Title: Lecture - Relativity, PHYS 604

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Collection/Series: Relativity (Core), PHYS 604, November 12 - December 11, 2024

Subject: Cosmology, Strong Gravity

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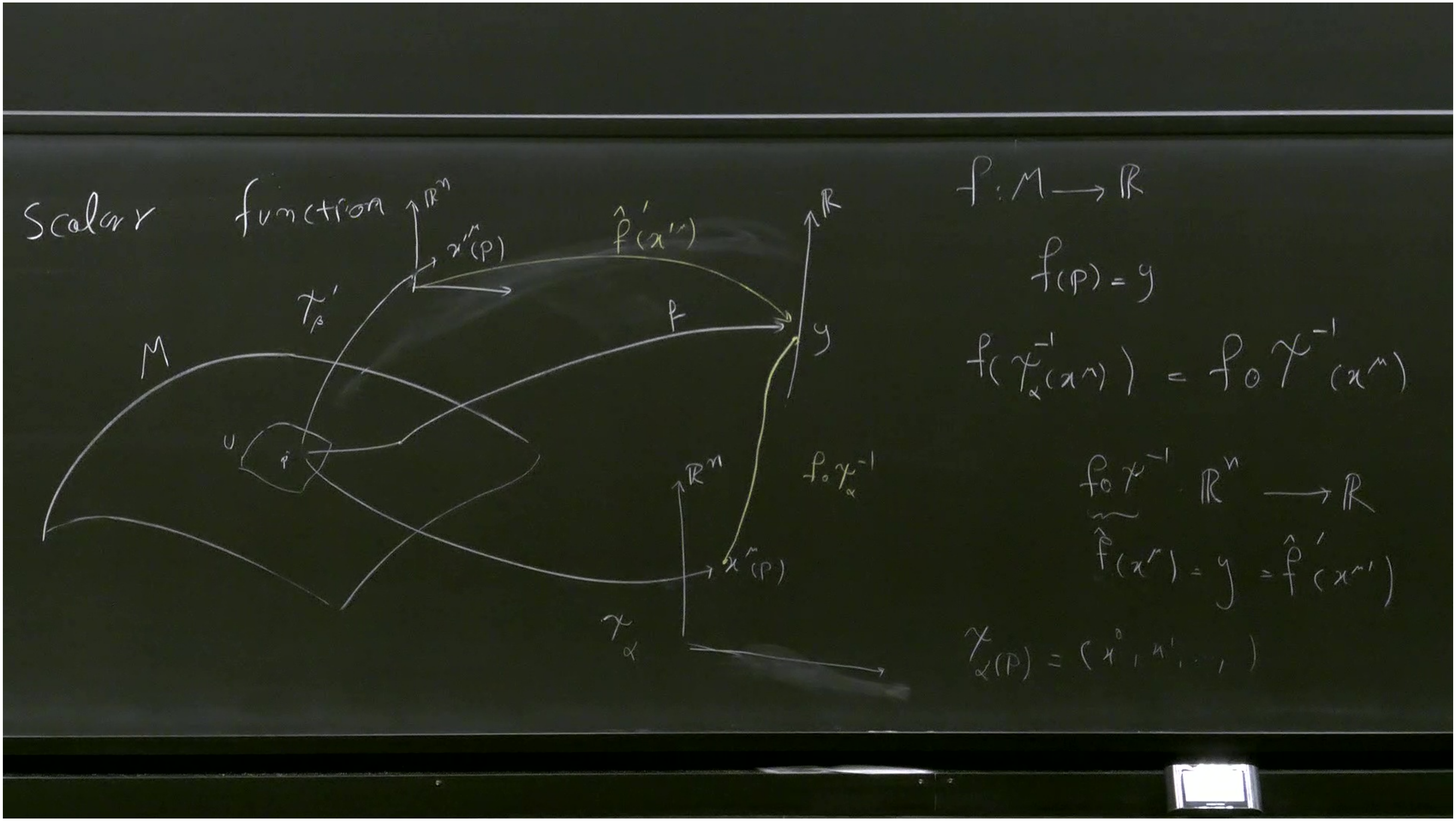
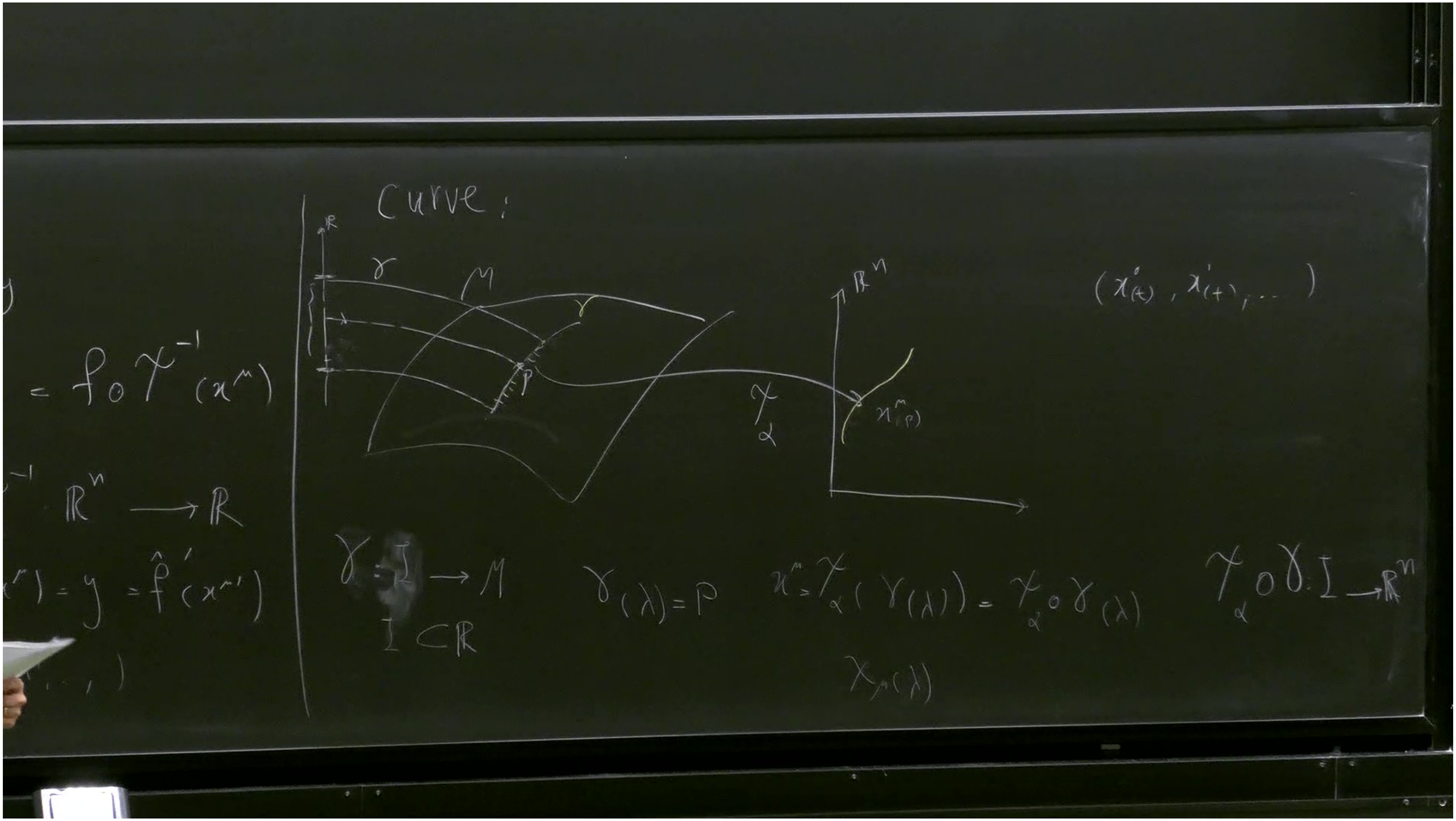
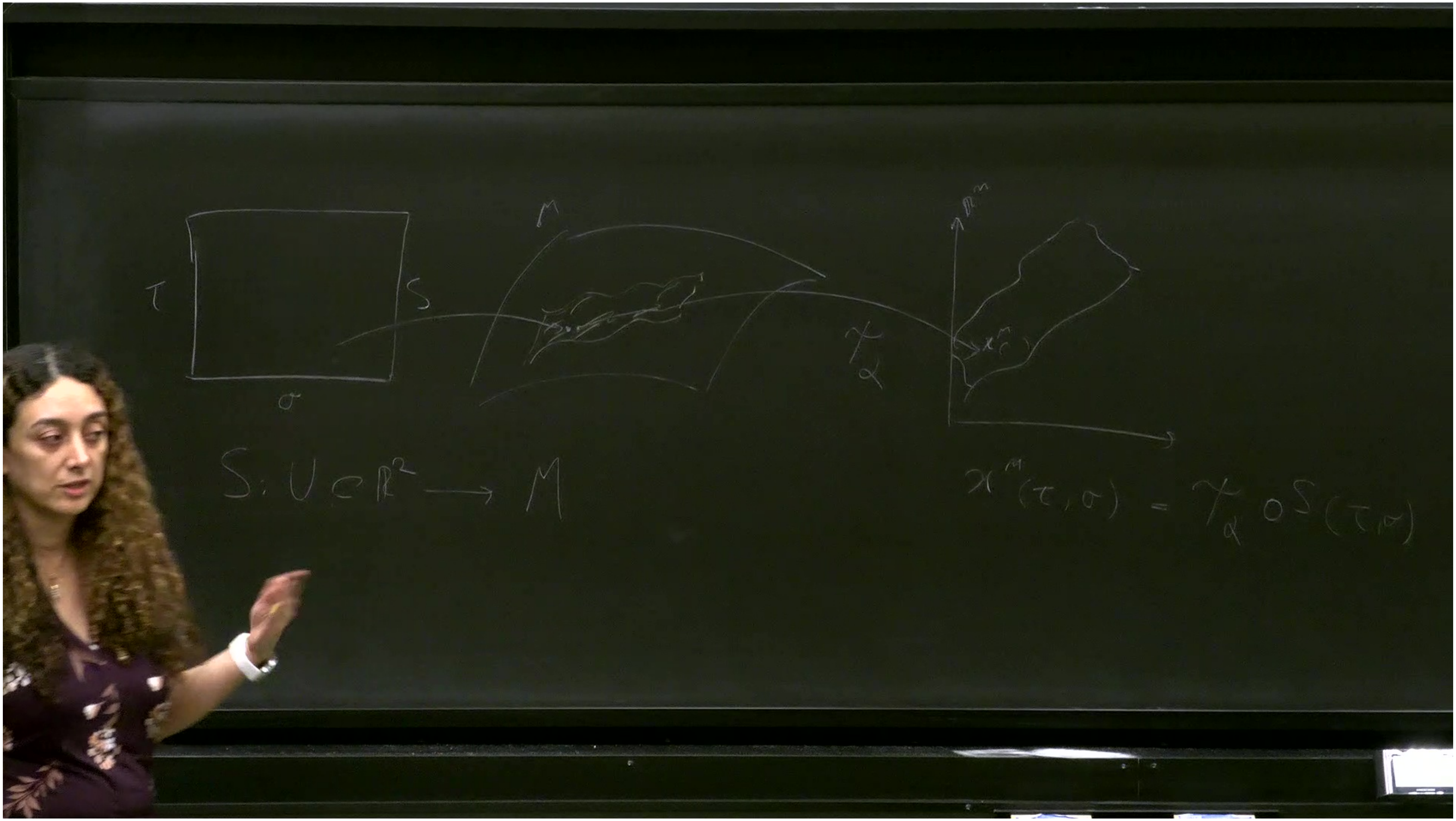
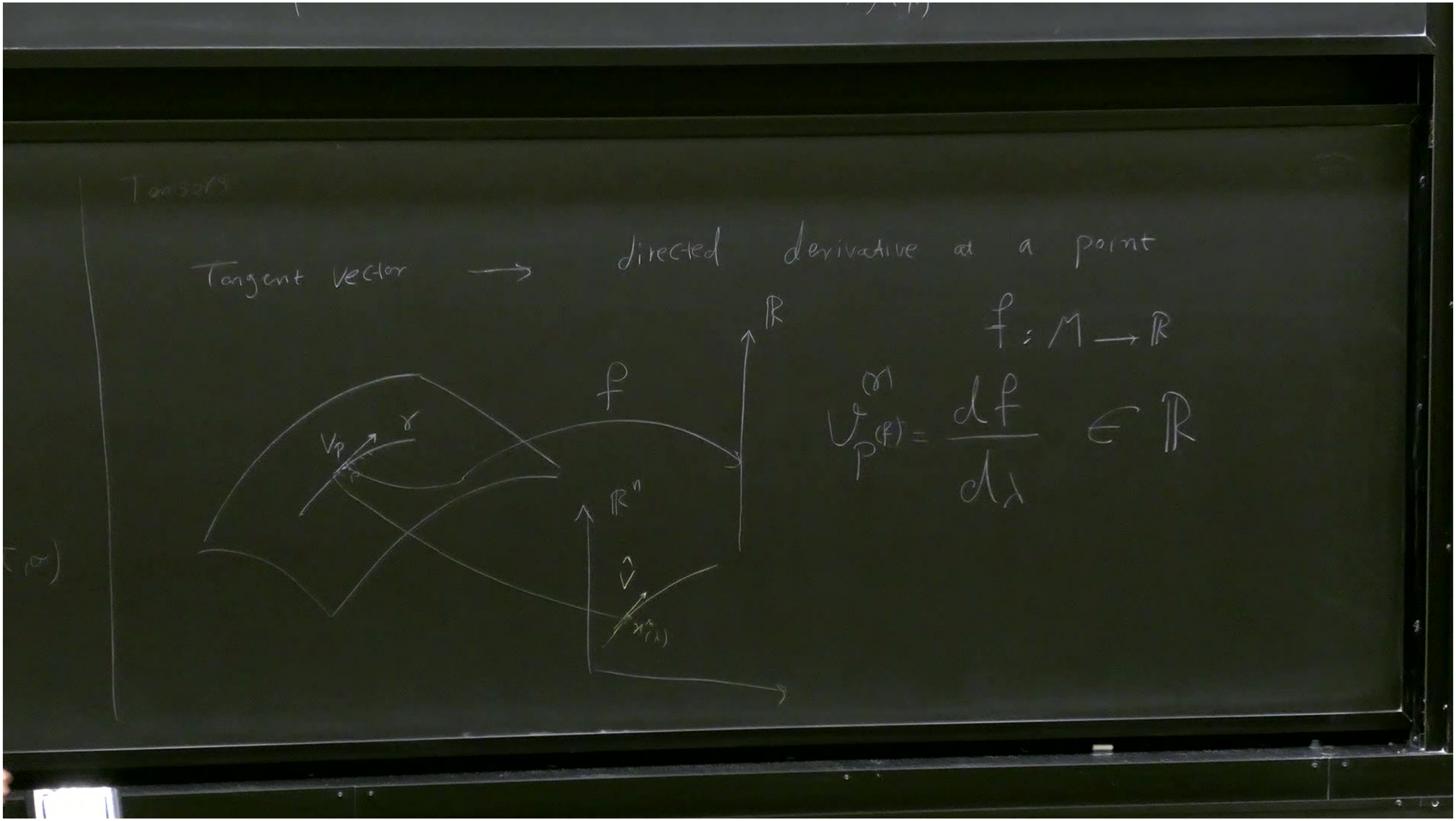
Abstract:


Scalar function
$f: M \to \mathbb{R}$
$f(p) = y$
$f(\chi_\alpha^{-1}(x^\mu)) = f \circ \chi^{-1}(x^\mu)$
$f \circ \chi^{-1} : \mathbb{R}^n \to \mathbb{R}$
$\hat{f}(x^\mu) = y = \hat{f}'(x^{\mu'})$
$\chi_\alpha(p) = (x^0, x^1, \ldots,)$
M
U
p
χ_β'
$\mathbb{R}^n$
$x^{\mu'}(p)$
$\hat{f}'(x^{\mu'})$
f
$\mathbb{R}$
y
$f \circ \chi_\alpha^{-1}$
$\mathbb{R}^n$
$x^\mu(p)$
χ_α

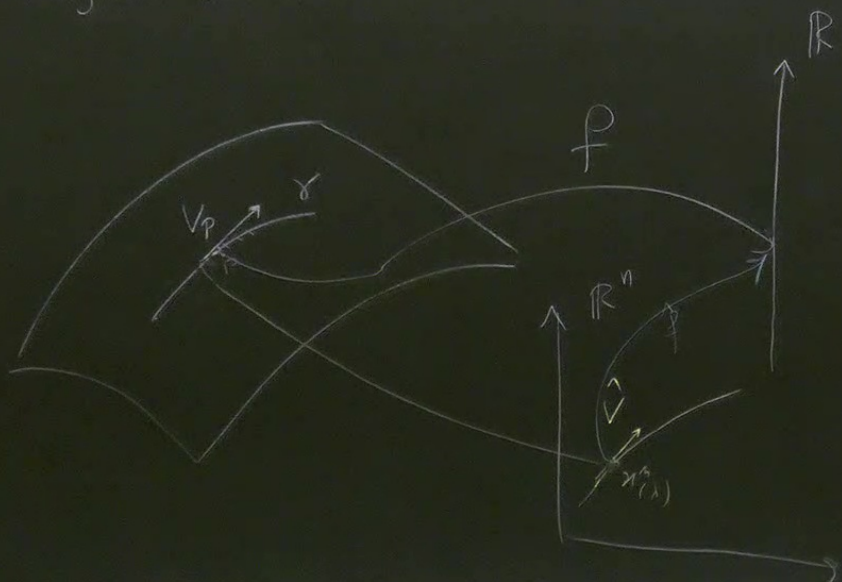

Curve:
$\mathbb{R}$
γ
M
$\mathbb{R}^n$
$(x^0(t), x^1(t), \ldots)$
$= f \circ \chi^{-1}(x^\mu)$
p
χ_α
$x^\mu(p)$
$\mathbb{R}^n \longrightarrow \mathbb{R}$
$\gamma : I \to M$
$I \subset \mathbb{R}$
$\gamma(\lambda) = p$
$x^\mu = \chi_\alpha(\gamma(\lambda)) = \chi_\alpha \circ \gamma(\lambda)$
$x^\mu(\lambda)$
$\chi_\alpha \circ \gamma : I \to \mathbb{R}^n$
$= y = \hat{f}'(x^{\mu'})$


M
τ
S
σ
S: U ⊂ ℝ² ⟶ M
φα
x^μ(τ, σ) = φα ∘ S(τ, σ)


Tensors
Tangent vector → directed derivative at a point
$f : M \to \mathbb{R}$
$v_p^{(\gamma)}(f) = \frac{df}{d\lambda} \in \mathbb{R}$
V_p
γ
f
$\mathbb{R}$
$\mathbb{R}^n$
$\hat{V}$
$x^\mu(\lambda)$

Tensors

Tangent vector $\longrightarrow$ directed derivative at a point



$$f : M \longrightarrow \mathbb{R}$$

$$v_p^{(\gamma)}(f) = \frac{df}{d\lambda} \in \mathbb{R}$$

$$\frac{d\hat{f}(x^\mu)}{d\lambda}\Bigg|_p = \frac{dx^\mu}{d\lambda}^{(\gamma)} \partial_\mu \hat{f}(x^\mu)\Bigg|_p$$

$$V_p^{(\gamma)}(f) = \frac{dx^\mu}{d\lambda}\,\partial_\mu \hat{f}$$

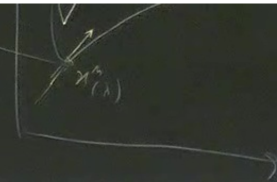
$\hat{f} \to f \circ \varphi$

$$V_p^{(\gamma)} : \mathcal{F} \longrightarrow \mathbb{R}$$

$$\tilde{D}(af + bg) = a\tilde{D}(f) + b\tilde{D}(g)$$

$$\tilde{D}(fg) = f\tilde{D}(g) + g\tilde{D}(f)$$

Def. let $\mathcal{F}$ be a collection of C^∞ scalar functions. A tangent vector



$$\hat{V}(f) = \frac{d f(x^\mu)}{d\lambda}\Big|_p = \frac{d x^\mu}{d\lambda}\, \partial_\mu f(x$$
$$= \hat{V} \cdot \nabla \hat{f}$$

$$(af + bg) = a\tilde{D}(f) + b\tilde{D}(g)$$

$$\tilde{D}(fg) = f\tilde{D}(g) + g\tilde{D}(f)$$

...lar functions. A tangent vector V at point $P \in M$ is a map $V : \tilde{f} \to \mathbb{R}$

$$V_p^{(\gamma)}(f) = \frac{dx^\mu}{d\lambda}\, \partial_\mu \hat{f}$$

$\hat{f} \to f \circ x$

$$V_p^{(\gamma)} : \mathcal{F} \longrightarrow \mathbb{R}$$

$$\tilde{D}(af+bg) = a\tilde{D}(f) + b\tilde{D}(g)$$

$$\tilde{D}(fg) = f\tilde{D}(g) + g\tilde{D}(f)$$

Def. let $\mathcal{F}$ be a collection of C^∞ scalar functions. A tangent vector

such that V is linear and obeys Leibniz rule.

Theorem / Definition

The set of all tangent Vectors at P forms a

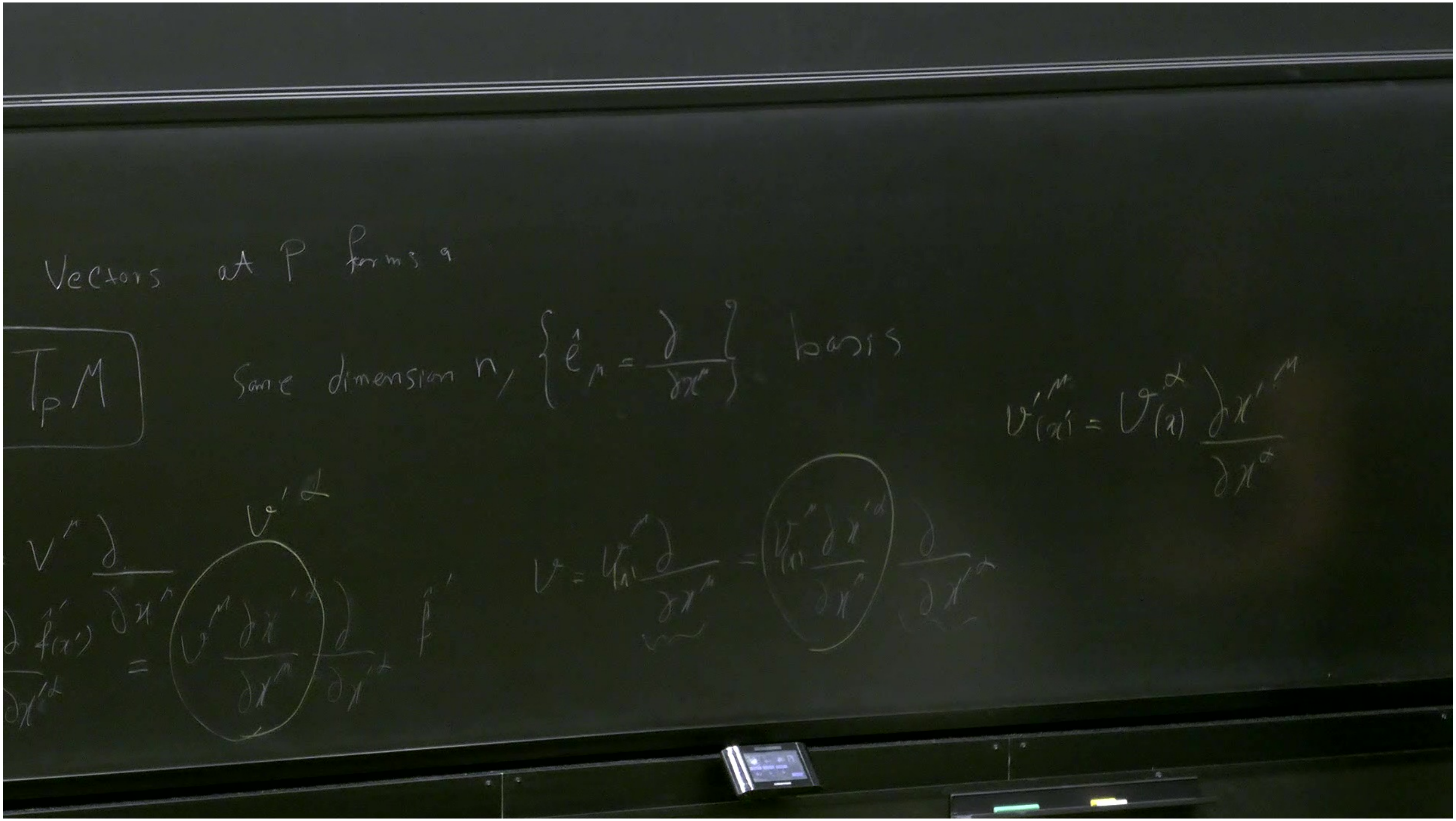
$\boxed{\text{Tangent Vector Space} \quad T_p M}$

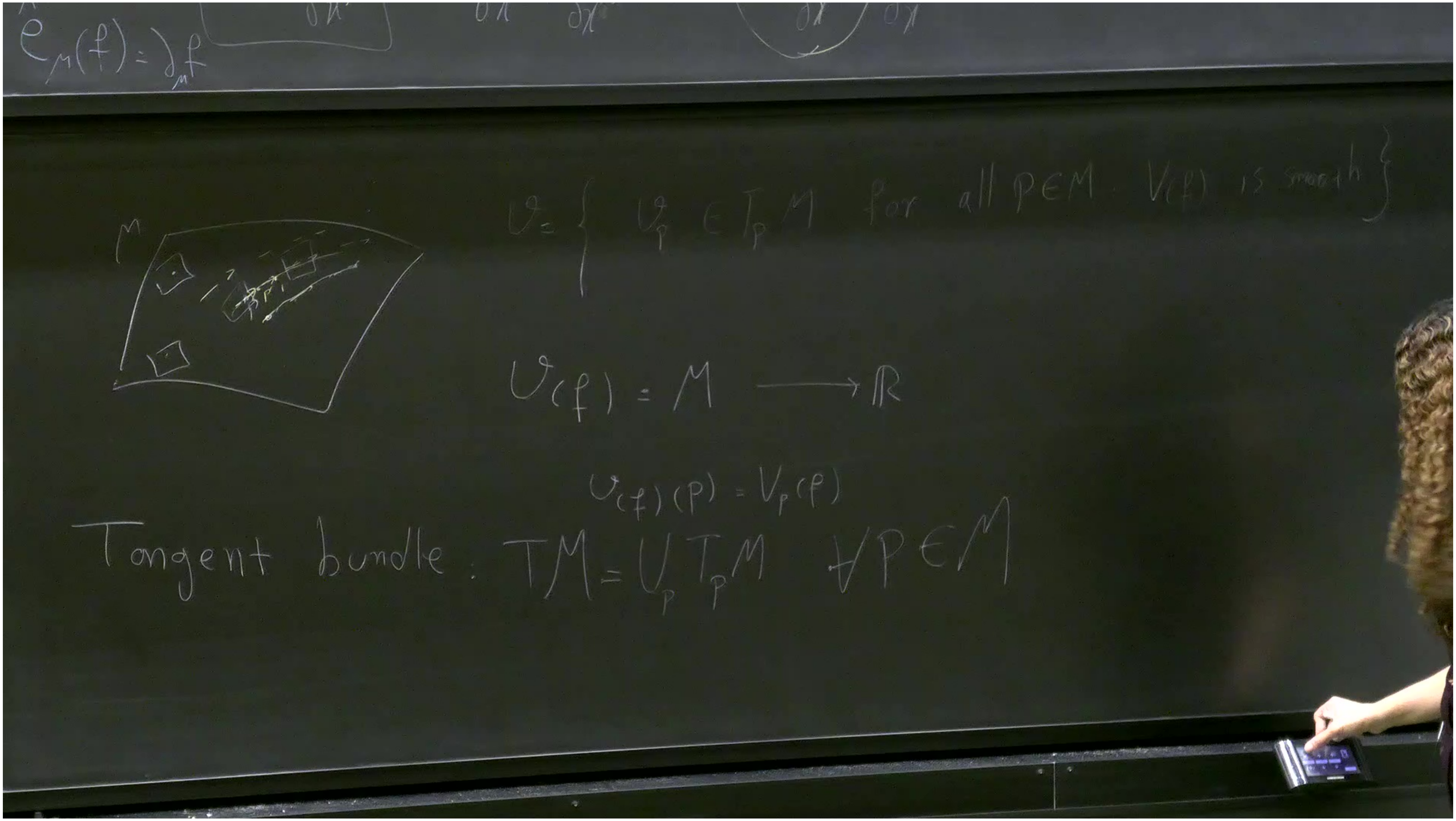
Same dimension n, $\left\{ \hat{e}_\mu = \frac{\partial}{\partial x^\mu} \right\}$ basis

x^μ : $V = V^\mu \hat{e}_\mu = V^\mu \frac{\partial}{\partial x^\mu}$

$V(f) = V^\mu \frac{\partial \hat{f}(x^\mu)}{\partial x^\mu} = V^\mu \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial f(x')}{\partial x'^\alpha} = \underbrace{V^\mu \frac{\partial x'^\alpha}{\partial x^\mu}}_{V'^\mu} \frac{\partial}{\partial x'^\alpha} \hat{f}'$

$V = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial}{\partial x'^\alpha}$


Vectors at P forms a
T_pM
Same dimension n, $\{\hat{e}_\mu = \frac{\partial}{\partial x^\mu}\}$ basis
$v'^\mu(x') = v^\alpha(x) \frac{\partial x'^\mu}{\partial x^\alpha}$
v'^α
$v^\mu \frac{\partial}{\partial x^\mu}$
$\frac{\partial \hat{f}(x')}{\partial x'^\alpha} = \left(v^\mu \frac{\partial x'^\alpha}{\partial x^\mu}\right) \frac{\partial}{\partial x'^\alpha} \hat{f}$
$v = v^\mu(x) \frac{\partial}{\partial x^\mu} = v^\mu(x) \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial}{\partial x'^\alpha}$



$e_\mu(f) = \partial_\mu f$
$v = \{ v_p \in T_p M$ for all $p \in M$. $V(f)$ is smooth $\}$
M
$v(f) = M \longrightarrow \mathbb{R}$
$v(f)(p) = V_p(f)$
Tangent bundle: $TM = \bigcup_p T_p M \quad \forall p \in M$

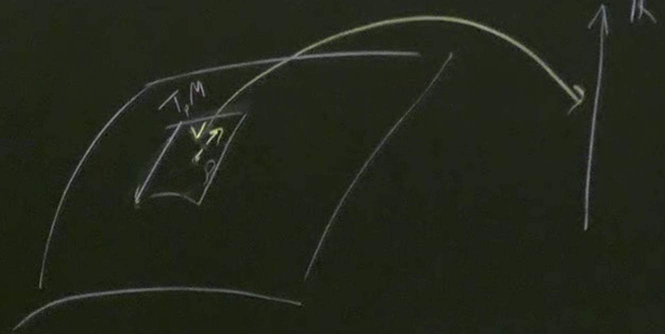
Cotangent Vector (1-form) ω at $p \in M$ is linear map $\omega : T_pM$

$\langle \omega | u \rangle = \alpha$

$\omega(V) \in \mathbb{R}$

$\vec{u}^{T} \; \vec{v} = \alpha'$

$1 \times n \qquad n \times 1$



$\omega(V_1 + V_2) = \omega(V_1) + \omega(V_2)$

$\omega \in T_p^{*}M$ cotangent vector space basis T_pM: $\hat{e}_\nu = \frac{\partial}{\partial x^\nu}$ or ∂_ν

T^*_pM: $\underbrace{\hat{e}^\mu(\hat{e}_\nu) = \delta^\mu}_{\text{basis}}{}_\nu$

coordinate basis dx^μ

$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$$

$$df(V) \equiv V(f)$$

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

$$V(f) = \frac{d f(x^\mu)}{d\lambda}\Big|_p = \frac{dx^\mu}{d\lambda}\,\partial_\mu f(x^\mu)\Big|_p = \hat{v}\cdot\nabla f$$

$$\omega = \omega_\mu \, dx^\mu = \left(\omega_\mu \frac{\partial x^\mu}{\partial x'^\alpha}\right) dx'^\alpha$$

$$= \omega'_\alpha \, dx'^\alpha$$

$$\omega'_\alpha = \omega_\mu \frac{\partial x^\mu}{\partial x'^\alpha}$$

$$(x^0, \ldots, x^{n-1}, \omega_0, \ldots, \omega_{n-1})$$

Cotangent bundle $= T^*M = \bigcup T_p^* M$

Tensor : A tensor of type (k, l) of Rank $k+l$ is a multi linear Map

$$T : \underbrace{T_p^* \times \dots \times T_p^*}_{k} \times \underbrace{T_p \times \dots \times T_p}_{l} \longrightarrow \mathbb{R}$$

$(2,1)$

$$T(\omega, Z, V) \in \mathbb{R} \qquad \partial_\mu \quad dx^\mu$$

$$T = T^{\mu\nu}{}_{\alpha}\, \partial_\mu \otimes \partial_\nu \otimes dx^\alpha$$

$$T(a w_1 + b w_2, cZ, V_1 + V_2) = a\, T(w_1, cZ, V_1 + V_2) + b\, T(w_2, cZ, V_1 + V_2)$$

is a multi linear map

$$\partial_\mu \otimes \partial_\nu \otimes dx^\alpha$$

$$\delta^\gamma_\mu \qquad \delta^\sigma_\nu \qquad \delta^\alpha_\beta$$

$$T(w, z, V) = \left(T^{\mu\nu}{}_\alpha\, \partial_\mu \otimes \partial_\nu \otimes dx^\alpha\right)\left(w_\gamma dx^\gamma\, z_\sigma dx^\sigma\, V^\beta \partial_\beta\right)$$

$$= T^{\gamma\sigma}{}_\beta\, w_\gamma z_\sigma V^\beta$$

Tensor : A tensor of type (k, l) of Rank $k+l$ is a multi linear Map

$$T : \underbrace{T_p^* \times \cdots \times T_p^*}_{k} \times \underbrace{T_p \times \cdots \times T_p}_{l} \longrightarrow \mathbb{R}$$

$(2,1)$

$$T(\omega, Z, V) \in \mathbb{R}$$

$\partial_\mu \quad dx^\mu$

$$T = T^{\mu\nu}{}_{\alpha}\, \partial_\mu \otimes \partial_\nu \otimes dx^\alpha$$

$$\frac{\partial x'^{\beta}}{\partial y^{\mu}\, \partial x'^{\alpha}}$$

$$T(a\omega_1 + b\omega_2, cZ, V_1 + V_2) = a\,T(\omega_1, cZ, V_1 + V_2) + b\,T(\omega_2, cZ, V_1 + V_2)$$

$T($