$

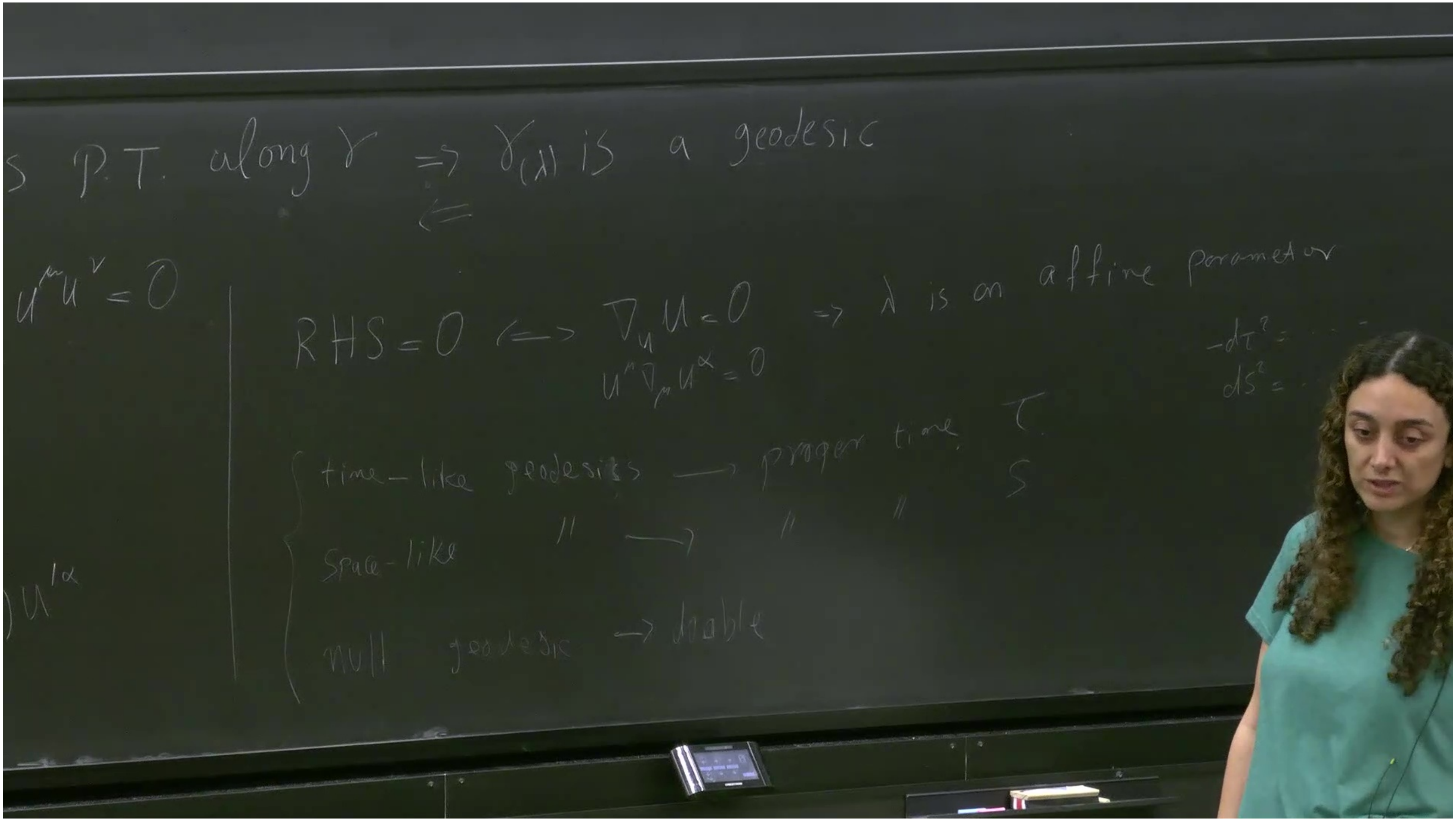
$$\nabla_u u^\alpha = 0 \Rightarrow \frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0$$

$$u'^\sigma = \frac{dx^\sigma}{d\lambda'} = \frac{d\lambda}{d\lambda'} \frac{dx^\sigma}{d\lambda} = \frac{d\lambda}{d\lambda'} u^\sigma$$

$\lambda = f(\lambda')$ $\qquad$ $\nabla_{u'} u'^\alpha = f(\lambda') u'^\alpha$

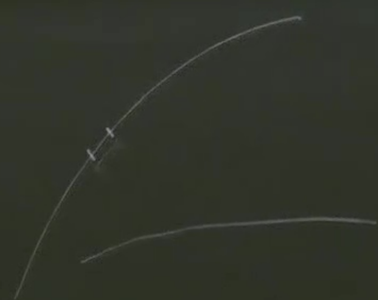
RHS $= 0 \Longleftrightarrow \nabla_u u = 0$ $\Rightarrow$ λ is an

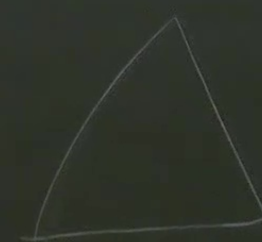
$u^\mu \nabla_\mu u^\alpha = 0$


s P.T. along γ ⇒ γ(λ) is a geodesic
⇐
u^μ u^ν = 0
RHS = 0 ⟺ ∇_u u = 0 ⇒ λ is an affine parameter
u^μ ∇_μ u^α = 0
-dτ² = ...
ds² = ...
time-like geodesics → proper time τ
space-like // → // // s
null geodesic → doable
u'^α

$g(u,u) < 1$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$
$$= -dt^2 + dx^2$$





$$(1, 0, 0, 0)$$
$$(1, 0, 0, 0)$$

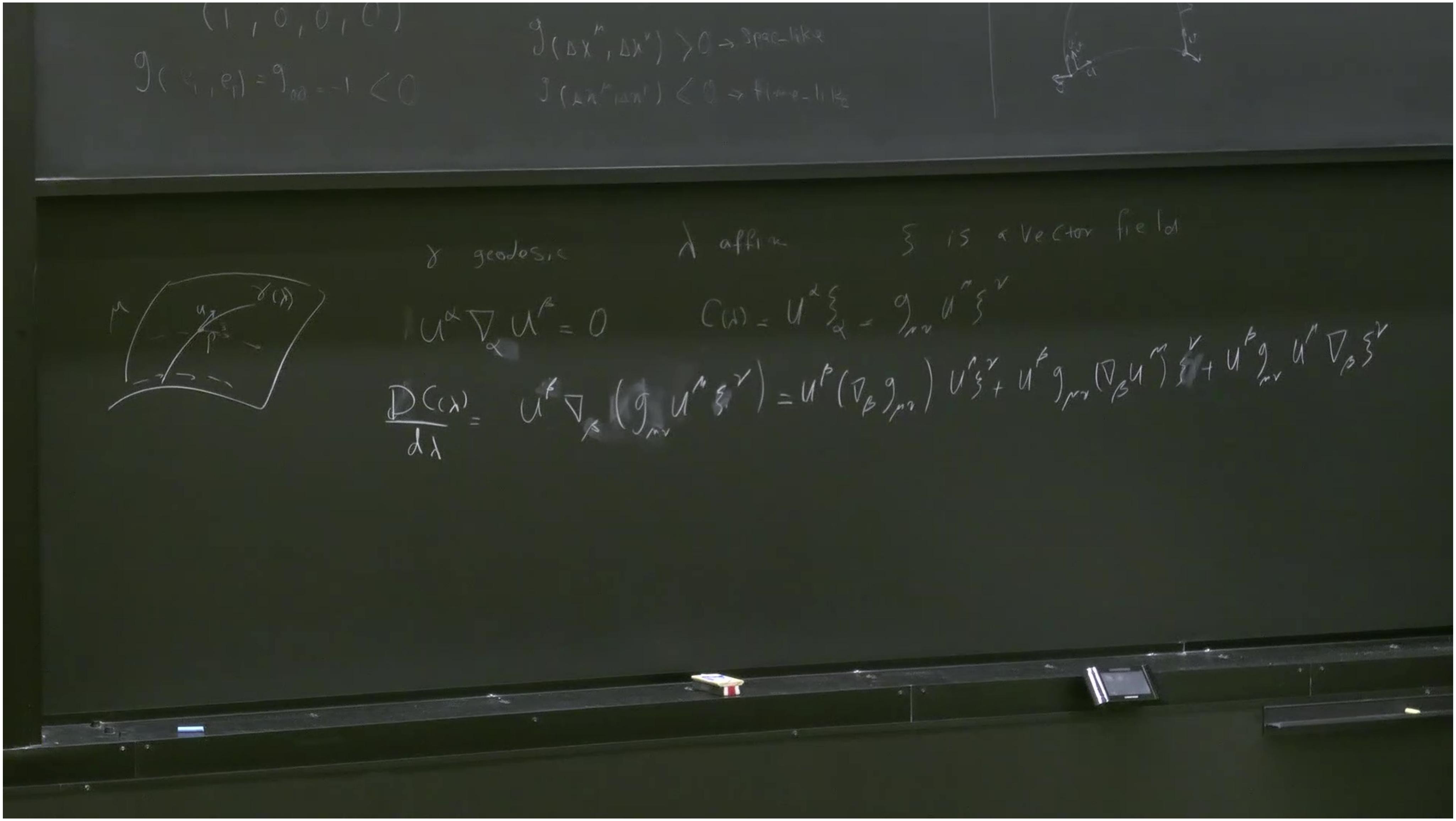
$$g(e_1, e_1) = g_{00} = -1 < 0$$

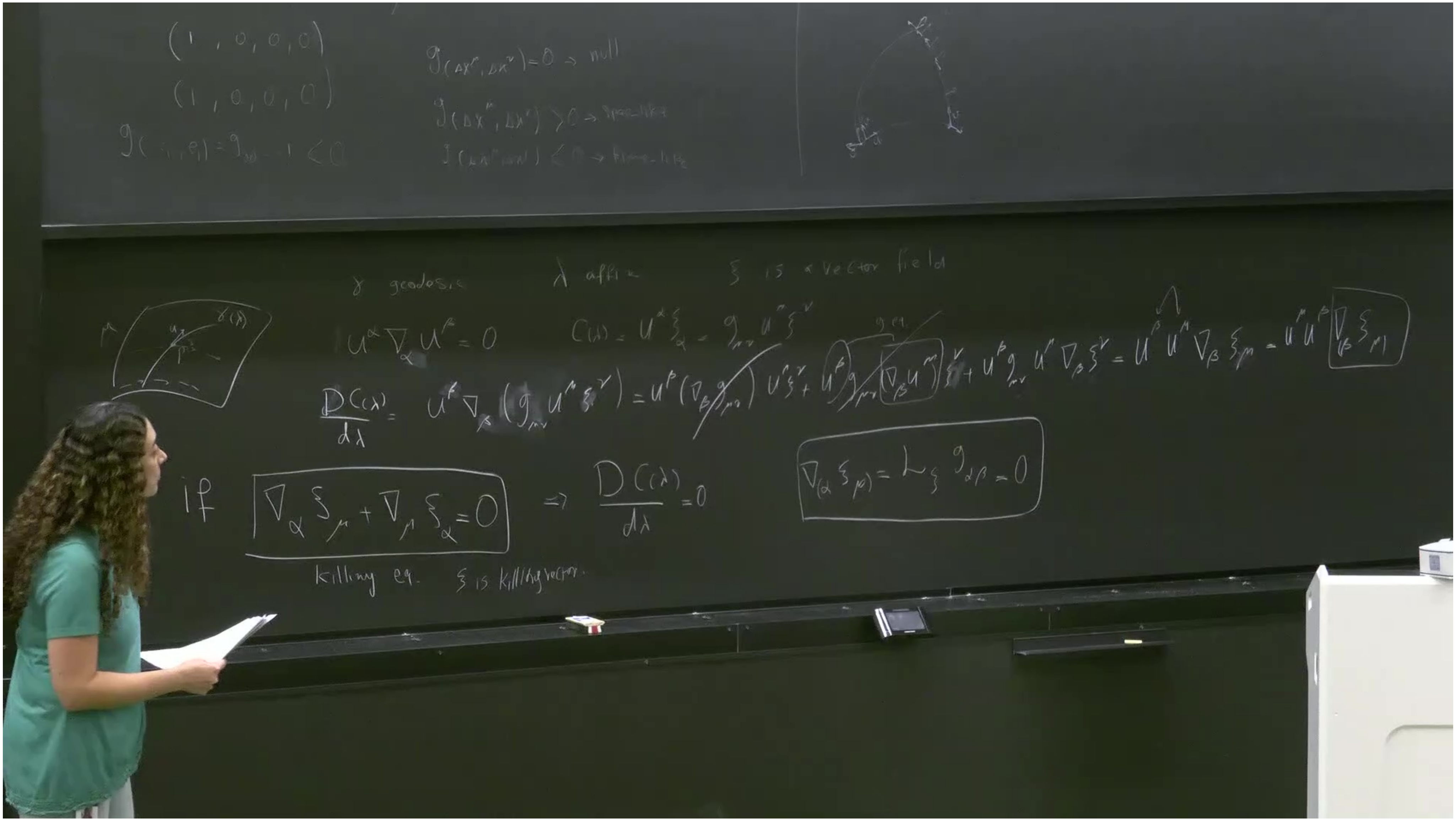
$g(\Delta x^\mu, \Delta x^\nu) = 0 \rightarrow$ null

$g(\Delta x^\mu, \Delta x^\nu) > 0 \rightarrow$ spac-like

$g(\Delta x^\mu, \Delta x^\nu) < 0 \rightarrow$ time-like

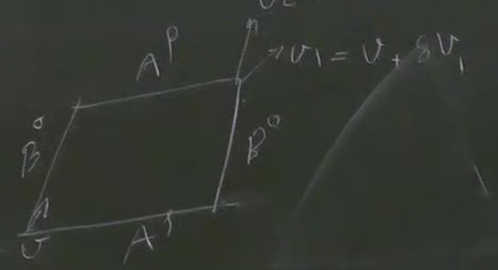

D v·w / dλ = 0
v·w = g_μν v^μ w^ν


g(e₀, e₀) = g_00 = -1 < 0
g(Δx^μ, Δx^ν) > 0 → space-like
g(Δx^μ Δx^ν) < 0 → time-like
γ geodesic
λ affine
ξ is a vector field
u^α ∇_α u^β = 0
C(λ) = u^α ξ_α = g_μν u^μ ξ^ν
DC(λ)/dλ = u^β ∇_β (g_μν u^μ ξ^ν) = u^β (∇_β g_μν) u^μ ξ^ν + u^β g_μν (∇_β u^μ) ξ^ν + u^β g_μν u^μ ∇_β ξ^ν


γ geodesic
λ affine
ξ is a vector field
if
killing eq.
ξ is killing vector.

$v \to$ P.T $\to v + \delta v$

$\delta v_1 - \delta v_2 \to R^{\alpha}{}_{\beta\sigma\gamma}$



let D

$\frac{D\, v \cdot w}{d\lambda} =$