Title: Lecture - Relativity, PHYS 604

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Abstract:

Remarks on Tensors: $\mathrm{Rank}(T) = t$ $\mathrm{Rank}(S) = s$

i) $T + S$ T & S of the same type $t = s$

ii) Tensor product $T \otimes S$ $\underset{(1,1)}{T} = T^{\alpha}{}_{\beta}\, \partial_\alpha \otimes$

iii) Contraction $\underset{(2,1)}{T} = T^{\alpha\beta}{}_{\mu}\, \partial_\alpha \otimes \partial_\beta \otimes dx^\mu$

$= S$

$$T = T^{\alpha}{}_{\beta}\, \partial_\alpha \otimes dx^{\beta}$$
$(1,1)$

$$S = S^{\mu}\, \partial_\mu$$
$(1,0)$

$$T \otimes S = T^{\alpha}{}_{\beta} S^{\mu}\, \partial_\alpha \otimes \partial_\mu \otimes dx^{\beta}$$
$(2,1)$

$\otimes dx^{\mu}$

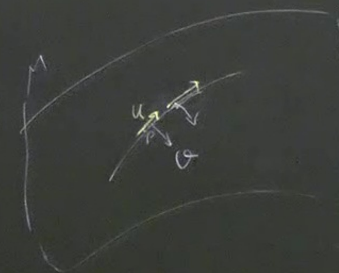
$$\tilde{T} = T^{\alpha\beta}{}_{\alpha}\, \partial_\beta$$

$$(T.S)^{\alpha \dots \mu}{}_{\gamma \dots \nu} = T^{\alpha \dots \kappa \dots \mu}{}_{\gamma \dots}\, S^{\sigma \dots}{}_{\kappa\, \delta \dots}$$

$s + t - 2$ # of contractions

(Anti) Sym. Tensor $S_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} \pm T_{\nu\mu}) = T_{(\mu\nu)}$, $T_{[\mu\nu]}$

How to differentiate Tensors on M



i) Lie derivative $\xrightarrow{\text{needs}}$ vector field

$$L_u f = \frac{df}{d\lambda} = f_{;\alpha} u^\alpha$$

$$L_u v^\alpha = v^\alpha_{;\mu} u^\mu - u^\alpha_{;\beta} v^\beta \longrightarrow \text{Some Tensor}$$

$(0,1) \longrightarrow (0,1)$

ii) Exterior derivatives

iii) Covariant derivative $\longrightarrow$ connection

$$\phi \longrightarrow d\phi(V) \in \mathbb{R}$$

$$\underbrace{d\phi}_{\omega}(V) \equiv V(\phi) \in \mathbb{R}$$

$$\underset{(0,0)}{\phi} \longrightarrow \underset{(0,1)}{d\phi}$$

$$d\phi = \omega = \omega_\alpha dx^\alpha = \phi_{,\alpha} dx^\alpha$$

$$\phi \longrightarrow \partial_\alpha \phi \rightarrow \nabla_\alpha \phi$$

V^α or V_α

$V^\alpha{}_{,\beta}(x)$

$\downarrow ?$

$T^\alpha{}_\beta$

$V'^\alpha{}_{,\mu}(x') = \frac{\partial}{\partial x'^\mu} V$

$= \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial}$

$T'^\alpha{}_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial x'^\alpha}{\partial x^\beta}$

Covariant derivative: $\boxed{\nabla_\mu V^\alpha} = V^\alpha{}_{,\mu} + \Gamma^\alpha{}_{\mu\beta} V^\beta$

$\Gamma' = \cdots (\Gamma_{\cdots})$

Γ is called

$$\partial\phi = W = \omega_\alpha dx^\alpha = \phi_{,\alpha} dx^\alpha$$

$$v'^{\alpha}{}_{,\mu}(x') = \frac{\partial}{\partial x'^\mu} v'^\alpha(x') = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\alpha}{\partial x^\beta} v^\beta(x)\right) = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x'^\alpha}{\partial x^\beta} v^\beta(x)\right)$$

$$= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\nu \partial x^\beta} v^\beta(x) + \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\alpha}{\partial x^\beta} v^\beta{}_{,\nu}(x)}$$

$$T'^{\alpha}{}_{\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\alpha}{\partial x^\beta} T^\beta{}_\nu$$

$$\rightarrow \bigcirc^\alpha{}_{\mu\beta} \longrightarrow \Gamma^\alpha{}_{\mu\beta}$$

$$\alpha \quad {}_{\mu\beta} V^\beta$$

Γ is called connection.

$$\nabla_\alpha w_\beta = w_{\beta,\alpha} - \Gamma^\sigma_{\alpha\beta} w_\sigma$$

$$\nabla_\alpha T^\mu{}_\beta = T^\mu{}_{\beta,\alpha} + \Gamma^\mu_{\alpha\sigma} T^\sigma_\beta - \Gamma^\sigma_{\alpha\beta} T^\mu{}_\sigma$$

$$(1,1) \xrightarrow{\nabla_\alpha} (1,2)$$

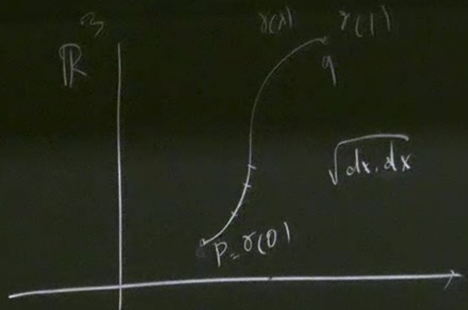
Γ is called connection.

Torsion

$$T^{\alpha}{}_{\beta\gamma} = -2\Gamma^{\alpha}{}_{[\beta\gamma]} = -(\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta})$$

$$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta} \rightsquigarrow \text{sym}$$

1. $\nabla_\mu f = \partial_\mu f$
2. $\nabla \to$ linear & Libnitz
3. ∇ commutes with contraction
4. $[\nabla_\mu, \nabla_\nu] f = -T^{\alpha}{}_{\mu\nu} \nabla_\alpha f$



$$d(p,q) = \int_0^1 d\lambda \sqrt{\frac{dx}{d\lambda}\cdot\frac{dx}{d\lambda}}$$

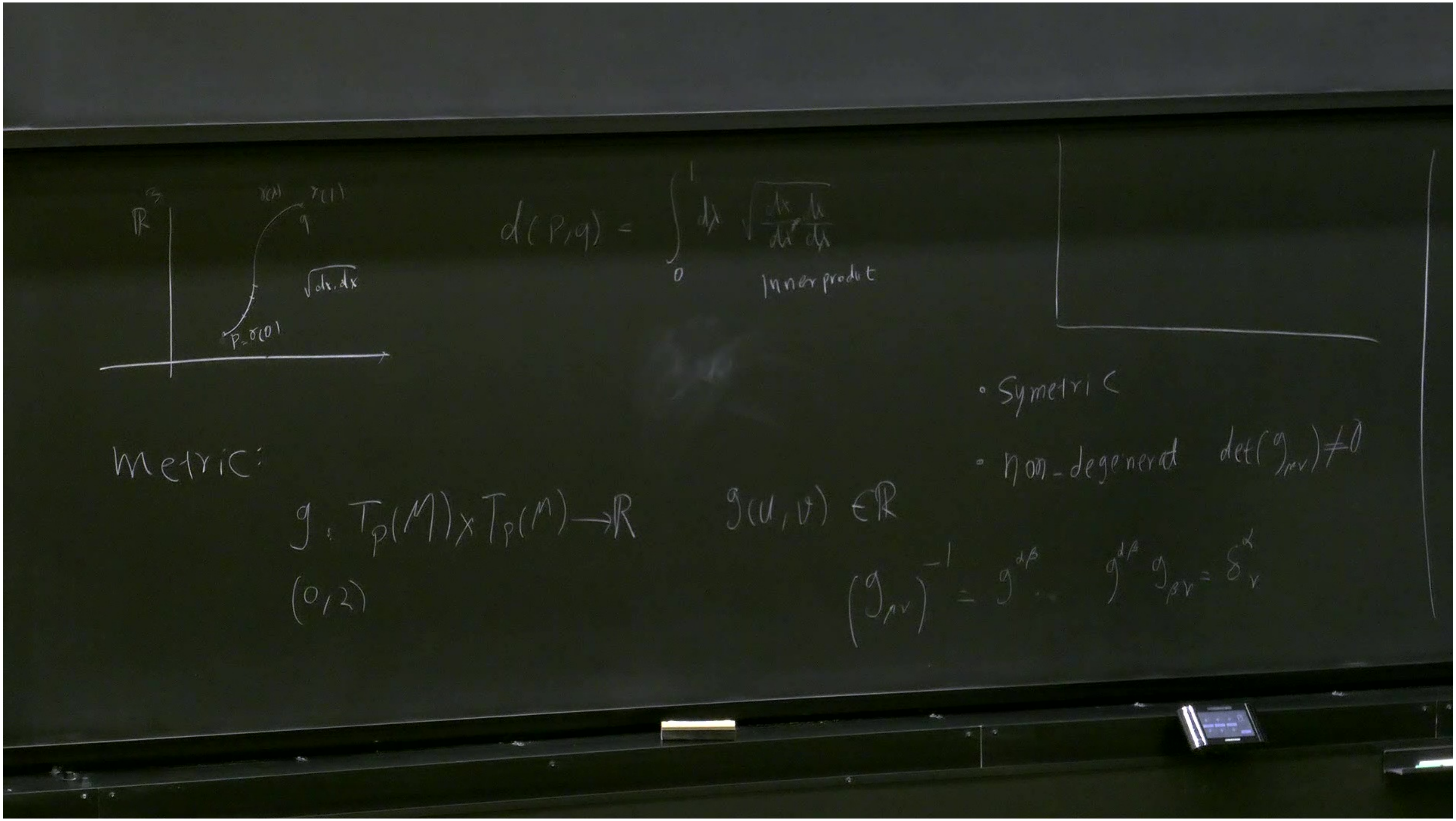
inner product

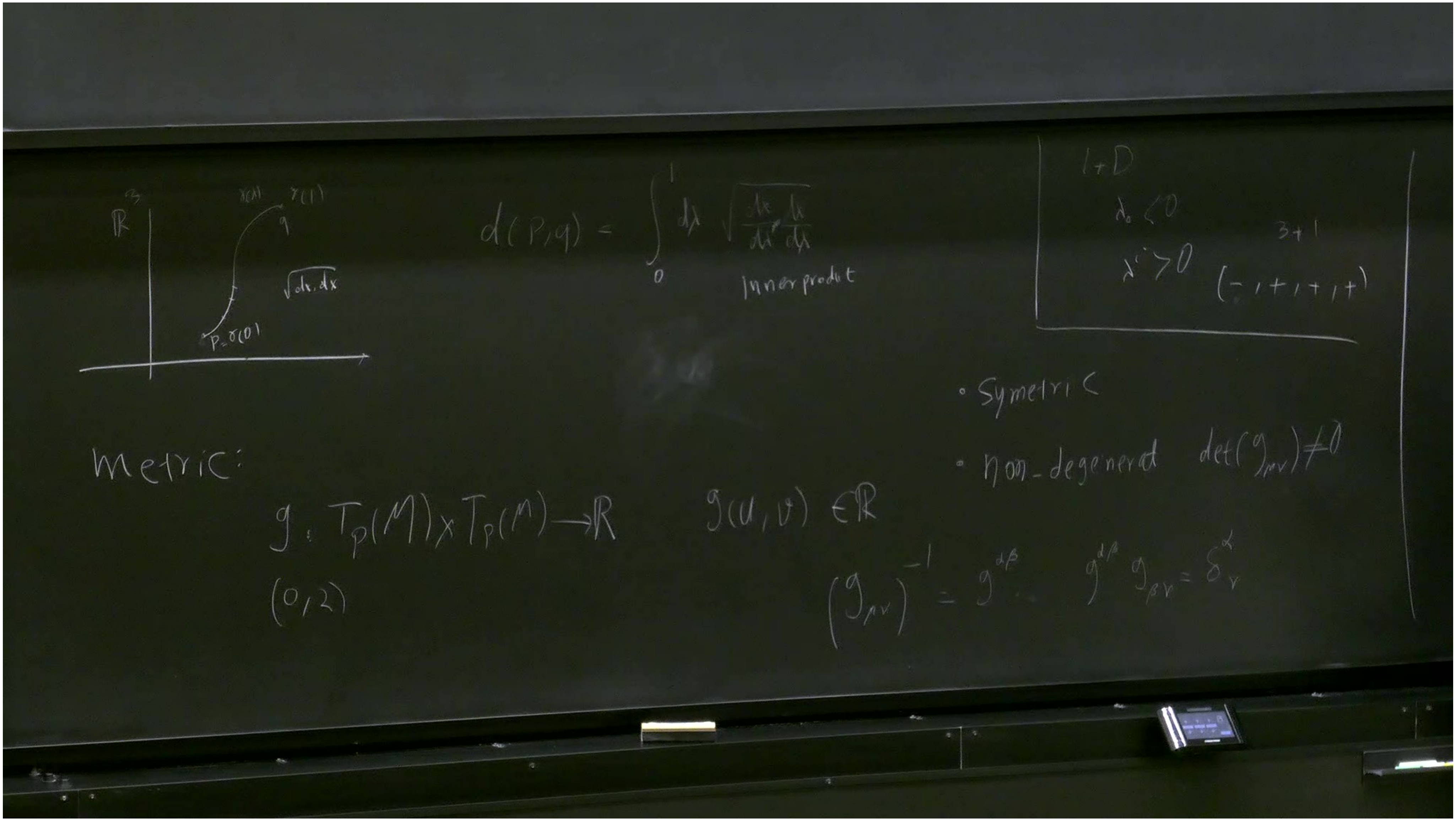
Metric:

$$g: T_p(M)\times T_p(M)\longrightarrow\mathbb{R} \qquad g(u,v)\in\mathbb{R}$$

$(0,2)$

- Symetric
- non-degenerat $\det($


$\mathbb{R}^3$
$\gamma(\lambda)$
$\gamma(1)$
q
$\sqrt{dx \cdot dx}$
$p = \gamma(0)$
$d(p,q) = \int_0^1 d\lambda \sqrt{\frac{dx}{d\lambda} \cdot \frac{dx}{d\lambda}}$
inner product
Metric:
$g : T_p(M) \times T_p(M) \to \mathbb{R}$
$g(u, v) \in \mathbb{R}$
$(0,2)$
• Symetric
• non-degenerat $\det(g_{\mu\nu}) \neq 0$
$(g_{\mu\nu})^{-1} = g^{\alpha\beta}$
$g^{\alpha\beta} g_{\beta\nu} = \delta^{\alpha}_{\nu}$



$\mathbb{R}^3$
$\gamma(\lambda)$ $\gamma(1)$
q
$\sqrt{dx \cdot dx}$
$p = \gamma(0)$
$d(p,q) = \int_0^1 d\lambda \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx}{d\lambda}}$
inner product
1+D
$\lambda_0 < 0$
$\lambda^i > 0$
3+1
$(-,+,+,+)$
• Symetric
• non-degenerate $\det(g_{\mu\nu}) \neq 0$
Metric:
$g : T_p(M) \times T_p(M) \rightarrow \mathbb{R}$
$g(u,v) \in \mathbb{R}$
$(0,2)$
$(g_{\mu\nu})^{-1} = g^{\alpha\beta}$
$g^{\alpha\beta} g_{\beta\nu} = \delta^\alpha_\nu$

$+,+)$

$\neq 0$

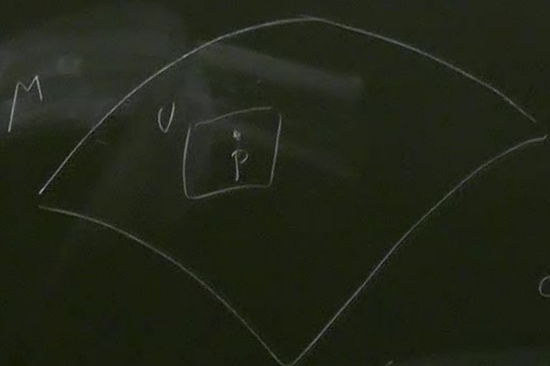
$M_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu}$ $\rightarrow$ metricity

$M_{\alpha\mu\nu} = 0$ $\rightarrow$ Γ special metric compatible

$(0,2)$ $\qquad (g_{\mu\nu}) = g \ldots \quad g_{\beta\nu} = \delta_\nu$

if $\quad M=0 \ \& \ T=0 \quad \Rightarrow \quad \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right)$

Christoffel symbols or Levi-Civita



Theorem: $p \in M \Rightarrow \exists \ x \longrightarrow x'$ such that $g_{\mu\nu}(x_p) = \eta_{\mu\nu}$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad X^T G X \stackrel{?}{=} Y^T \eta Y$$

$$X = \begin{pmatrix} dx^0 \\ dx^1 \\ \vdots \end{pmatrix}$$

$\gamma\delta$

civitia

$(x_\rho) = \eta_{\mu\nu}$

$$= \begin{pmatrix} -\lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix}$$

$$X^T G X = X^T P^T D P X = X^T \underbrace{P^T L^T \overset{\Lambda}{\eta} L P}_{G} X$$

$$P^T P = 1$$

$$G = P^T D P$$

$$D = \overset{L^T}{\begin{pmatrix} \sqrt{\lambda_0} & & \\ & \sqrt{\lambda_1} & \\ & & \ddots \end{pmatrix}} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_0} & & \\ & \sqrt{\lambda_1} & \\ & & \ddots \end{pmatrix}$$

$$y^\alpha = \Lambda^\alpha{}_\beta \, x^\beta \qquad \Lambda = LP$$

$$\longrightarrow \quad g_{\mu\nu}(x_p) = \eta_{\mu\nu}\big|_p$$

$$g_{\mu\nu,\alpha}(x_p) = 0$$

$$g_{\mu\nu,\alpha\beta} \neq 0$$