Title: Lecture - Relativity, PHYS 604

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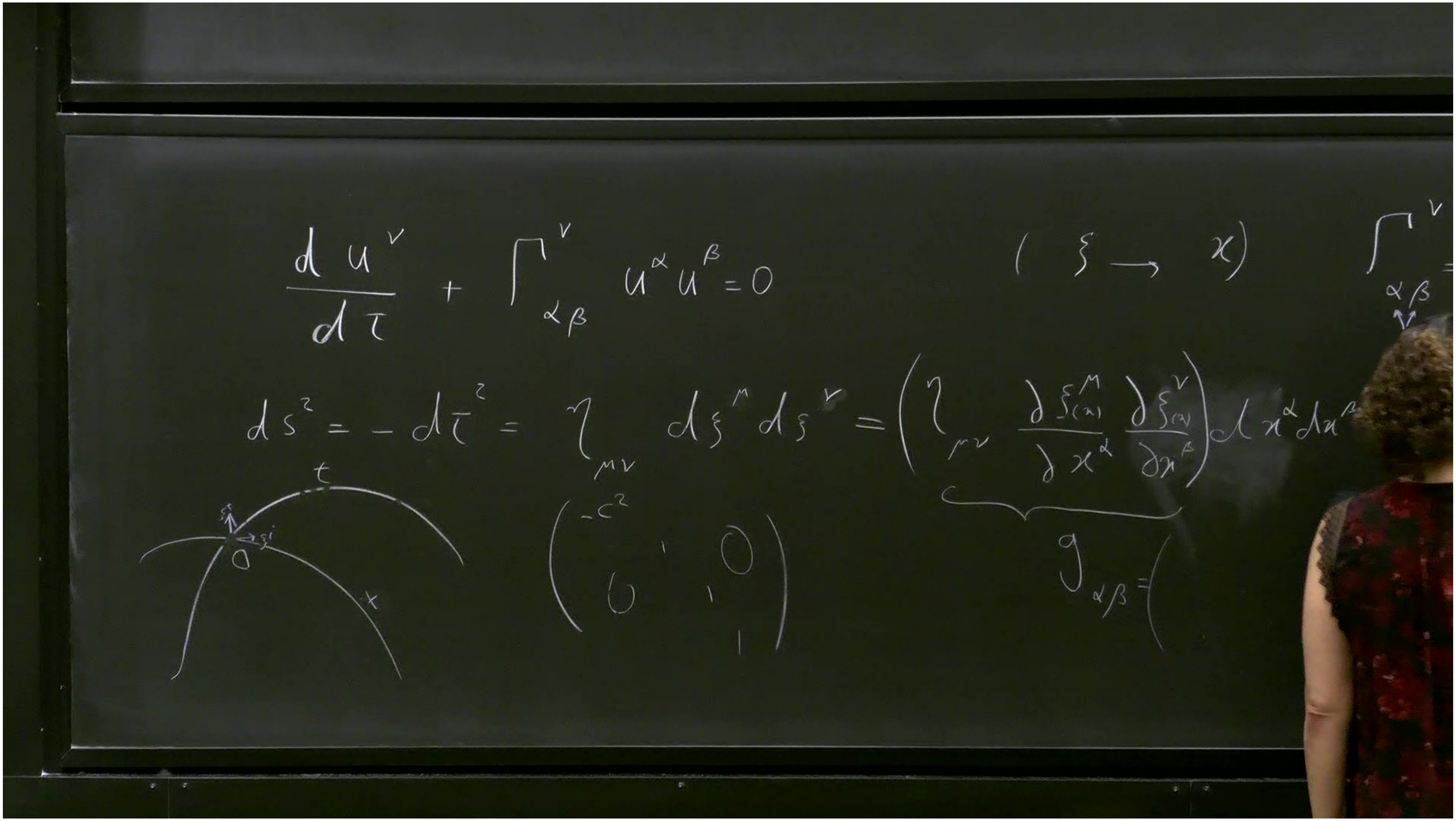
Collection/Series: Relativity (Core), PHYS 604, November 12 - December 11, 2024

Subject: Cosmology, Strong Gravity

Date: November 13, 2024 - 10:45 AM

URL: https://pirsa.org/24110023

Abstract:


$\frac{du^{\nu}}{d\tau} + \Gamma^{\nu}_{\alpha\beta} u^{\alpha} u^{\beta} = 0$
$(\xi \to x)$
$\Gamma^{\nu}_{\alpha\beta}$
$ds^2 = -d\tau^2 = \eta_{\mu\nu} d\xi^{\mu} d\xi^{\nu} = \Big(\underbrace{\eta_{\mu\nu} \frac{\partial \xi^{\mu}_{(x)}}{\partial x^{\alpha}} \frac{\partial \xi^{\nu}_{(x)}}{\partial x^{\beta}}}_{g_{\alpha\beta}}\Big) dx^{\alpha} dx^{\beta}$
$g_{\alpha\beta} = ($
t
ξ^{0}
ξ^{i}
O
x

$$\Gamma^{\nu}_{\alpha\beta} = \frac{\partial x^{\nu}}{\partial \xi^{\mu}} \frac{\partial^2 \xi^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} = \Gamma^{\nu}_{\beta\alpha}$$

$d x^{\alpha} d x^{\beta}$

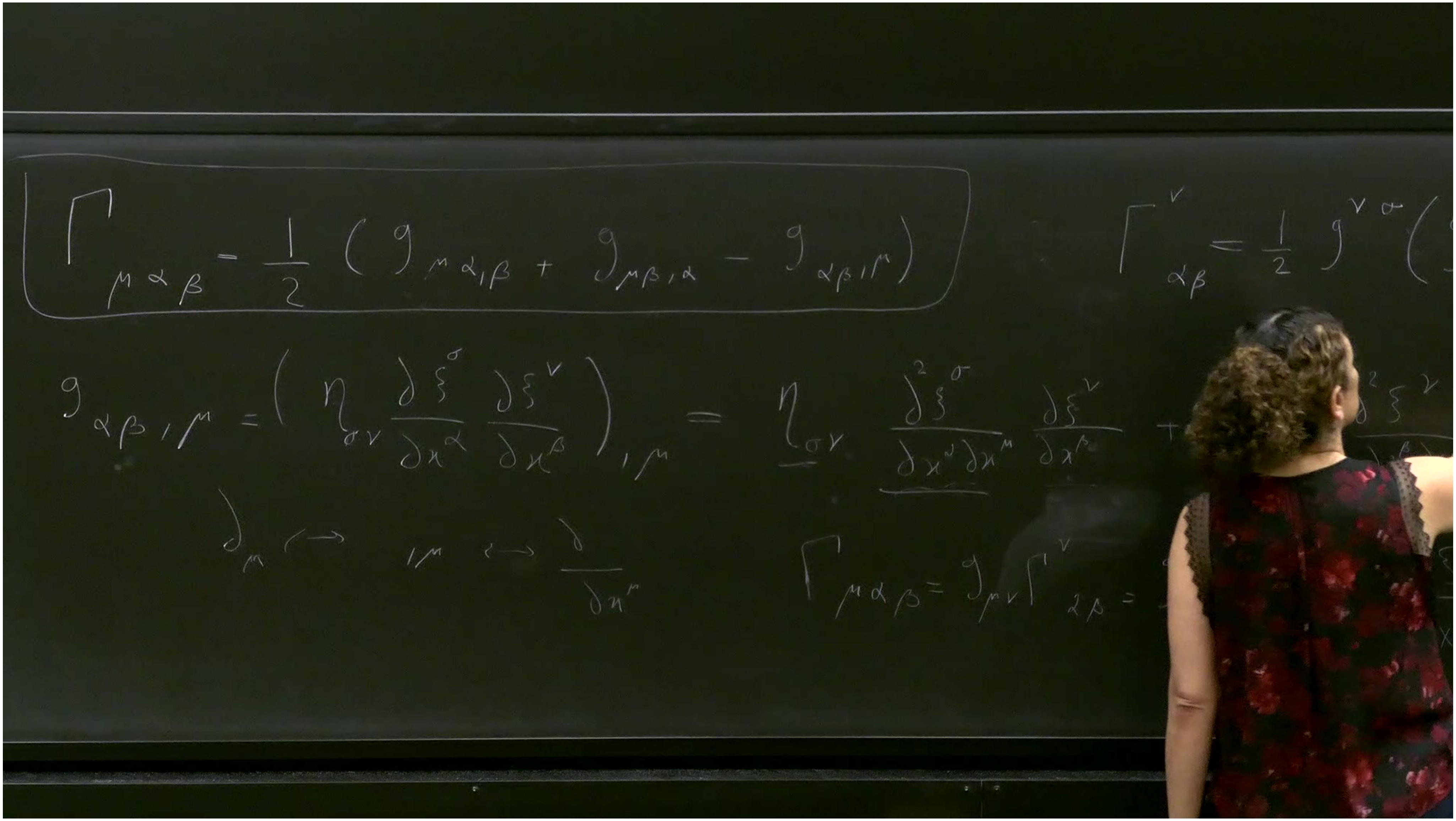
$$g^{\alpha\beta} \equiv \left(g_{\alpha\beta} \right)^{-1}$$

$$\Gamma_{\mu\alpha\beta} = g_{\mu\sigma} \Gamma^{\sigma}_{\beta\alpha}$$

g_{01}

$$g_{\alpha\beta} = g_{\beta\alpha} = g_{(\alpha\beta)}$$

$$\Gamma^{\mu}_{\alpha\beta} = g^{\mu\nu} \Gamma_{\nu\alpha\beta}$$



$\boxed{\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)}$
$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2} g^{\nu\sigma}(\dots$
$g_{\alpha\beta,\mu} = \left(\eta_{\sigma\nu}\frac{\partial\xi^{\sigma}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}\right)_{,\mu} = \eta_{\sigma\nu}\frac{\partial^2\xi^{\sigma}}{\partial x^{\alpha}\partial x^{\mu}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}} + \dots$
$\partial_{\mu} \longleftrightarrow {}_{,\mu} \longleftrightarrow \frac{\partial}{\partial x^{\mu}}$
$\Gamma_{\mu\alpha\beta} = g_{\mu\nu}\Gamma^{\nu}_{\alpha\beta} = \dots$

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2} g^{\nu\sigma}\left(g_{\sigma\alpha/\beta} + g_{\sigma\beta/\alpha} - g_{\alpha\beta/\sigma}\right)$$

$$+ \eta_{\sigma\nu}\frac{\partial\xi^{\sigma}}{\partial x^{\alpha}}\frac{\partial^{2}\xi^{\nu}}{\partial x^{\beta}\partial x^{\mu}} = \Gamma_{\beta\alpha\mu} + \Gamma_{\alpha\beta\mu}$$

$$\beta = g_{\mu\nu}\left(\frac{\partial x^{\nu}}{\partial\xi^{\sigma}}\frac{\partial^{2}\xi^{\sigma}}{\partial x^{\alpha}\partial x^{\beta}}\right) = \left(\eta_{\gamma\iota}\frac{\partial\xi^{\gamma}}{\partial x^{\mu}}\frac{\partial\xi^{\iota}}{\partial x^{\nu}}\right)\left(\frac{\partial x^{\nu}}{\partial\xi^{\sigma}}\frac{\partial^{2}\xi^{\sigma}}{\partial x^{\alpha}\partial x^{\beta}}\right) = \eta_{\gamma\sigma}\frac{\partial\xi^{\gamma}}{\partial x^{\mu}}\frac{\partial^{2}\xi^{\sigma}}{\partial x^{\alpha}\partial x^{\beta}}$$

$$\frac{\partial\xi^{\iota}}{\partial\xi^{\sigma}} = \delta^{\iota}_{\sigma}$$

Newtonian Limit of geodesic eq., $g_{tt} \leftrightarrow \phi$

- slow motion $|v^i| << 1$
- stationary $g_{\mu\nu,t} = 0$
- weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$ $\boxed{|h << 1}$

$$\frac{\partial \xi^\lambda}{\partial \xi^\sigma} = \delta^\lambda_\sigma$$

ϕ

$O(v^2, hv, h^2)$

keep only 1st order v, h

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$$u^0 = \frac{dt}{d\tau} = \gamma_v = 1 + O(v^2)$$

$$u^i = \frac{dx^i}{d\tau} = \gamma v^i = v^i$$

$\boxed{h \ll 1}$

$$\frac{du^\mu}{dt} + \Gamma^\mu_{\alpha\beta} u^\mu u^\nu = 0$$

$$\frac{\partial \xi^\sigma}{\partial x^\alpha \partial x^\beta}\Big) = \Big(\Gamma \frac{\partial \xi}{\partial x^\mu} \frac{\partial x^\nu}{\partial \xi^\sigma} \Big)\Big(\frac{\partial \xi}{\partial x^\alpha \partial x^\beta} \Big) \qquad \frac{\partial x^\mu}{\partial x^\alpha \partial x^\beta}$$

$$\frac{\partial \xi^\iota}{\partial \xi^\sigma} = \delta^\iota_\sigma$$

ϕ

$O(v^2, hv, h^2)$

keep only 1st order v, h

$h \ll 1$

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad u^0 = \frac{dt}{d\tau} = \gamma_v = 1 + O(v^2)$$

$$u^i = \frac{dx^i}{d\tau} = \gamma v^i = v^i$$

$$\frac{du^\mu}{dt} + \Gamma^\mu_{\alpha\beta} u^\mu u^\nu = 0 \Rightarrow \frac{du^\mu}{dt} + \Gamma^\mu_{00} u^0 u^0 = 0$$

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2}\eta^{\nu\sigma}\left(h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma}\right) \sim O(h) \qquad \Gamma^{\nu i}_{tt} = \frac{1}{2}\eta^{\nu\sigma}(\ldots$$

$$\frac{du^0}{dt} + \cancel{\Gamma^{0}_{00}}\,(u^0)^0 = 0$$

$$\frac{dv^i}{dt} + \Gamma^{i}_{00} = 0 \Rightarrow \frac{dv^i}{dt} = -\Gamma^{i}_{00} = \frac{1}{2}h_{tt,}{}^{i}$$

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2}\eta^{\nu\sigma}\left(h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma}\right) \sim O(h)$$

$$\Gamma^{\nu}_{tt} = \frac{1}{2}\eta^{\nu\sigma}(\ldots$$

$$\frac{du^0}{dt} + \cancel{\Gamma^{0}_{00}}\,(u^0)^0 = 0$$

$$\frac{dv^i}{dt} + \Gamma^{i}_{00} = 0 \Rightarrow \frac{dv^i}{dt} = -\Gamma^{i}_{00} = \frac{1}{2}h_{tt,}{}^{i}$$

$$\frac{d\vec{v}}{dt} = F_g/m = -\vec{\nabla}\phi$$

$$\ldots\left(h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma} \right) \sim O(h)$$

$$\Gamma^{\nu}_{tt} = \frac{1}{2}\, \eta^{\nu\sigma}\left(\cancel{h_{\sigma t,t}} + \cancel{h_{\sigma t,t}} - h_{tt,\sigma} \right)$$

$$(u^0)^{0} = 0$$

$$= 0 \Rightarrow \frac{dv^i}{dt} = -\Gamma^{i}_{00} = \frac{1}{2} h_{tt,}{}^{i} = \frac{\partial}{\partial x^i}\left(\frac{1}{2} h_{tt}\right) \Rightarrow \frac{d\vec{v}}{dt} = \vec{\nabla}\left(\frac{1}{2} h_{tt}\right) \Rightarrow h_{tt} = -2\phi$$

$$\frac{d\vec{v}}{dt} = F_g/m = -\vec{\nabla}\phi$$

$$\Gamma^{\nu}_{tt} = \frac{1}{2}\eta^{\nu\sigma}\left(\cancel{h_{\sigma t,t}} + \cancel{h_{\sigma t,t}} - h_{tt,\sigma}\right) = -\frac{1}{2}\eta^{\nu i} h_{tt,i}$$

$$= \frac{\partial}{\partial x^i}\left(\frac{1}{2}h_{tt}\right) \Rightarrow \frac{d\vec{v}}{dt} = \vec{\nabla}\left(\frac{1}{2}h_{tt}\right) \Rightarrow h_{tt} = -2\phi \Rightarrow g_{tt} = \eta_{tt} + h_{tt} = -c^2 - 2\phi = -c^2\left(1 + 2\phi/c^2\right)$$

$$\phi_{(r)} = \frac{GM}{r}$$

M

$$\frac{2\phi}{c^2} = \frac{2GM}{c^2 r^2} << 1$$

Proton 10^{-39}

Earth 10^{-9}

Sun 10^{-6}

Neutron Stars $10^{-2} - 10^{-1}$

B.H $\sim 10^{-1} - 1$

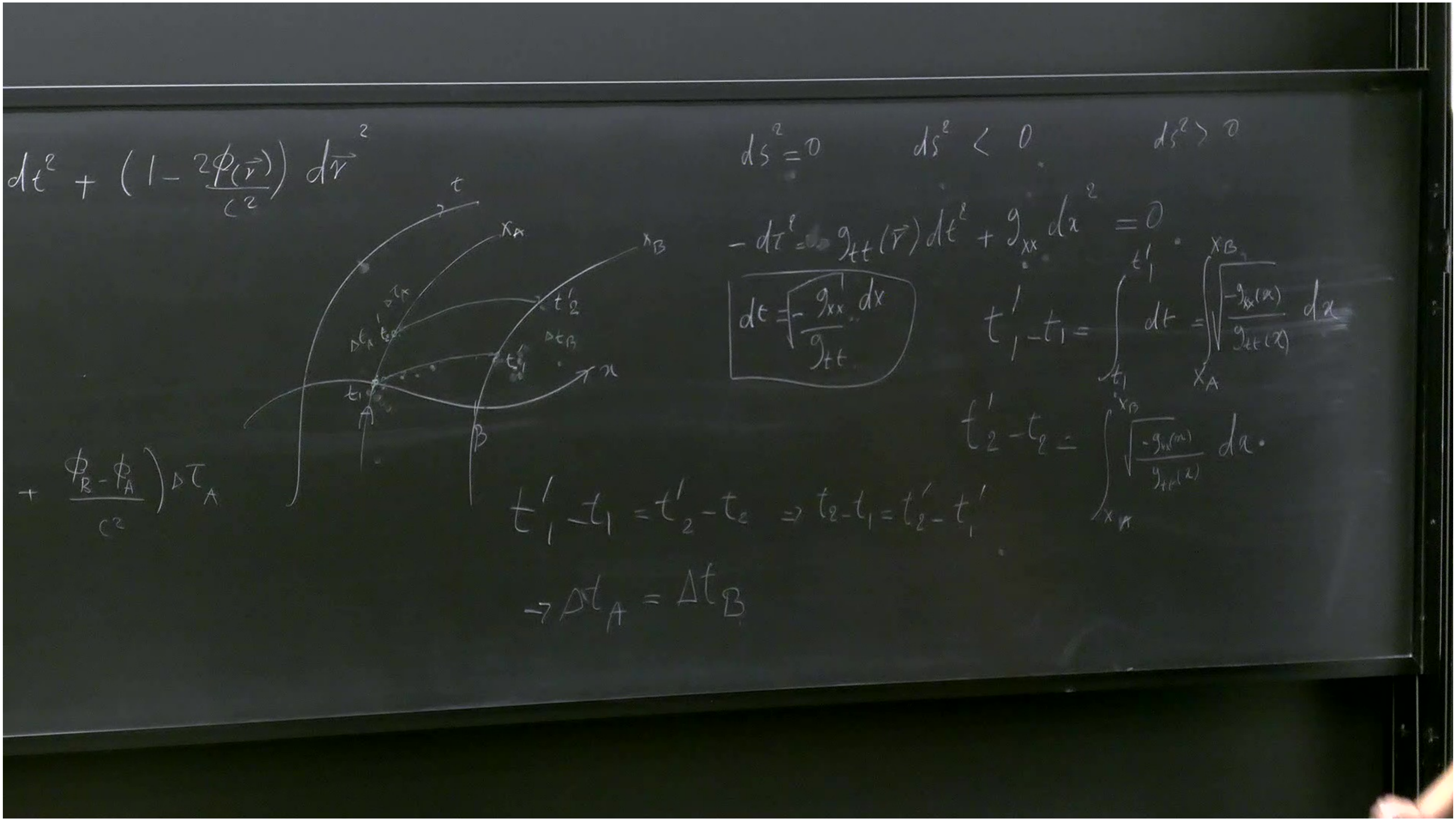
$$ds^2 = -c^2\left(1 + \frac{2\phi(\vec{r})}{c^2}\right)dt^2 + \left(1 - \frac{2\phi(\vec{r})}{c^2}\right)d\vec{r}^{\,2}$$

$$\Delta\tau_B = ($$

B $\cdot$ ϕ_B

A $\cdot$ ϕ_A

$$\Delta\tau_B \simeq \left(1 + \frac{\phi_B - \phi_A}{c^2}\right)\Delta\tau_A$$


$dt^2 + \left(1 - \frac{2\phi(\vec{r})}{c^2}\right) d\vec{r}^{\,2}$
$ds^2 = 0$
$ds^2 < 0$
$ds^2 > 0$
$-d\tau^2 = g_{tt}(\vec{r})\, dt^2 + g_{xx}\, dx^2 = 0$
$dt = \sqrt{-\frac{g_{xx}}{g_{tt}}}\, dx$
$t_1' - t_1 = \int_{t_1}^{t_1'} dt = \int_{x_A}^{x_B} \sqrt{\frac{-g_{xx}(x)}{g_{tt}(x)}}\, dx$
$t_2' - t_2 = \int_{x_A}^{x_B} \sqrt{\frac{-g_{xx}(x)}{g_{tt}(x)}}\, dx$
$+ \frac{\phi_B - \phi_A}{c^2}\Big) \Delta\tau_A$
$t_1' - t_1 = t_2' - t_2 \Rightarrow t_2 - t_1 = t_2' - t_1'$
$\rightarrow \Delta t_A = \Delta t_B$
t
x_A
x_B
x
t_1
t_2
t_1'
t_2'
Δt_A
Δt_B
$\Delta\tau_A$
A
B

$$\Delta\tau_A \simeq d\tau_A \simeq g_{tt}(x_A)\,\Delta t_A$$

$$d\tau_A^{2} = g_{tt}\, dt_A^{2}$$

$$\Delta\tau_B \simeq g_{tt}(x_B)\,\Delta t_B$$

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \frac{g_{tt}(x_B)}{g_{tt}(x_A)}\,\frac{\cancel{\Delta t_A}}{\cancel{\Delta t_B}} = \frac{-c^2(1+2\phi}{-c^2(1+2}$$

$A: \phi_A$

$\rightarrow \Delta t_A = \Delta t_B$

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{g_{tt}(x_B)}{g_{tt}(x_A)}}\;\frac{\cancel{\Delta t_A}}{\cancel{\Delta t_B}} = \sqrt{\frac{\cancel{-c^2}\left(1+2\phi_B/c^2\right)}{\cancel{-c^2}\left(1+2\phi_A/c^2\right)}} = \left(1+\phi_B/c^2\right)\left(1-\phi_A/c^2\right) = 1 + \frac{\phi_B-\phi_A}{c^2}$$