Title: Lecture - QFT II, PHYS 603

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Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

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$G = SU(2)$ Gauge Group

Gauge Field $A \longrightarrow \{A^a_\mu(x)\}$

Covariant Derivative $D_\mu A_\nu = \partial_\mu A_\nu - i[A_\mu, A_\nu]$

Action $S_{YM}[A] = -\frac{1}{2g^2}\int dx\, \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$

Field Strength $F_{\mu\nu} = [D_\mu, D_\nu] =$

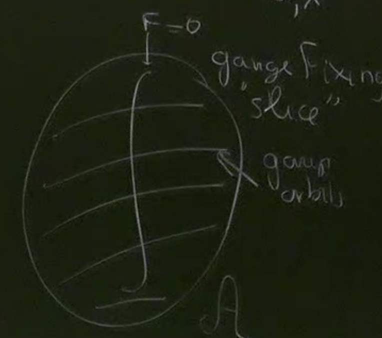
Lorentz Gauge

Gauge Fixing $F[A]$ $\quad$ $F[A] = \{F^a[A](x) = \partial^\mu A^a_\mu(x)\}$

$$\int D[A]\, e^{i S_{YM}[A]}\, \delta[F[A]] \cdot ?$$

Dirac

$F[A] = 0 \qquad \prod_{a,x} \delta[F^a[A](x)]$



? .

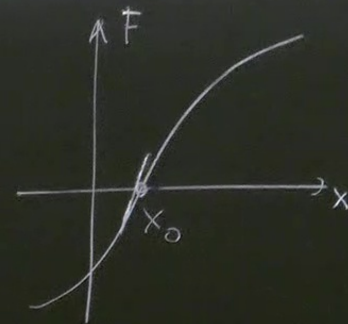
How to write 1 as an integral ?

(1) $\int_{-\infty}^{+\infty} dx \, \delta(x - x_0) = 1$

$F(x) \quad \mathbb{R} \to \mathbb{R}$

$F'(x) > 0$

↖ derivative



only one x_0, $F(x_0) = 0$

$1 = \int dx \, \delta(F(x)) \, F'(x_0) = \boxed{\int dx \, \delta(F(x)) \, |F'(x)| = 1}$ if $F' < 0$

(2) $F(x) \quad \mathbb{R}^N \to \mathbb{R}^N$ N variables

$x = (x_1 \dots x_N)$

$F(x) = (F^1(x) \dots F^N(x))$

a unique point in $\mathbb{R}^N$

$F'(x) = \left\{ \frac{\partial F^i}{\partial x_j}(x) \right\} > 0$, $F(x_0) = 0$

$N \times N$ matrix

$F \quad \mathbb{R}^N \to \mathbb{R}^N$

$$\boxed{\int d^N x \, \delta(x - x_0) = \int dx^N \, \delta(F(x)) \left| \det(F'(x)) \right| = 1}$$

N-dim "Dirac Funct"

$F'(x) = \left(F'^{ij}(x) \right) \quad (F')^{ij} = \frac{\partial F^i}{\partial x_j}$

N×N matrix   partial derivative

(3) F Group $G \longrightarrow \mathbb{R}^N$

dim N $SU(2) \to \mathbb{R}^3$

what is $F'(g)$?

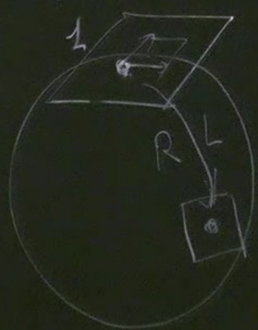
Lie Group theory

1

Lie Group theory

G has an invariant measure

Haar measure. $\mathbb{1} \to \mathbb{1} + i\,\delta\alpha + \dots = \mathbb{1}(\mathbb{1} + \delta_3)$



$\delta\alpha = \alpha_a t^a$

$d\mu(g) = \prod_a d\alpha_a$

$\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$

$g \xrightarrow{g'} g'g$ or $g \cdot g'$

$\qquad\quad L \qquad\quad R$

$$F \quad G \to \mathbb{R}^N \qquad N = \text{dim of the group}$$

$$g \to F(g) = \left(F_a(g)\right)_{a=1,N} \qquad \delta\alpha = \delta\alpha_b t^b$$

$$g \to g(1 + i\,\delta\alpha) \quad \text{inf. deformation of } g$$

$$F_a(g(1+i\,\delta\alpha)) - F_a(g) = \delta F_a = \boxed{\frac{\partial F_a(g(1+i\alpha))}{\partial \alpha_b}} \cdot \delta\alpha_b + \dots$$

define $\boxed{F'_{ab}(g) = \left.\frac{\partial F_a(g(1+i\alpha))}{\partial \alpha_b}\right|_{\alpha=0}}$

$N \times N$ matrix

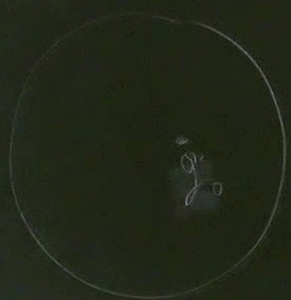
$F: G \longrightarrow \mathbb{R}^N$ a unique g_0 s.t. $F(g)=0$

$$\int_G d\mu_{Haar}(g)\, \delta_{Haar}(g, g_0) = \boxed{\int_G d\mu(g)\, \delta[F(g)]\, \left|\det[F'(g)]\right| = 1}$$

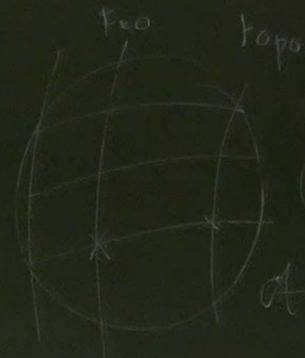
δ-function on the group

δ-function on $\mathbb{R}^N$

$N \times N$ matrix



G



topology of the group

Gribov ambiguities

(not important)

N × N matrix

$[F'(g)]] = 1$

N × N matrix

an assumption

Gauge Fixing

$G = SU(2)$; $\mathcal{G} = \times_{x \in M} G_x$

$\boxed{\text{start from } A}$ $F[A] \neq 0$

gauge field configuration

gauge transf. $A \to A_g = g A g^{-1} + \partial g \cdot g^{-1}$

<u>a unique g_0</u> such that A_{g_0} satisfies

my gauge fixing condition $F[A_g] = 0$

$$\boxed{1 = \int_{\mathcal{G}} D[g]\, \delta[F[A_g]]\, \left|\det(F'[A_g])\right|}$$

all gauge transformations

N×N Matrix

$= SU(2) \to \mathcal{G} = \underset{x \in M}{\times} G_x$

$\neq 0$

$g^{-1} + \partial g \cdot g^{-1}$

g_0 satisfies

$F[A_g] = 0$

$] \, \left| \det(F'[A_g]) \right|$ *

$$Z = \int D[A] \exp(i S_{YM}[A])$$ *

$$= \int_{\mathcal{A}} D[A] \int_{\mathcal{G}} D[g] \, \delta[F(A_g)] \left| \det[F'[A_g]] \right| \underbrace{\exp(-i S_{YM}[A])}$$

$$= \int_{\mathcal{G}} D[g] \int_{\mathcal{A}} D[A] \quad ''$$

$$D[A] = D[A_g], \quad S_{YM}[A] = S[A_g]$$

gauge invariance

$N \times N$ matrix

$G = SU(2) \Rightarrow \mathcal{G} = \times_{x \in M} G_x$

$] \neq 0$

$g^{-1} + \partial g \cdot g^{-1}$ most general gauge transf

A_{g_0} satisfies

$F[A_g] = 0$

$] \, | \det (F'[A_g]) |$ *

③

$$Z = \int D[A] \exp(i S_{YM}[A]) \; *$$

$$= \int_{\mathcal{A}} D[A] \int_{\mathcal{G}} D[g] \, \delta[F[A_g]] \, \underbrace{\left| \det [F'[A_g]] \right| \exp(-i S_{YM}[A])}_{''}$$

$$= \int_{\mathcal{G}} D[g] \int_{\mathcal{A}} D[A] \quad ''$$

$$D[A] = D[A_g], \quad S_{YM}[A] = S[A_g]$$

gauge invariance — Haar meas on G — Lattice Theory (K. Wilson)

(2)

(4) g fixed, A_g is a dummy variable $A_g \to A$ Faddeev-Popov Determinant

$$Z = \int_{\mathcal{G}} D[g] \times \boxed{\int D[A]\, \delta[F[A]]\ \det[F'[A]]\ \exp(i S_{YM}[A])}$$

$$= \cancel{\mathrm{Vol}(\mathcal{G})} \times \boxed{}$$

Gauge Fixed correct Functional integral

Sect. 6.5 of notes

$F[A] \to F^a[A](x) = \partial^\mu A_\mu^a(x)$ $\quad A_\mu = A_\mu^a t_a,\ \delta\alpha = \delta\alpha^a t_a$

infinit. gauge transf.

$A' = A + \delta A$

$$A_\mu^a \to A_\mu^{a\,\prime} = A_\mu^a + \boxed{(D_\mu \delta\alpha)^a = A_\mu^a + \underbrace{\partial_\mu \delta\alpha^a - i[A_\mu, \delta\alpha]^a}_{\delta A_\mu^a(x)}}$$

$$F^a[A+\delta A](x) = \partial^\mu A_\mu^a(x) + \boxed{\partial^\mu \delta A_\mu^a(x)}$$

$$\delta F^a(x) = \partial^\mu \delta A_\mu^a(x) = \boxed{\partial^\mu \cdot D_\mu \cdot \delta\alpha^a} \Longrightarrow [F'(A)]^{ab}_{xy} = [\partial^\mu D_\mu]^{ab}_{xy}$$

Diff. op. which depends on A

Diff operator in X

$$F'[A] = \left[\partial^\mu D_\mu\right]^{ab}_{xy}$$

Diff operator

$$\partial^\mu D_\mu \cdot \alpha^a = \partial^\mu\left[\partial_\mu \alpha^a - i\left[A_\mu, \alpha\right]^a\right]$$

$$= \partial^\mu\partial_\mu \alpha^a - i\,\epsilon_{bc}{}^a\left(\partial^\mu A^b_\mu \cdot \alpha^c + A^b_\mu \partial^\mu \alpha^c\right)$$

$$\left[A_\mu, \alpha\right]^a = \epsilon_{bc}{}^a A^b_\mu \alpha^c$$

structure cst of $SU(2)$

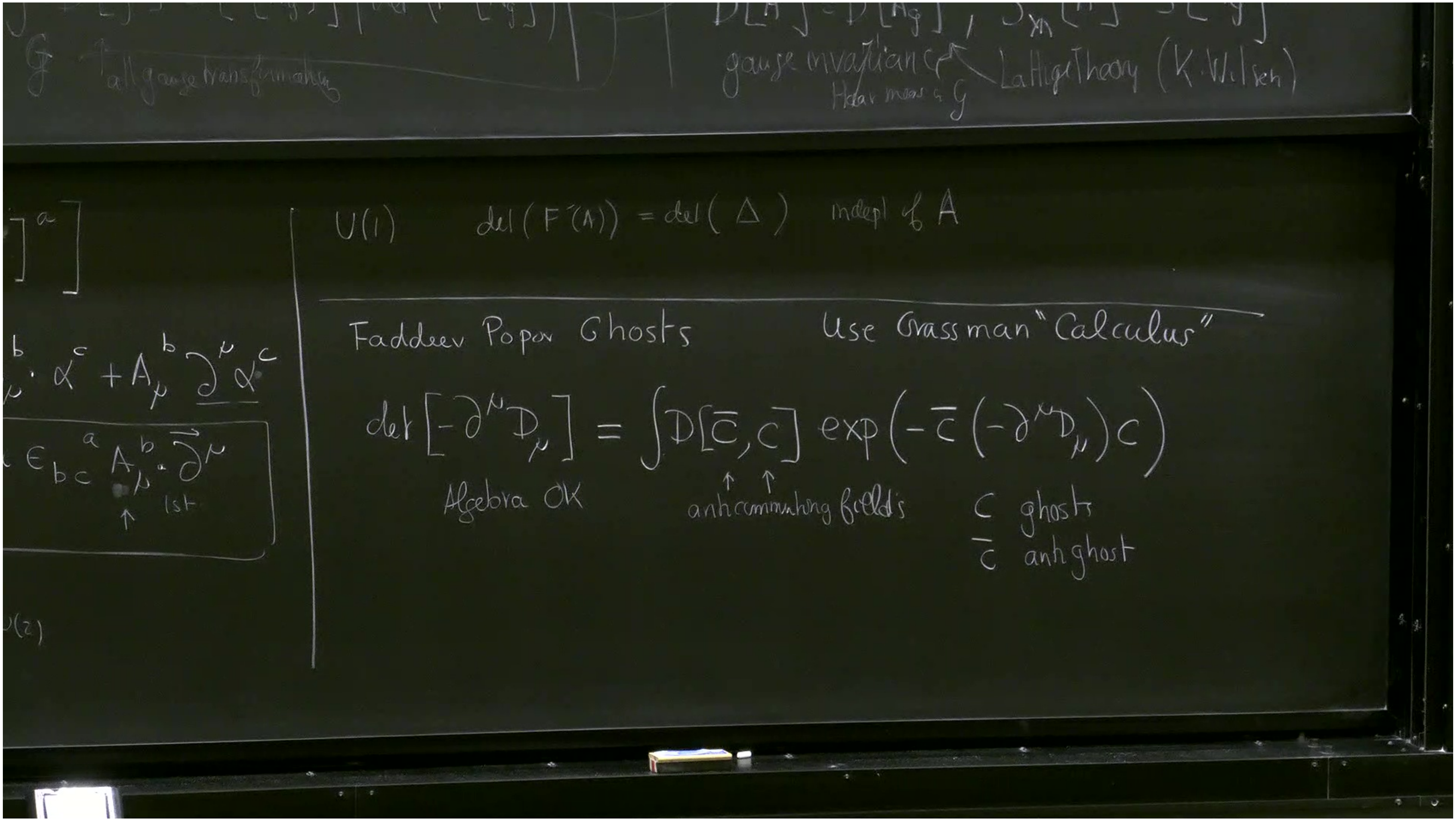
$$\left[\partial^\mu D_\mu\right]^{ac} = \delta^{ac}\Delta - i\,\epsilon_{bc}{}^a\left(\partial^\mu A^b_\mu\right) - i\,\epsilon_{bc}{}^a A^b_\mu \cdot \vec{\partial}^\mu$$

diff operator — 2nd — 0 order term — 1st

$$\boxed{\det\left[F'[A]\right] = \det\left[\partial^\mu D_\mu\right]}$$

depends on A

differential operator which depends on A — functions α $M \to Lie_{SU(2)}$


gauge invariance
Haar meas of G
Lattice Theory (K. Wilson)
all gauge transformations
U(1) det(F'(A)) = det(Δ) indep of A
Faddeev Popov Ghosts
Use Grassman "Calculus"
det[−∂^μ D_μ] = ∫D[c̄, c] exp(−c̄(−∂^μ D_μ)c)
Algebra OK
anticommuting fields
c ghosts
c̄ antighost

$$\mathcal{C} = \{ C^a(x) \} \qquad a = 1, 2, 3 \quad x \in M$$

$$\bar{\mathcal{C}} = \{ \bar{C}_a(x) \}$$

$$Z =$$

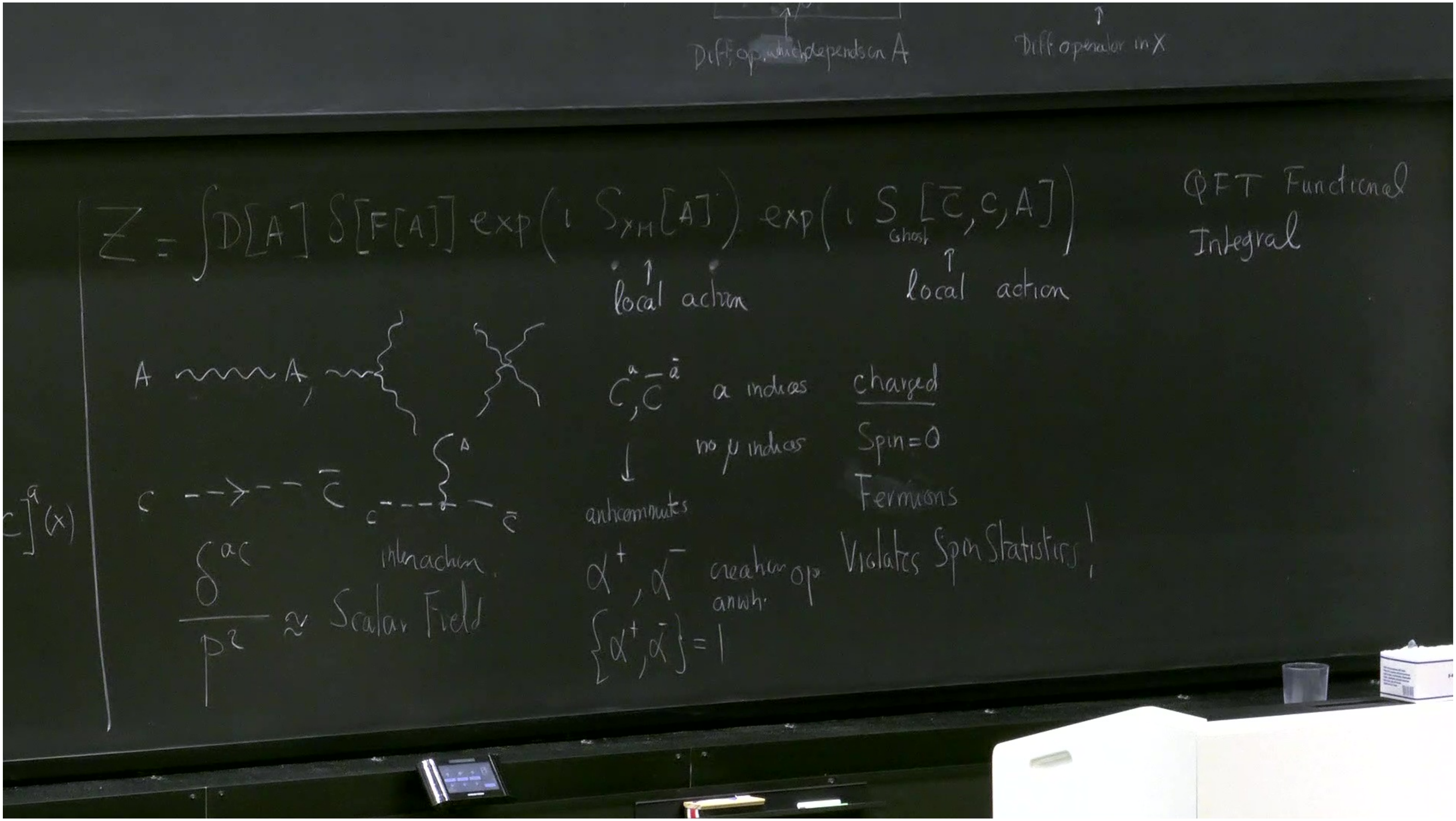
$$S[C, \bar{C}] = -\int d^4x \; \bar{C}_a(x) \, \partial^\mu D_\mu C^a(x)$$

Ghosts "Action"

$$= \int d^4x \; \bar{C}_a(x)(-\partial^\mu \partial_\mu) C^a(x) + i \, \bar{C}_a \, \partial^\mu [A_\mu, C]^a(x)$$

$$\boxed{= \int d^4x \, [\partial^\mu \bar{C}_a(x)][D_\mu C^a(x)]}$$

contains A


Diff op. which depends on A
Diff operator in X
QFT Functional Integral
$Z = \int D[A]\, \delta[F[A]] \exp(i S_{YM}[A]) \exp(i S_{Ghost}[\bar{c}, c, A])$
local action
local action
$c^a, \bar{c}^a$ a indices charged
no μ indices Spin=0
anticommutes
Fermions
Violates Spin Statistics!
α^+, α^- creation op. annih.
$\{\alpha^+, \alpha^-\} = 1$
interaction
$\frac{\delta^{ac}}{p^2} \approx$ Scalar Field
$c]^a(x)$

all gauge transformations

gauge invariance

Haar measure in G

Lattice Theory (K. Wilson)

$U(1)$ $\quad \det(F'(A)) = \det(\Delta)$ indep. of A

Faddeev Popov Ghosts

Use Grassman "Calculus"

$$\det\left[-\partial^\mu D_\mu\right] = \int D[\bar{c}, c] \exp\left(-\bar{c}\left(-\partial^\mu D_\mu\right)c\right)$$

Algebra OK

anticommuting fields

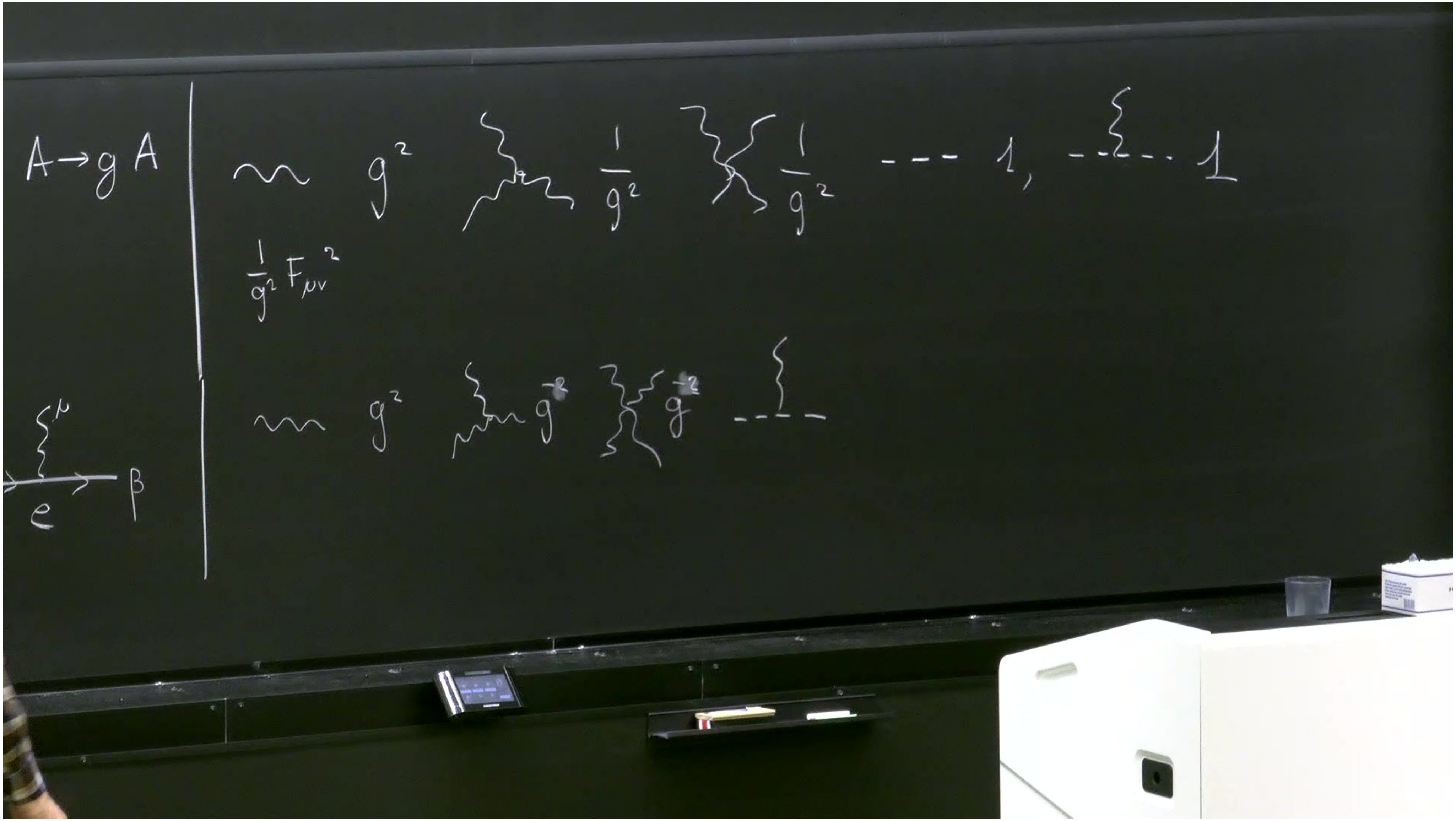
c ghosts

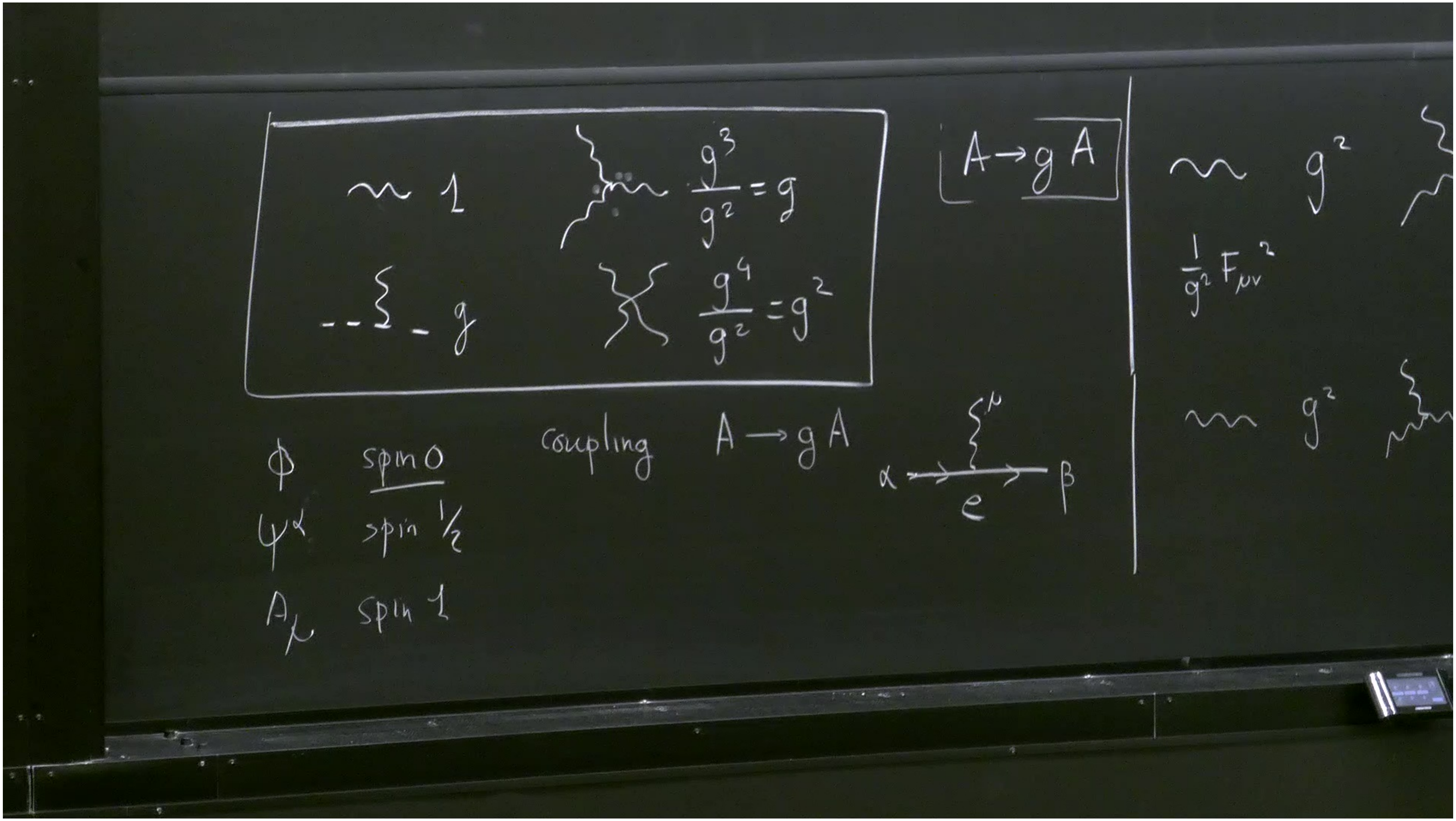
$\bar{c}$ antighost

$\alpha^c + A_\mu^b \partial^\mu \alpha^c$

$A_\mu^b \cdot \overrightarrow{\partial}^\mu$

1st


A → g A
g²
1/g²
1/g²
1, 1
1/g² F_μν²
g²
g⁻²
g⁻²
ν
e
β


1
g^3/g^2 = g
g
g^4/g^2 = g^2
A → g A
g^2
1/g^2 F_μν^2
g^2
φ spin 0
ψ^α spin 1/2
A_μ spin 1
coupling A → g A
μ
α e β