Title: Lecture - QFT II, PHYS 603

Speakers: Francois David

Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

Date: November 22, 2024 - 9:00 AM

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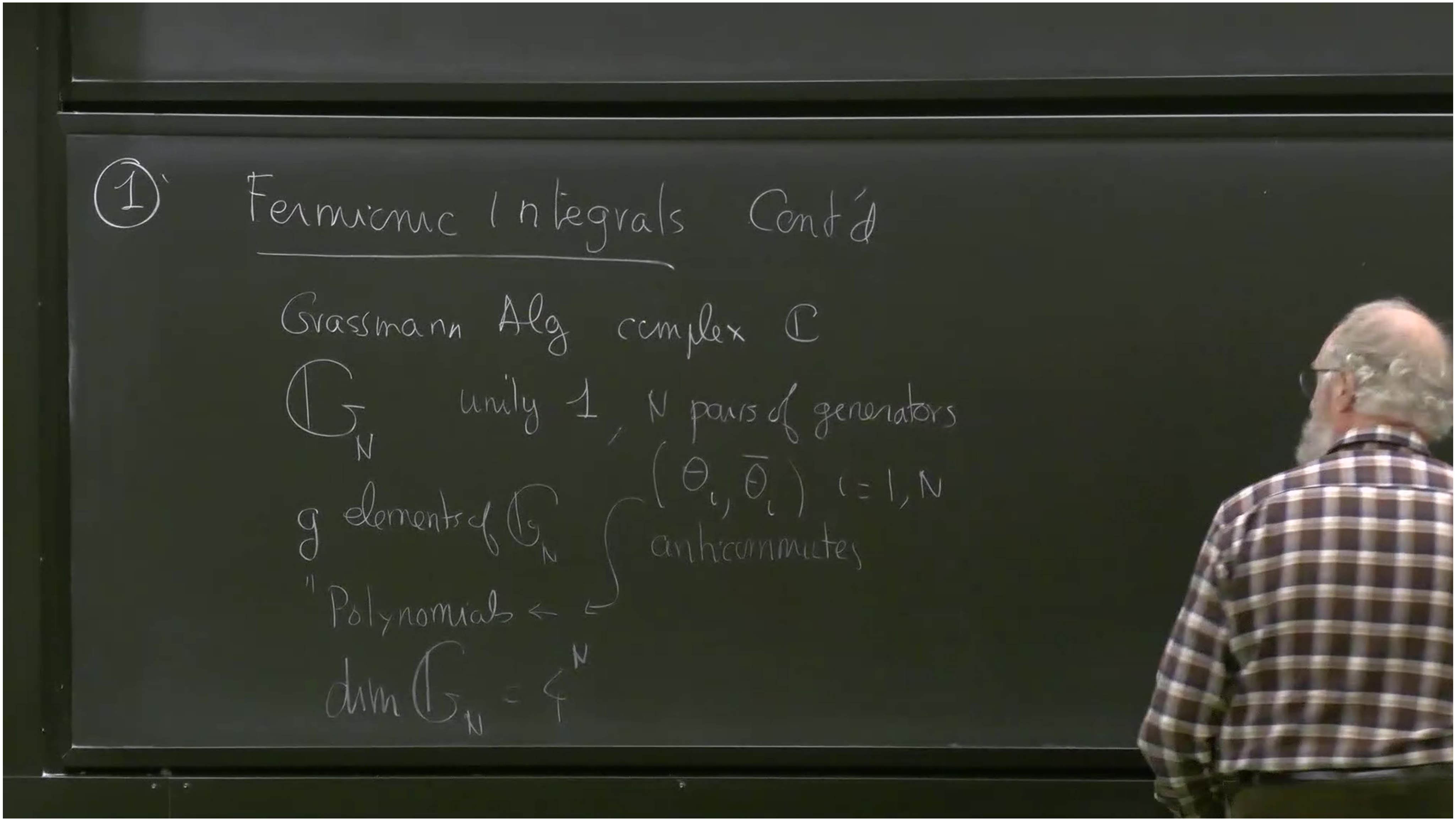


① Fermionic Integrals Cont'd

Grassmann Alg complex $\mathbb{C}$

$\mathbb{G}_N$ unity 1, N pairs of generators

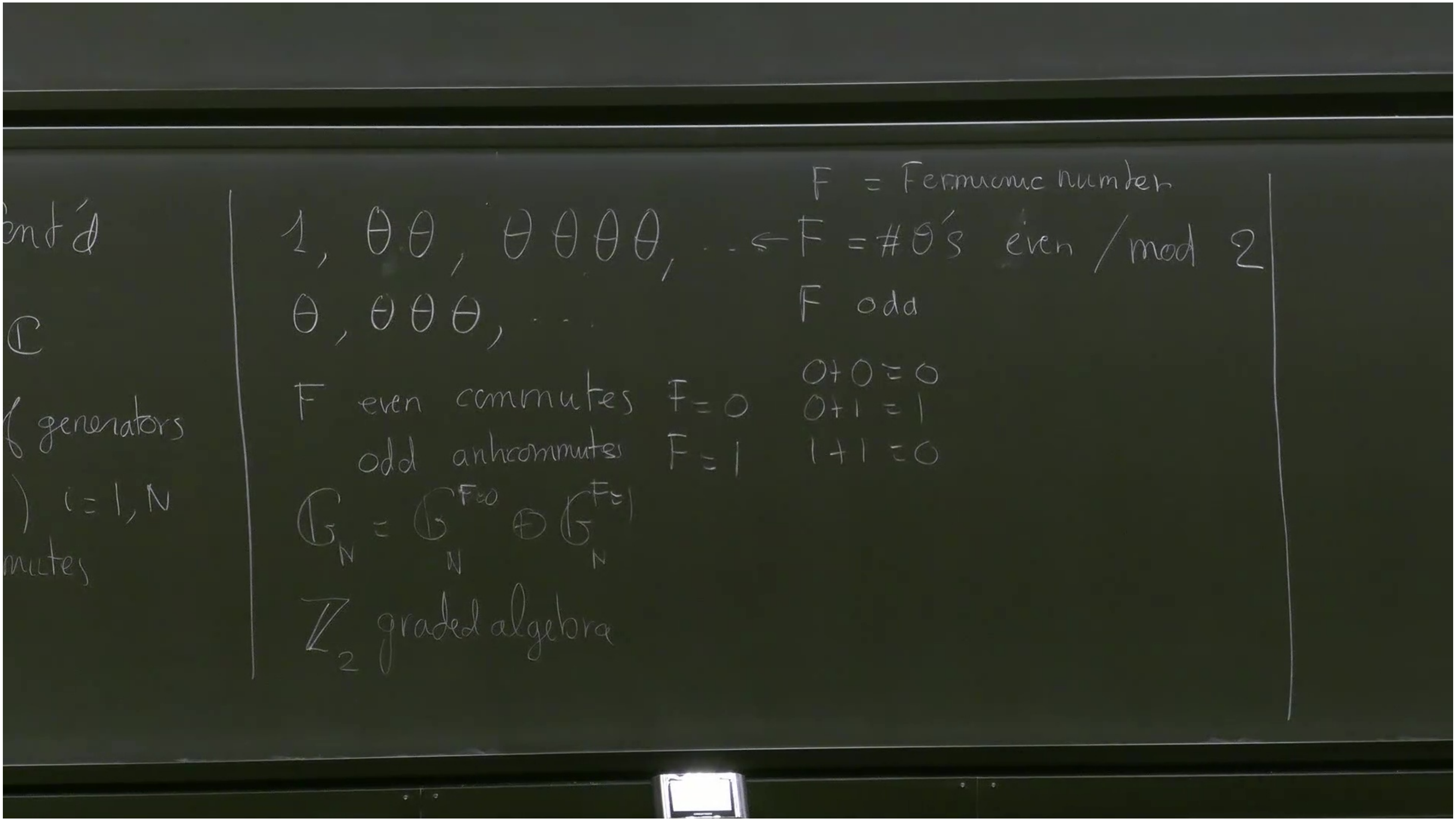
$(\theta_i, \bar{\theta}_i)$ $i = 1, N$

anticommutes

g elements of $\mathbb{G}_N$

"Polynomials ←

$\dim \mathbb{G}_N = 4^N$



nt'd

$\mathbb{C}$

f generators

) $i=1,N$

mutes

F = Fermionic number

$1,\ \theta\theta,\ \theta\theta\theta\theta,\ \ldots \leftarrow F = \#\theta\text{'s even} / \text{mod } 2$

$\theta,\ \theta\theta\theta,\ \ldots$ F odd

F even commutes $F=0$

odd anticommutes $F=1$

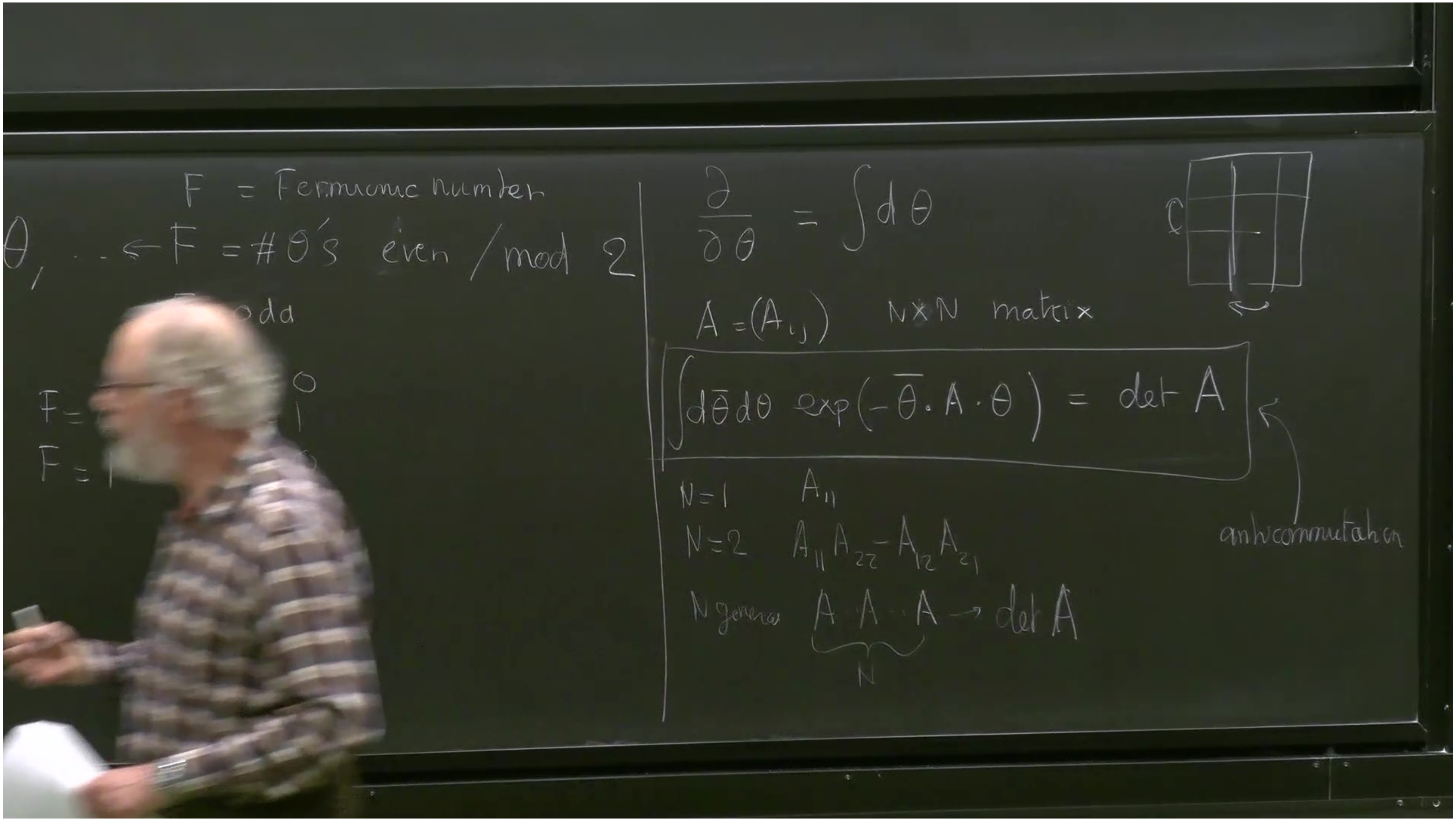
$0+0=0$

$0+1=1$

$1+1=0$

$$\mathbb{G}_N = \mathbb{G}_N^{F=0} \oplus \mathbb{G}_N^{F=1}$$

$\mathbb{Z}_2$ graded algebra



F = Fermionic number

$\theta, \dots \leftarrow F = \#\theta\text{'s}$ even / mod 2

odd

$F =$ 0

$F = 1$

$$\frac{\partial}{\partial\theta} = \int d\theta$$

$A = (A_{ij})$ N×N matrix

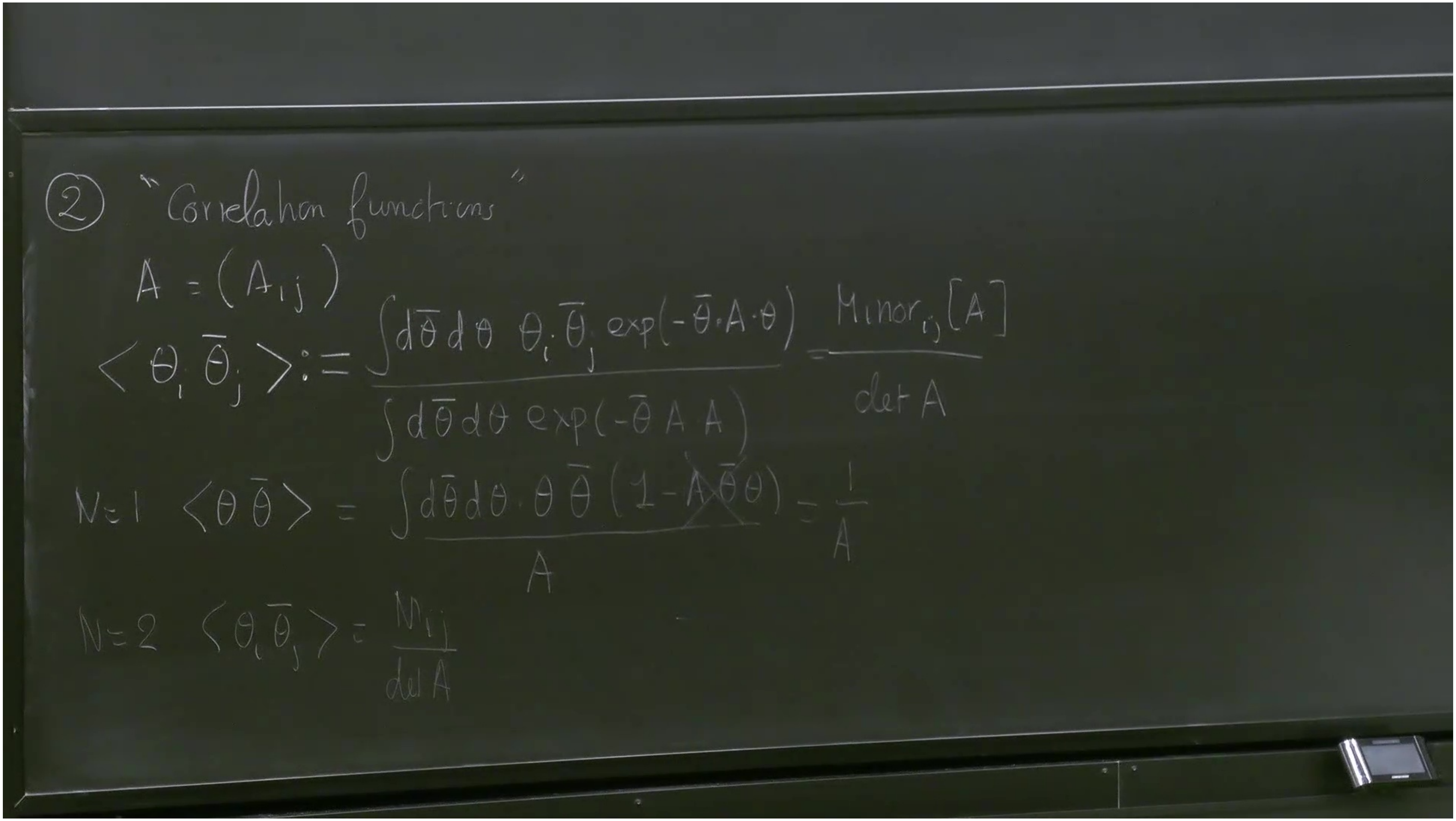
$$\boxed{\int d\bar{\theta}\, d\theta \, \exp(-\bar{\theta}\cdot A\cdot\theta) = \det A}$$

anticommutation

$N=1 \quad A_{11}$

$N=2 \quad A_{11}A_{22} - A_{12}A_{21}$

N general $\underbrace{A \cdot A \cdots A}_{N} \to \det A$



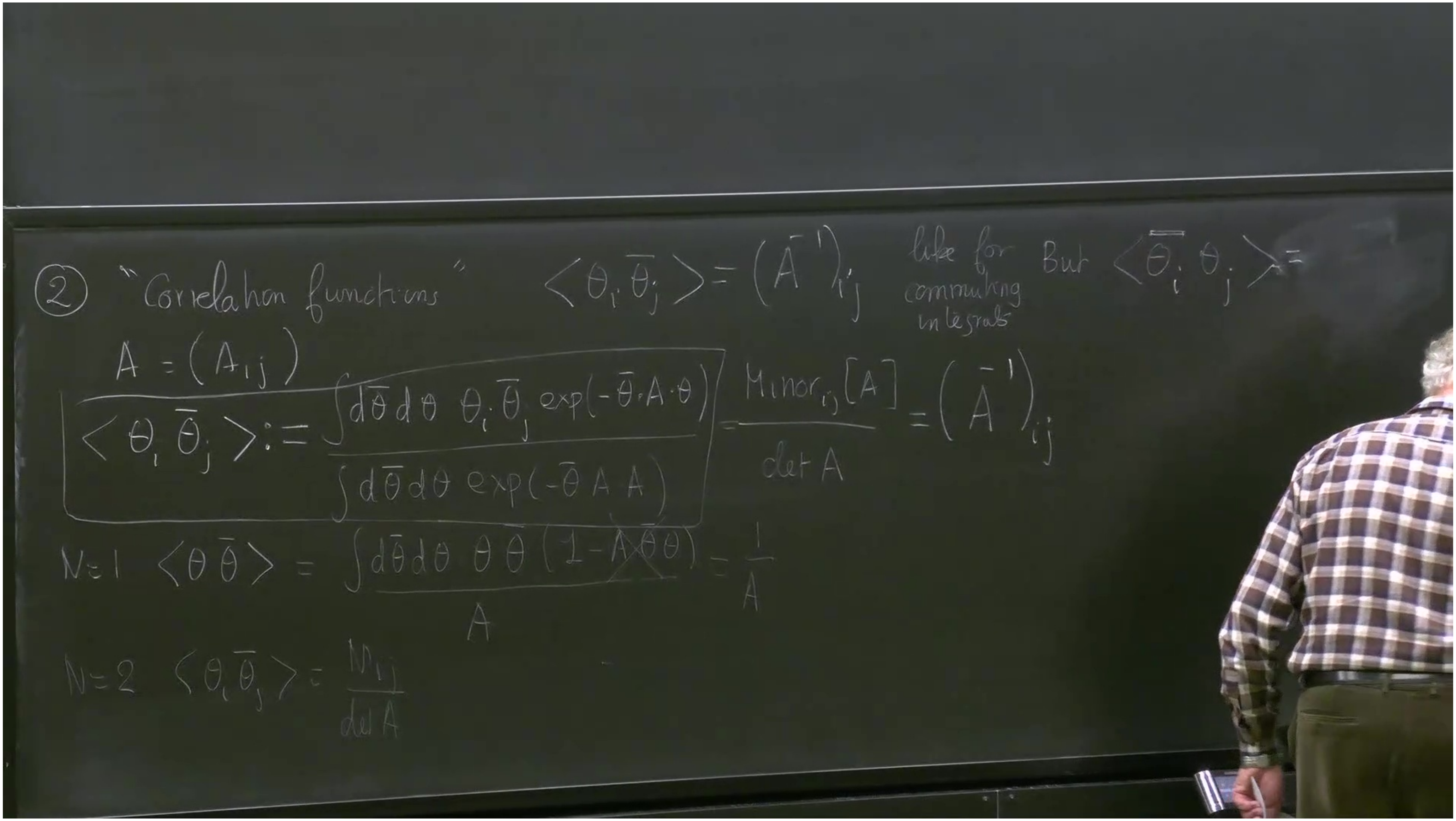
② "Correlation functions"

$A = (A_{ij})$

$$\langle \theta_i \bar{\theta}_j \rangle := \frac{\int d\bar{\theta}\, d\theta \;\, \theta_i \bar{\theta}_j \exp(-\bar{\theta}\cdot A \cdot \theta)}{\int d\bar{\theta}\, d\theta \exp(-\bar{\theta} A A)} = \frac{\mathrm{Minor}_{ij}[A]}{\det A}$$

$$N=1 \quad \langle \theta \bar{\theta} \rangle = \frac{\int d\bar{\theta}\, d\theta \cdot \theta \bar{\theta} \, (1 - \cancel{A\bar{\theta}}\theta)}{A} = \frac{1}{A}$$

$$N=2 \quad \langle \theta_i \bar{\theta}_j \rangle = \frac{M_{ij}}{\det A}$$



② "Correlation functions" $\langle \theta_i \bar{\theta}_j \rangle = (A^{-1})_{ij}$ like for commuting integrals But $\langle \bar{\theta}_i \theta_j \rangle =$

$A = (A_{ij})$

$$\boxed{\langle \theta_i \bar{\theta}_j \rangle := \frac{\int d\bar{\theta}\, d\theta\ \theta_i \bar{\theta}_j \exp(-\bar{\theta}\cdot A\cdot\theta)}{\int d\bar{\theta}\, d\theta \exp(-\bar{\theta} A A)}} = \frac{\text{Minor}_{ij}[A]}{\det A} = (A^{-1})_{ij}$$

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$$\boxed{\langle \theta_i \bar{\theta}_j \rangle = (A^{-1})_{ij}}$$

like for commuting integrals

But $\boxed{\langle \bar{\bar{\theta}}_i \theta_j \rangle = -(A^{-1})_{ji}}$

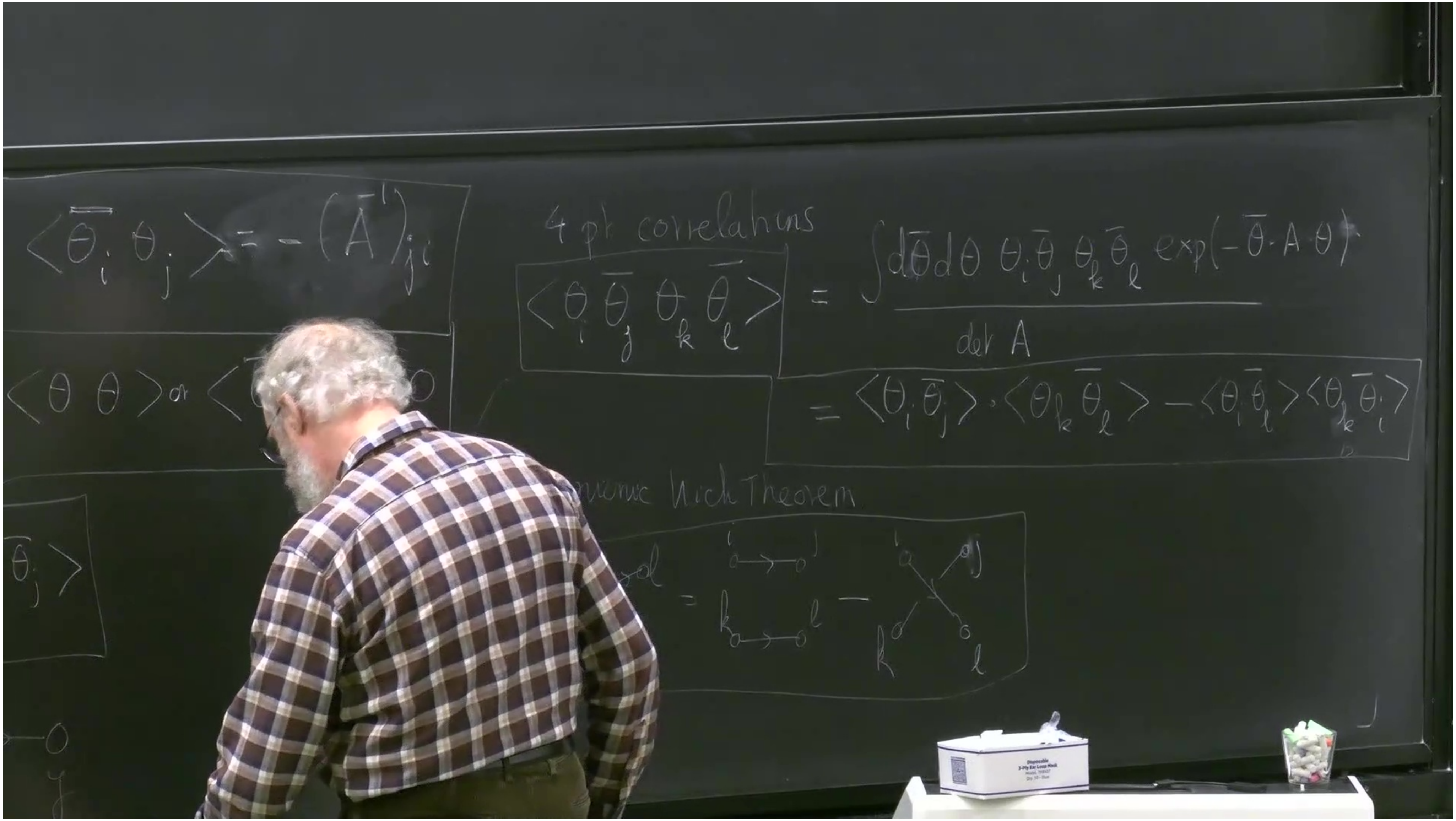
$$\boxed{\langle \theta\ \theta \rangle \text{ or } \langle \bar{\theta}\ \bar{\theta} \rangle = 0}$$

$$N=1 \quad \langle \theta \bar{\theta} \rangle = \frac{\int d\bar{\theta}\, d\theta \cdot \theta\, \bar{\theta}\, (1 - \cancel{A\bar{\theta}}\theta)}{A} = \frac{1}{A}$$

$$\boxed{\underset{i}{\circ}\longrightarrow\underset{j}{\circ} = \langle \theta_i \bar{\theta}_j \rangle}$$

$$N=2 \quad \langle \theta_i \bar{\theta}_j \rangle = \frac{M_{ij}}{\det A}$$

$$\underset{i}{\circ}\longleftarrow\underset{j}{\circ} = -\underset{i}{\circ}\longrightarrow\underset{j}{\circ}$$



$$\langle \bar{\theta}_i\, \theta_j \rangle = -(A^{-1})_{ji}$$

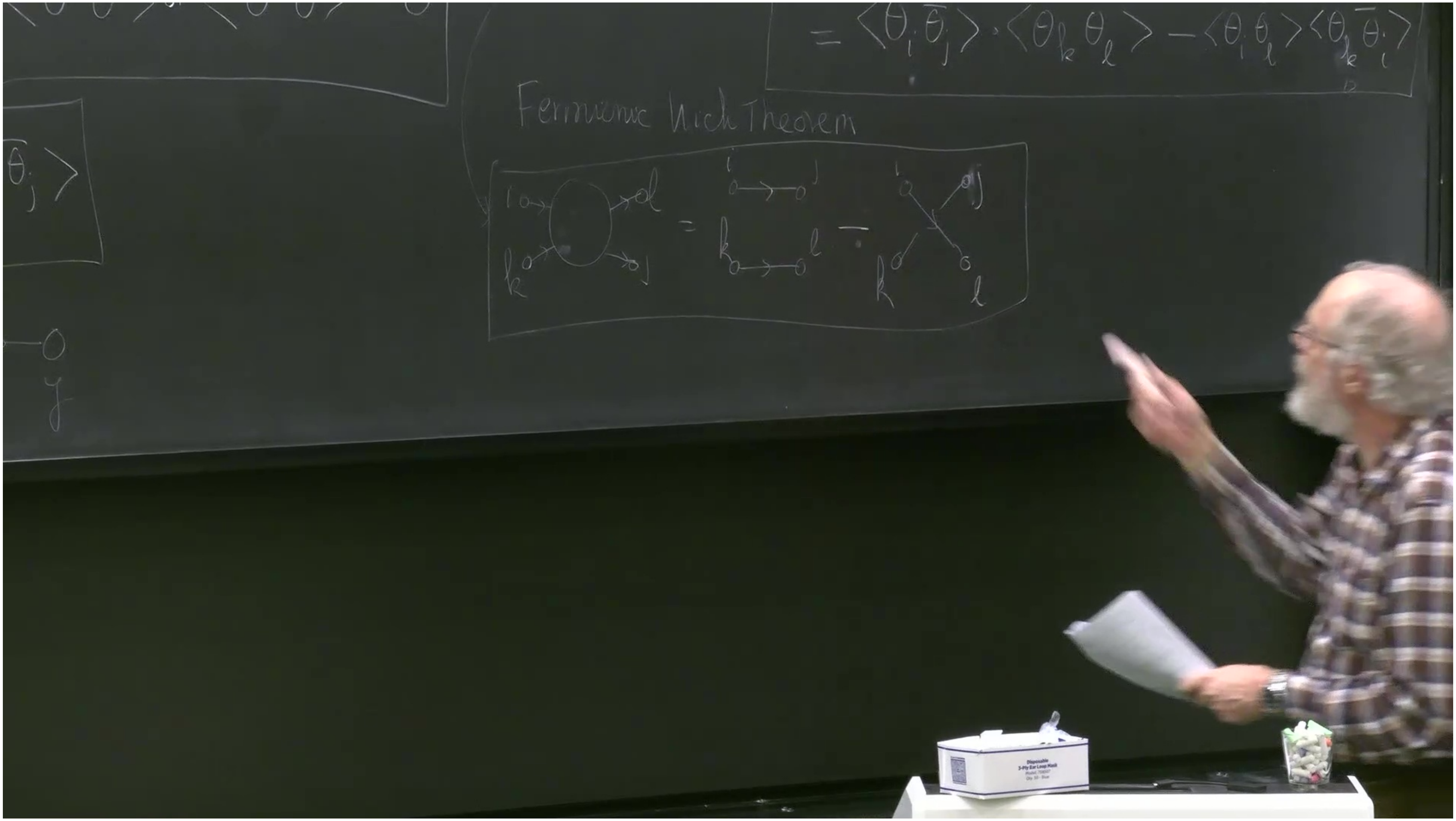
$$\langle \theta\ \theta \rangle \text{ or } \langle \bar{\theta}\ \ldots\ 0$$

4 pt correlations

$$\langle \theta_i\, \bar{\theta}_j\, \theta_k\, \bar{\theta}_l \rangle = \frac{\int d\bar{\theta}\, d\theta\ \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l \exp(-\bar{\theta}\cdot A\cdot\theta)}{\det A}$$

$$= \langle \theta_i \bar{\theta}_j \rangle \cdot \langle \theta_k \bar{\theta}_l \rangle - \langle \theta_i \bar{\theta}_l \rangle \langle \theta_k \bar{\theta}_i \rangle$$

…mionic Wick Theorem



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Fermionic Wick Theorem

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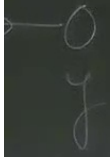
$$\bar{\theta}_j \rangle$$



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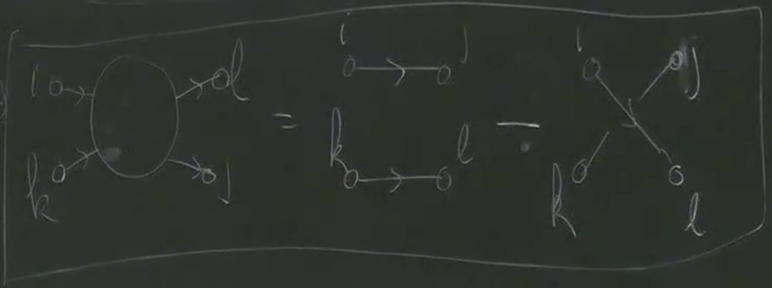
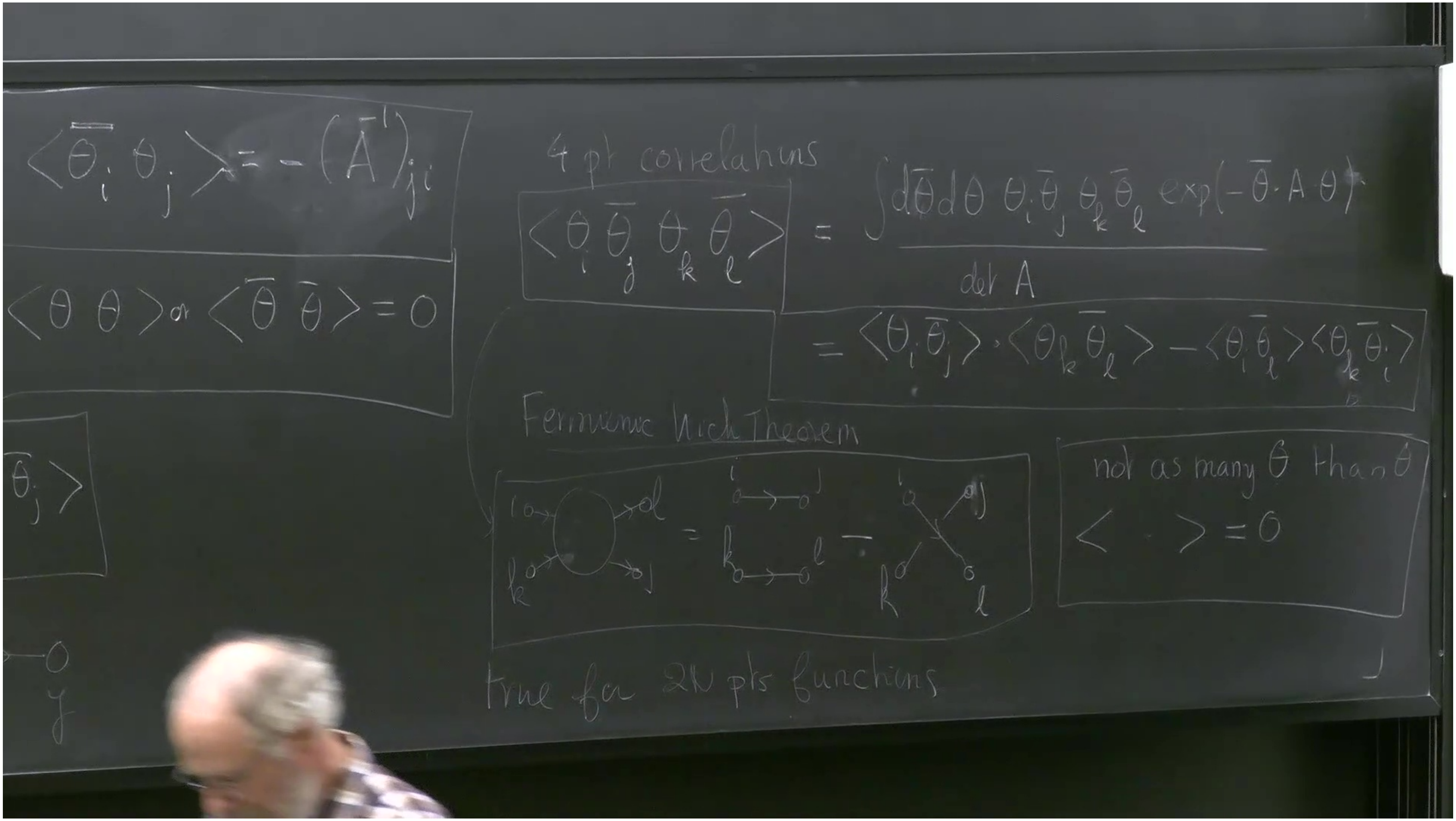
Fermionic Wick Theorem





$$\langle \bar\theta_i \theta_j \rangle = -(A^{-1})_{ji}$$

$$\langle \theta\,\theta \rangle \text{ or } \langle \bar\theta\,\bar\theta \rangle = 0$$

$\bar\theta_j \rangle$

4 pt correlations

$$\boxed{\langle \theta_i \bar\theta_j \theta_k \bar\theta_l \rangle} = \frac{\int d\bar\theta\, d\theta\; \theta_i \bar\theta_j \theta_k \bar\theta_l \exp(-\bar\theta\cdot A\cdot\theta)}{\det A}$$

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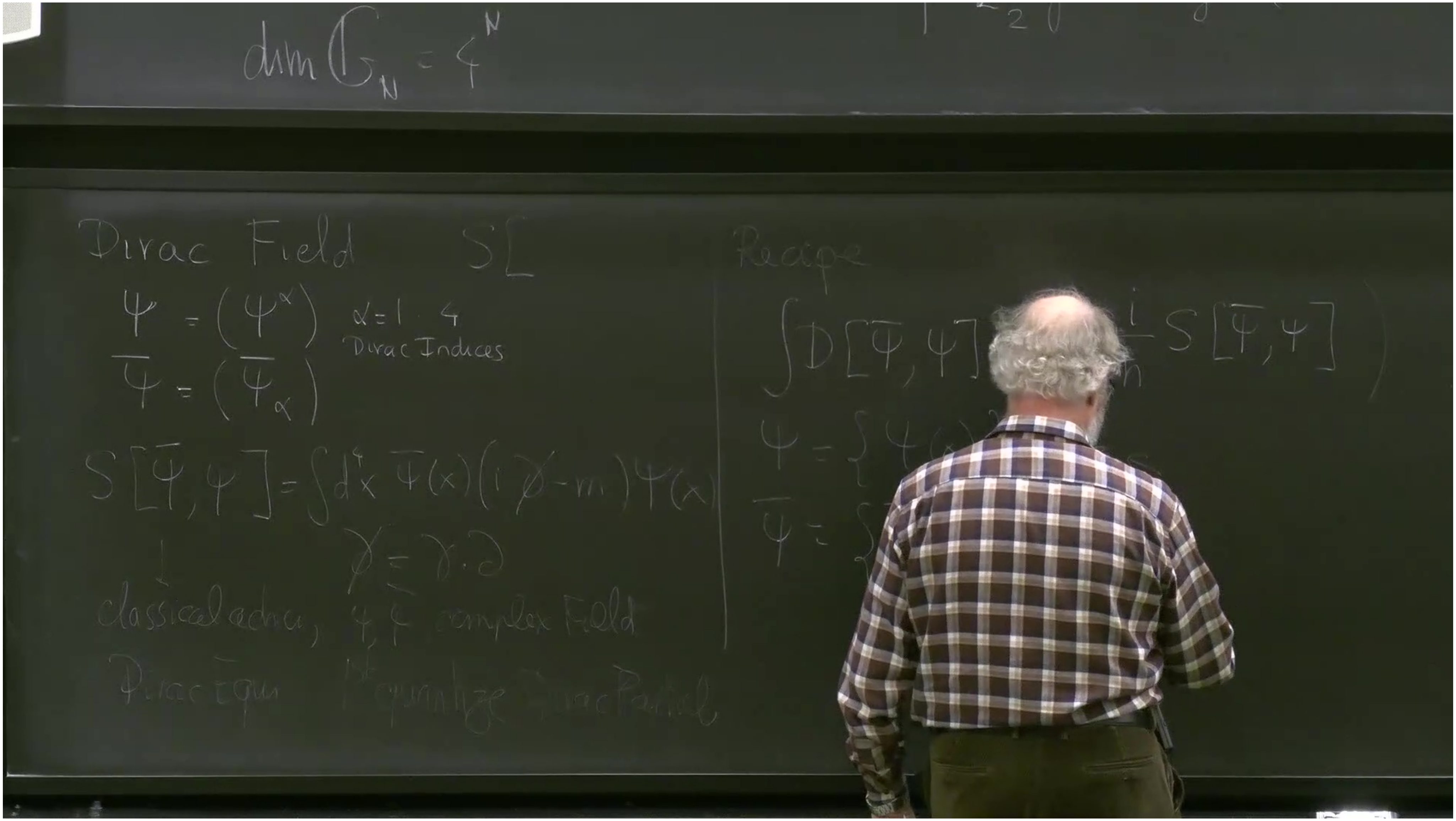
Fermionic Wick Theorem

true for 2N pts functions

not as many θ than $\bar\theta$

$$\langle \cdot \rangle = 0$$



dim $\mathbb{C}_N = 4^N$

Dirac Field   S[

$\psi = (\psi^\alpha)$ $\alpha = 1 \cdot 4$

Dirac Indices

$\overline{\psi} = (\overline{\psi}_\alpha)$

$S[\overline{\psi},\psi] = \int d^4x\, \overline{\psi}(x)(i\not{\partial} - m)\psi(x)$

$\not{\partial} = \gamma\cdot\partial$

↓

classical action, $\psi, \overline{\psi}$ complex Field

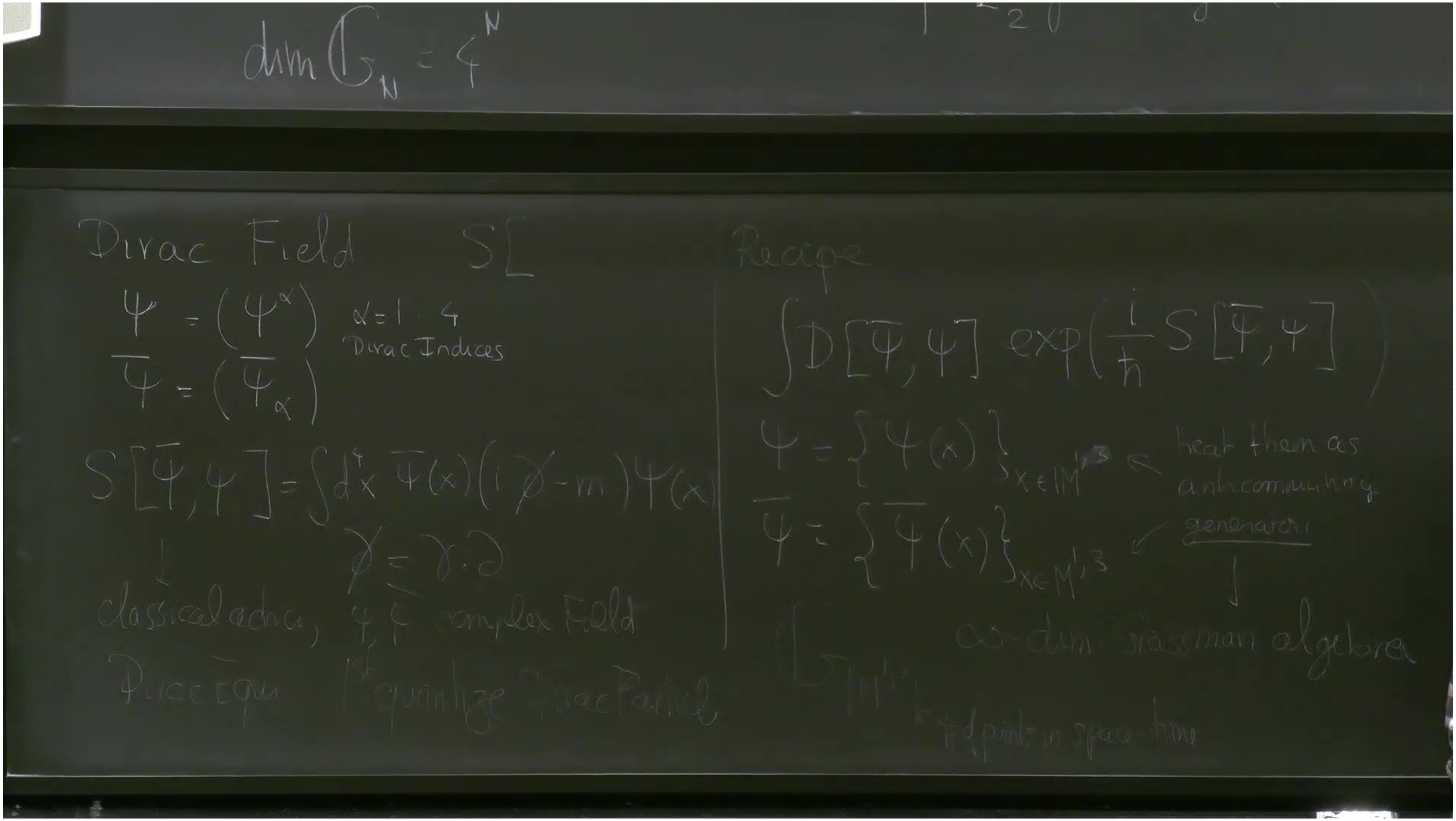
Dirac Equn   1st quantize Dirac Particle

Recipe

$\int D[\overline{\psi},\psi] \quad \frac{i}{\hbar} S[\overline{\psi},\psi]\,)$

$\psi = \{\psi(x)$

$\overline{\psi} = \{$



$\dim \mathcal{G}_N = 4^N$

Dirac Field $\quad S[$

$\psi = (\psi^\alpha)$ $\quad \alpha = 1 \ldots 4$ Dirac Indices

$\bar{\psi} = (\bar{\psi}_\alpha)$

$S[\bar{\psi}, \psi] = \int d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x)$

$\not{\partial} = \gamma \cdot \partial$

classical action, $\psi, \bar{\psi}$ complex Field

Dirac Equn $\quad$ 1st quantize Dirac Particle

Recipe

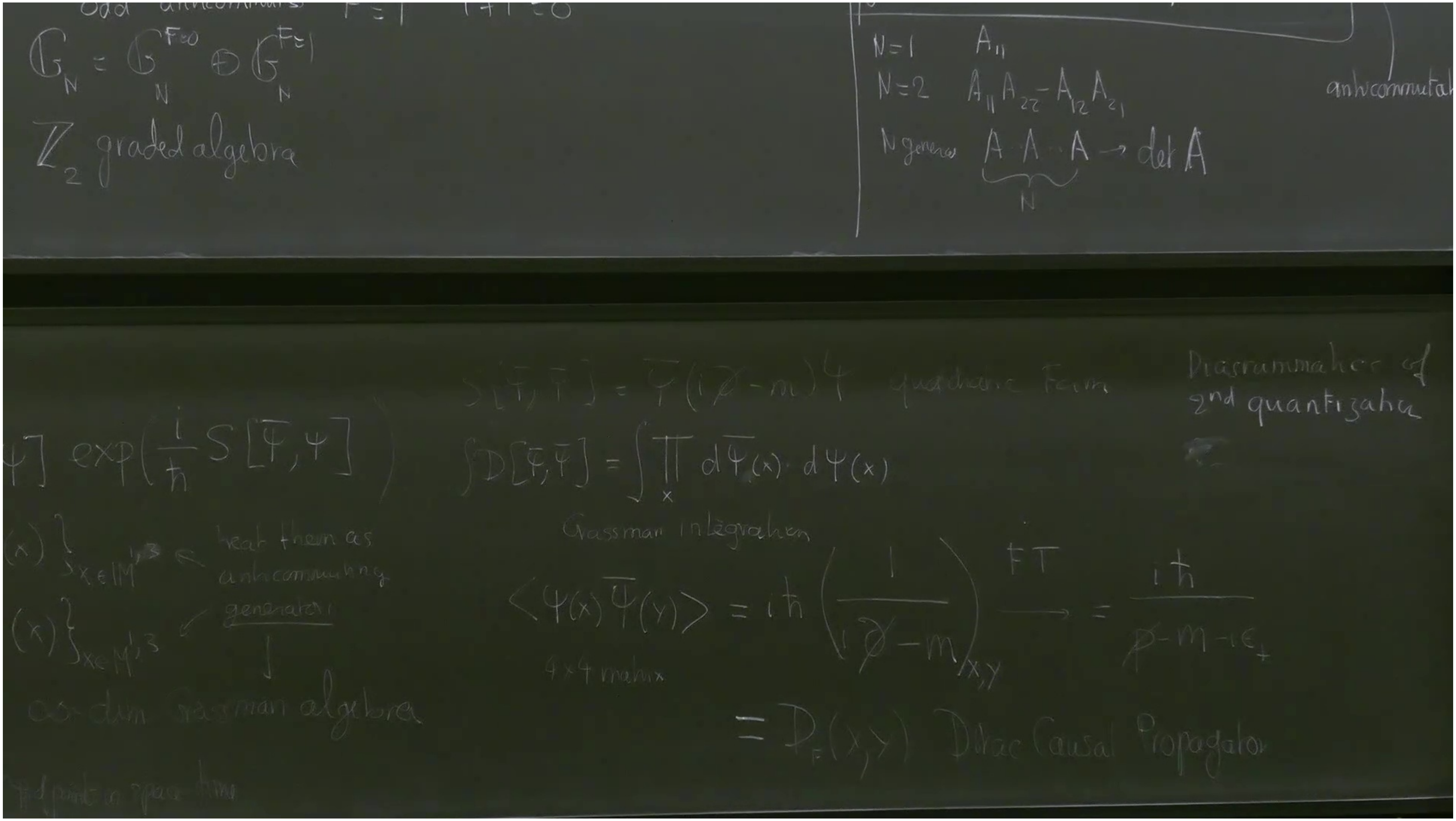
$\int D[\bar{\psi}, \psi] \exp\left(\frac{i}{\hbar} S[\bar{\psi}, \psi]\right)$

$\psi = \{\psi(x)\}_{x \in M^{1,3}}$ $\leftarrow$ treat them as anticommuting generators

$\bar{\psi} = \{\bar{\psi}(x)\}_{x \in M^{1,3}}$

$\mathcal{G}_{|M^{1,3}|}$ $\quad \infty$-dim Grassman algebra

# of points in space-time



odd anticommutes $F=1$

$\mathbb{G}_N = \mathbb{G}_N^{F=0} \oplus \mathbb{G}_N^{F=1}$

$\mathbb{Z}_2$ graded algebra

N=1 A_{11}

N=2 $A_{11}A_{22} - A_{12}A_{21}$

N generic $\underbrace{A \cdot A \cdots A}_{N} \to \det A$

anticommutat

$\psi] \exp\left(\frac{i}{\hbar} S[\bar{\psi}, \psi]\right)$

$(x)\}_{x \in M^{1,3}}$ ← treat them as anticommuting generators

$(x)\}_{x \in M^{1,3}}$

∞-dim Grassman algebra

and points in space-time

$S[\bar{\psi}, \psi] = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ quadratic Form

$\int D[\bar{\psi}, \psi] = \int \prod_x d\bar{\psi}(x) \cdot d\psi(x)$

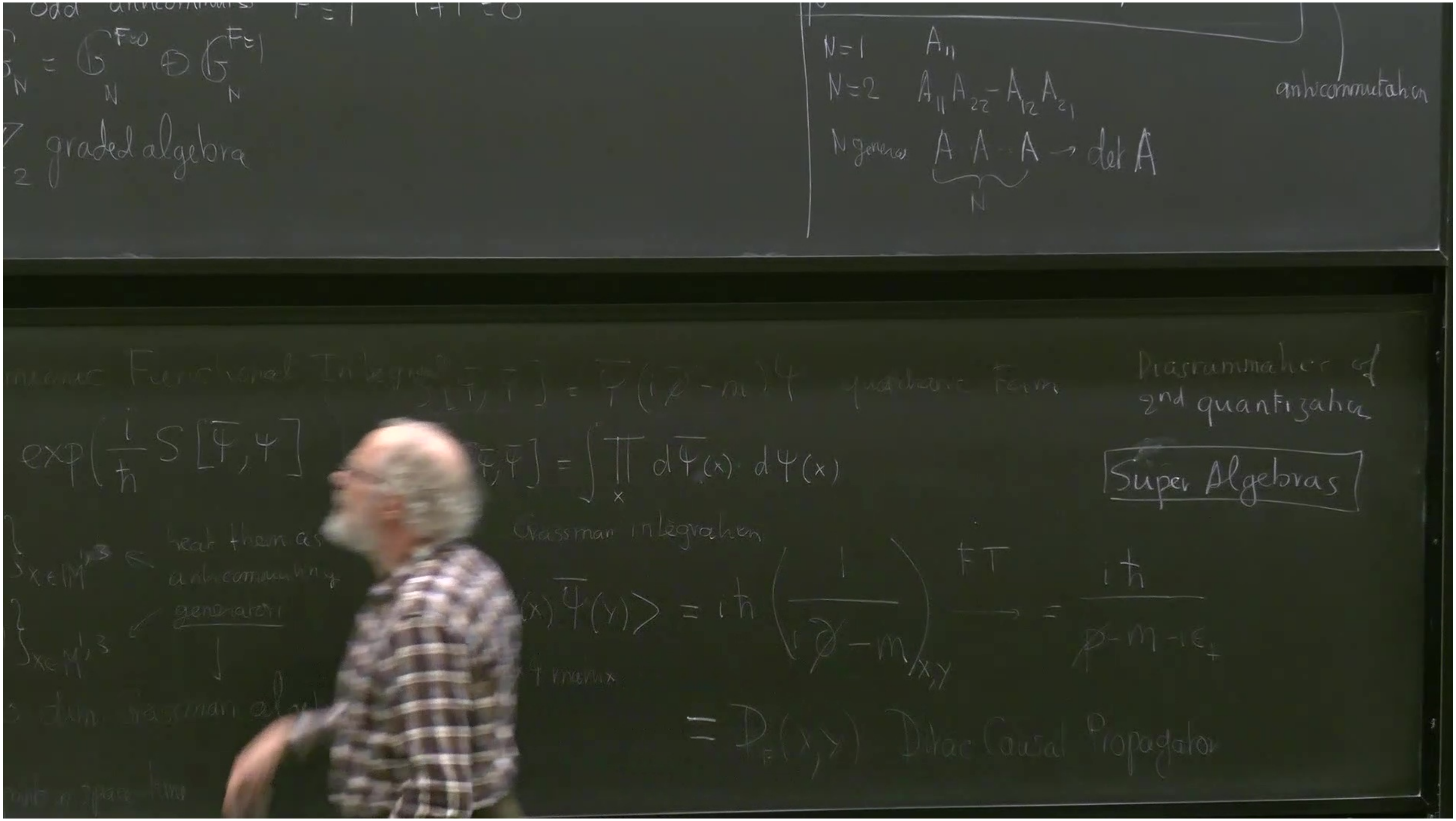
Grassman integration

$$\langle \psi(x)\bar{\psi}(y) \rangle = i\hbar \left(\frac{1}{i\partial\!\!\!/ - m}\right)_{x,y} \xrightarrow{FT} = \frac{i\hbar}{p\!\!\!/ - m - i\epsilon_+}$$

4×4 matrix

$= D_F(x,y)$ Dirac Causal Propagator

Diagrammatics of 2nd quantization



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N general $\underbrace{A \; A \cdots A}_{N} \rightarrow \det A$

anticommutation

Fermionic Functional Integral $S[\overline{\Psi}, \Psi] = \overline{\Psi}(i\not{\partial} - m)\Psi$ quadratic form

$\exp\left(\frac{i}{\hbar} S[\overline{\Psi}, \Psi]\right]$

$[\overline{\Psi}, \Psi] = \int \prod_x d\overline{\Psi}(x) \cdot d\Psi(x)$

Grassman integration

treat them as anticommuting generators

$\}_{x \in M^{1,3}}$

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dim Grassman alg

$(x)\overline{\Psi}(y)> = i\hbar \left(\frac{1}{i\not{\partial} - m}\right)_{x,y} \xrightarrow{FT} = \frac{i\hbar}{\not{p} - m - i\epsilon_+}$

4 matrix

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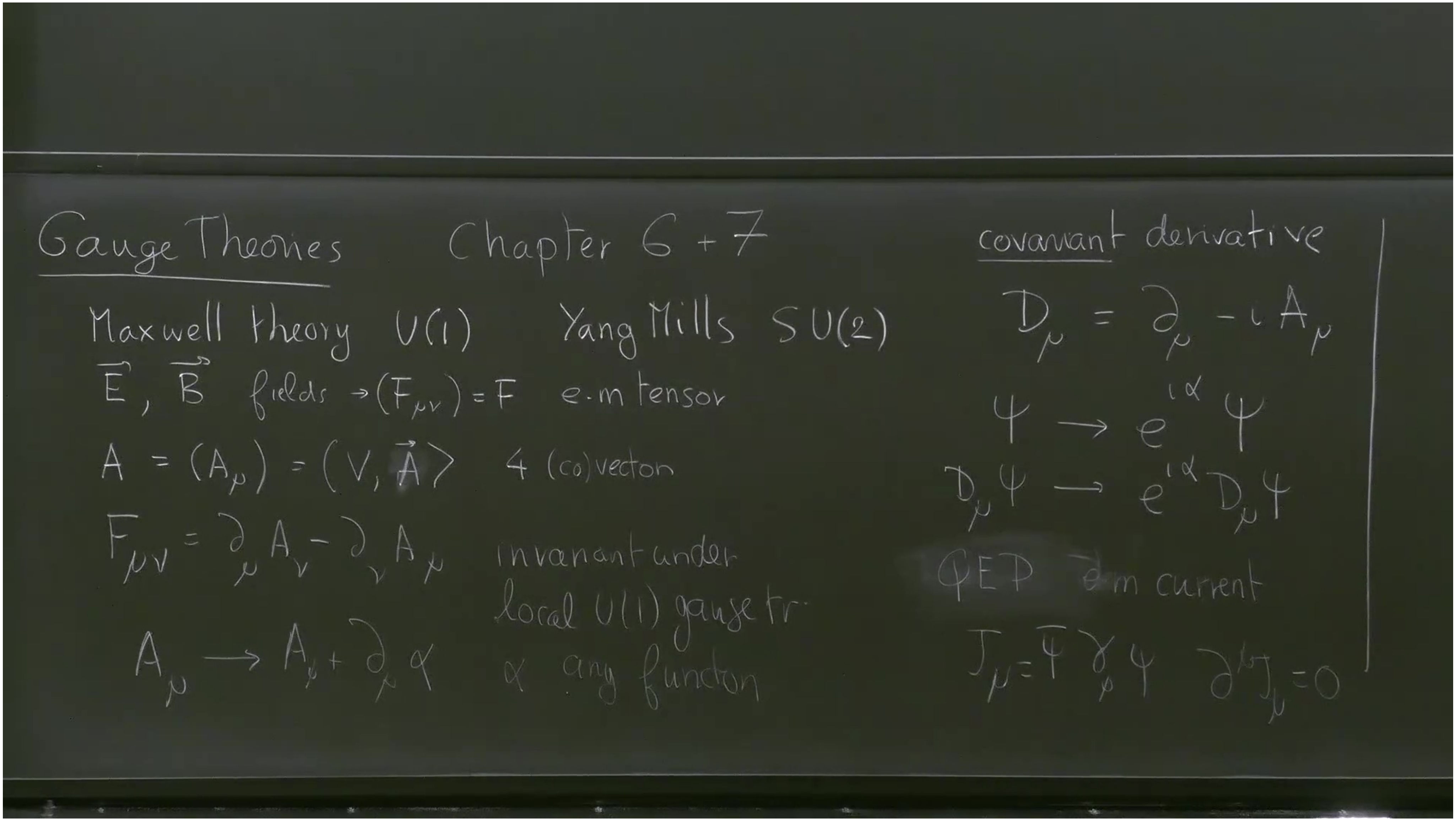
points in space-time

Diagrammatics of 2nd quantization

$\boxed{\text{Super Algebras}}$



Gauge Theories Chapter 6 + 7

Maxwell theory $U(1)$ Yang Mills $SU(2)$

$\vec{E}, \vec{B}$ fields $\rightarrow (F_{\mu\nu}) = F$ e.m tensor

$A = (A_\mu) = (V, \vec{A}\rangle$ 4 (co)vector

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ invariant under local $U(1)$ gauge tr.

$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha$ α any function

covariant derivative

$$D_\mu = \partial_\mu - i A_\mu$$

$$\psi \longrightarrow e^{i\alpha} \psi$$

$$D_\mu \psi \longrightarrow e^{i\alpha} D_\mu \psi$$

QED e.m current

$J_\mu = \bar{\psi} \gamma_\mu \psi$ $\partial^\mu J_\mu = 0$

ve

A_μ

$=0$

non-abelian symmetry? $SU(2) \rightarrow 3$ generators $\quad U = 1 + \epsilon$

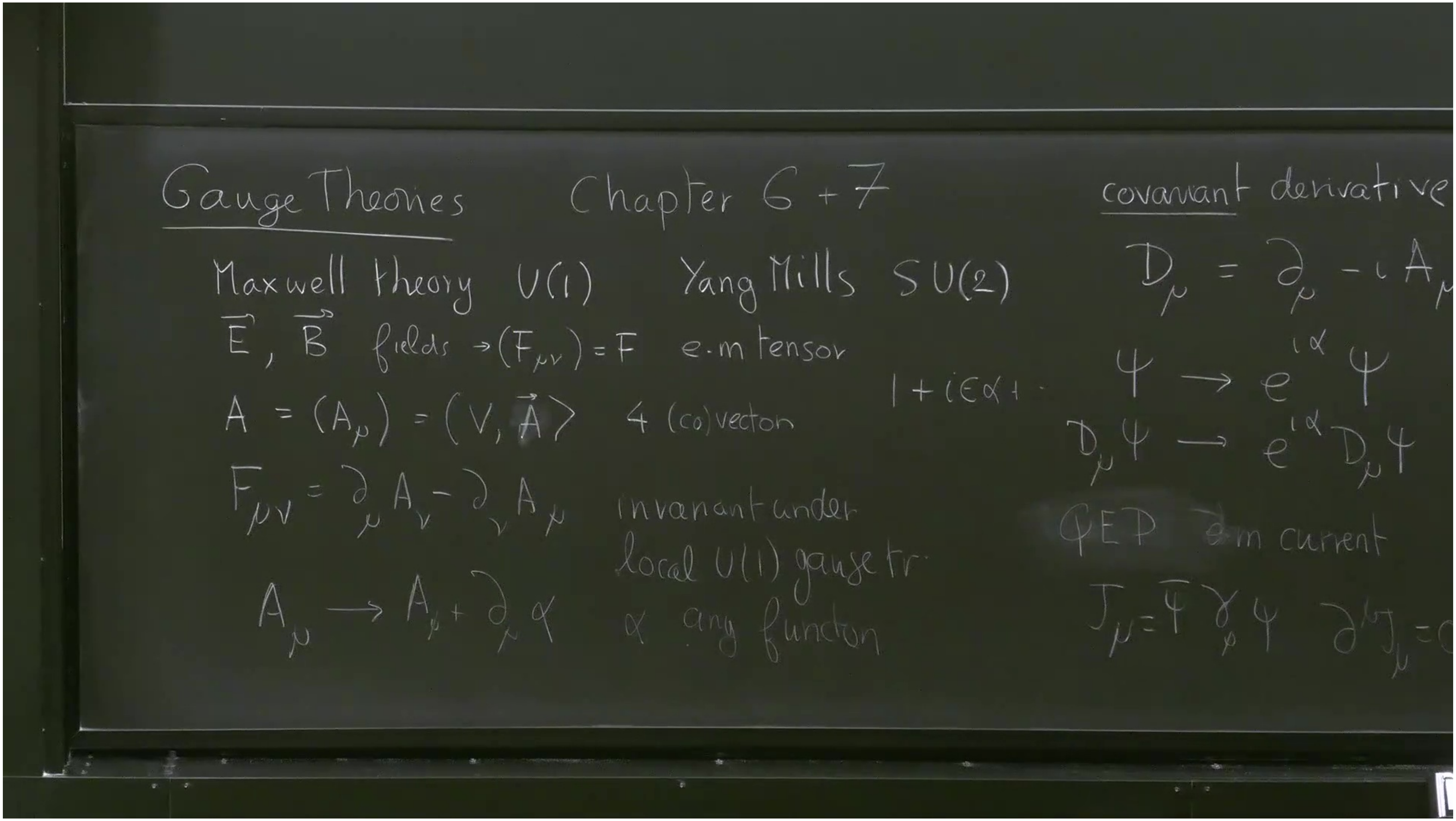
$\Psi = (\Psi^i) \quad i = 1, 2 \quad \Psi \rightarrow U\Psi$

U 2×2 unitary special matrix $U = U^{+}$ and $\det U = 1$

$G = SU(2)$ group $\rightarrow$ Lie Algebra g = Tracless hermitean matrices

generator $t_a \quad a = 1, 2, 3 \quad t_a = \frac{1}{2}\sigma_a$ Pauli matrices $[t_a, t_b] = \epsilon_{abc} t_c$

$[\ ,\ ]$ = Lie Bracket $\quad \epsilon_{abc}$ Fully antisymmetric tensor $\epsilon_{123} = 1$



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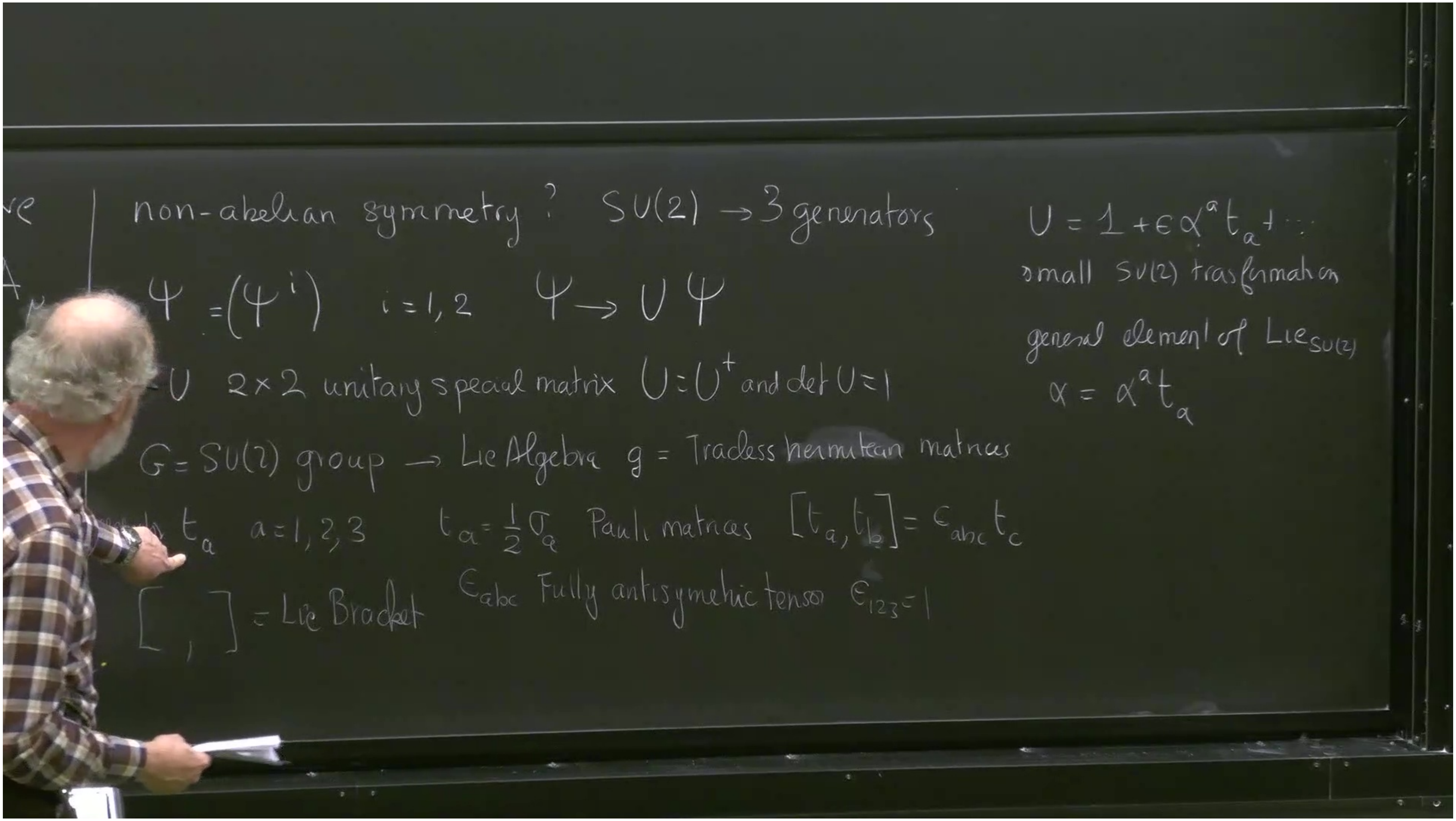
$1 + i\epsilon\alpha + \cdot$ $\psi \longrightarrow e^{i\alpha} \psi$

$D_\mu \psi \longrightarrow e^{i\alpha} D_\mu \psi$

QED → e.m current

$J_\mu = \bar{\Psi} \gamma_\mu \psi$ $\partial^\mu J_\mu =$



ve

A_μ

non-abelian symmetry? $SU(2) \rightarrow 3$ generators

$\Psi = (\Psi^i)$ $i = 1, 2$ $\Psi \rightarrow U \Psi$

U 2×2 unitary special matrix $U = U^{+}$ and $\det U = 1$

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t_a $a = 1, 2, 3$ $t_a = \frac{1}{2} \sigma_a$ Pauli matrices $[t_a, t_b] = \epsilon_{abc} t_c$

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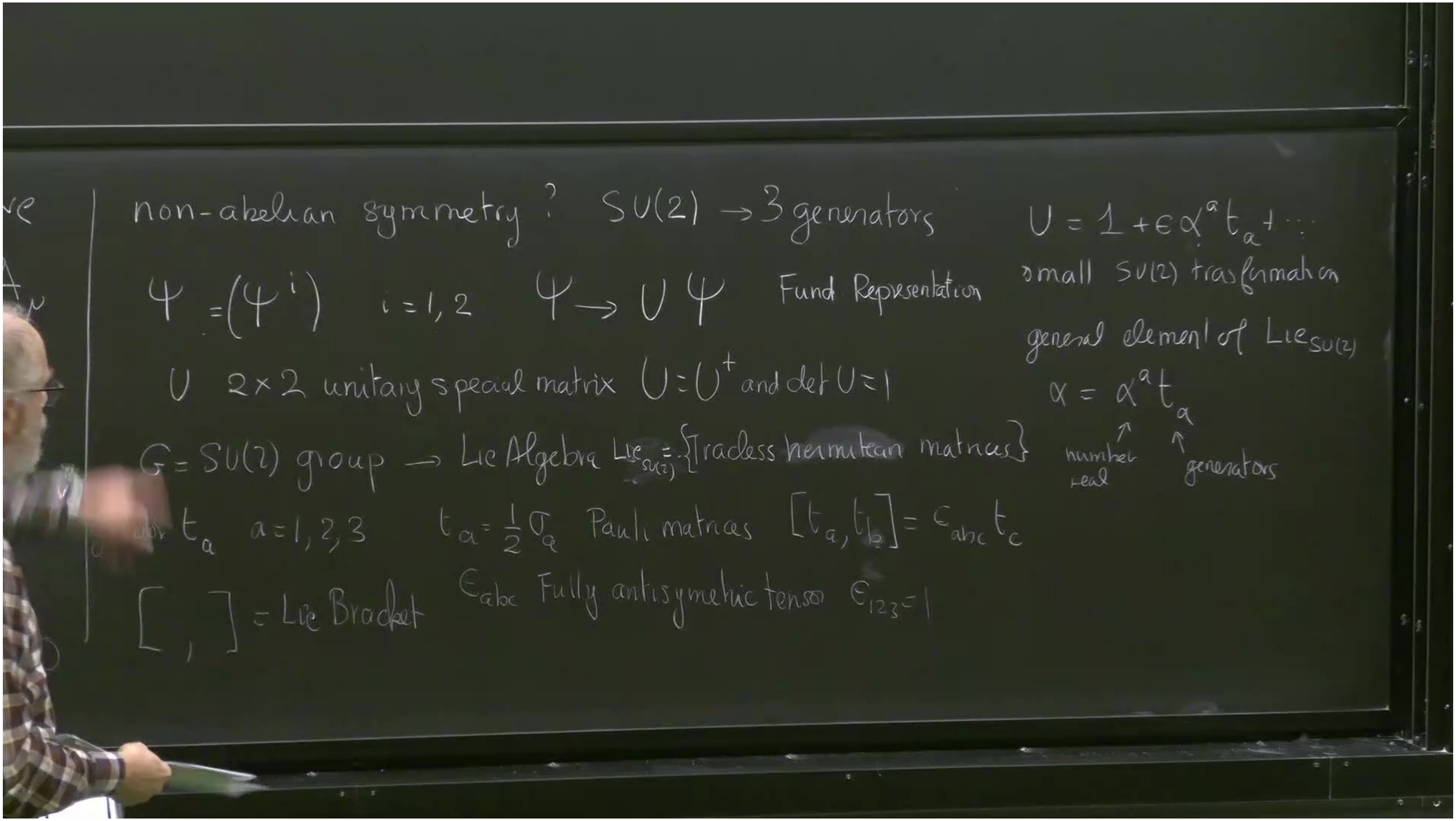
$U = 1 + \epsilon \alpha^a t_a + \cdots$

small SU(2) trasformation

general element of $\mathrm{Lie}_{SU(2)}$

$\alpha = \alpha^a t_a$



non-abelian symmetry ? SU(2) → 3 generators

$\Psi = (\Psi^i) \quad i = 1, 2 \quad \Psi \to U \Psi$ Fund Representation

U 2×2 unitary special matrix $U = U^+$ and $\det U = 1$

$G = SU(2)$ group → Lie Algebra $\mathrm{Lie}_{SU(2)} = \{$Tracless hermitean matrices$\}$

for $t_a \quad a = 1, 2, 3 \qquad t_a = \frac{1}{2}\sigma_a$ Pauli matrices $[t_a, t_b] = \epsilon_{abc} t_c$

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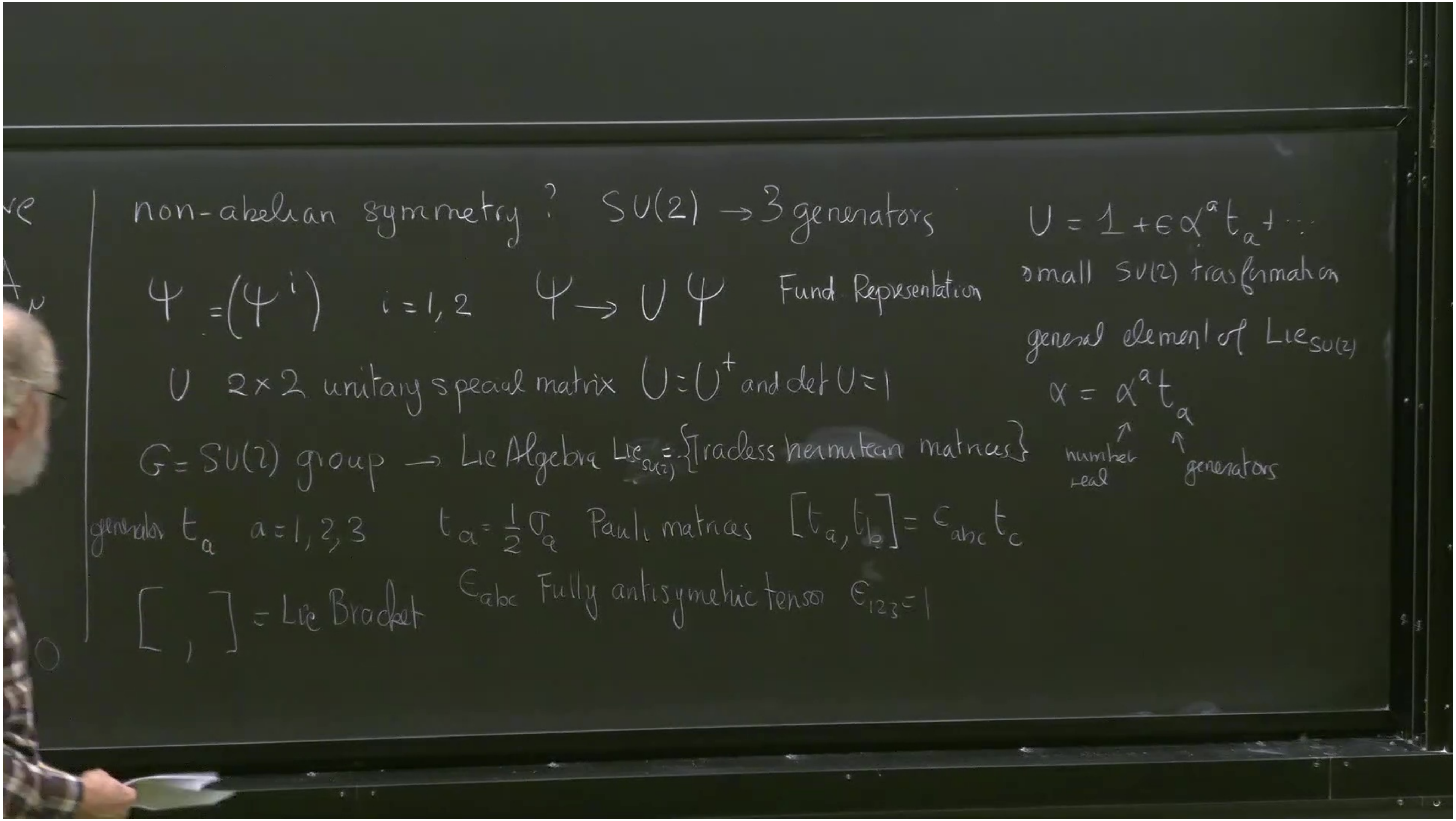

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$G = SU(2)$ group $\to$ Lie Algebra $\mathrm{Lie}_{SU(2)} = \{$traceless hermitean matrices$\}$

generator t_a $a = 1, 2, 3$ $t_a = \frac{1}{2}\sigma_a$ Pauli matrices $[t_a, t_b] = \epsilon_{abc} t_c$

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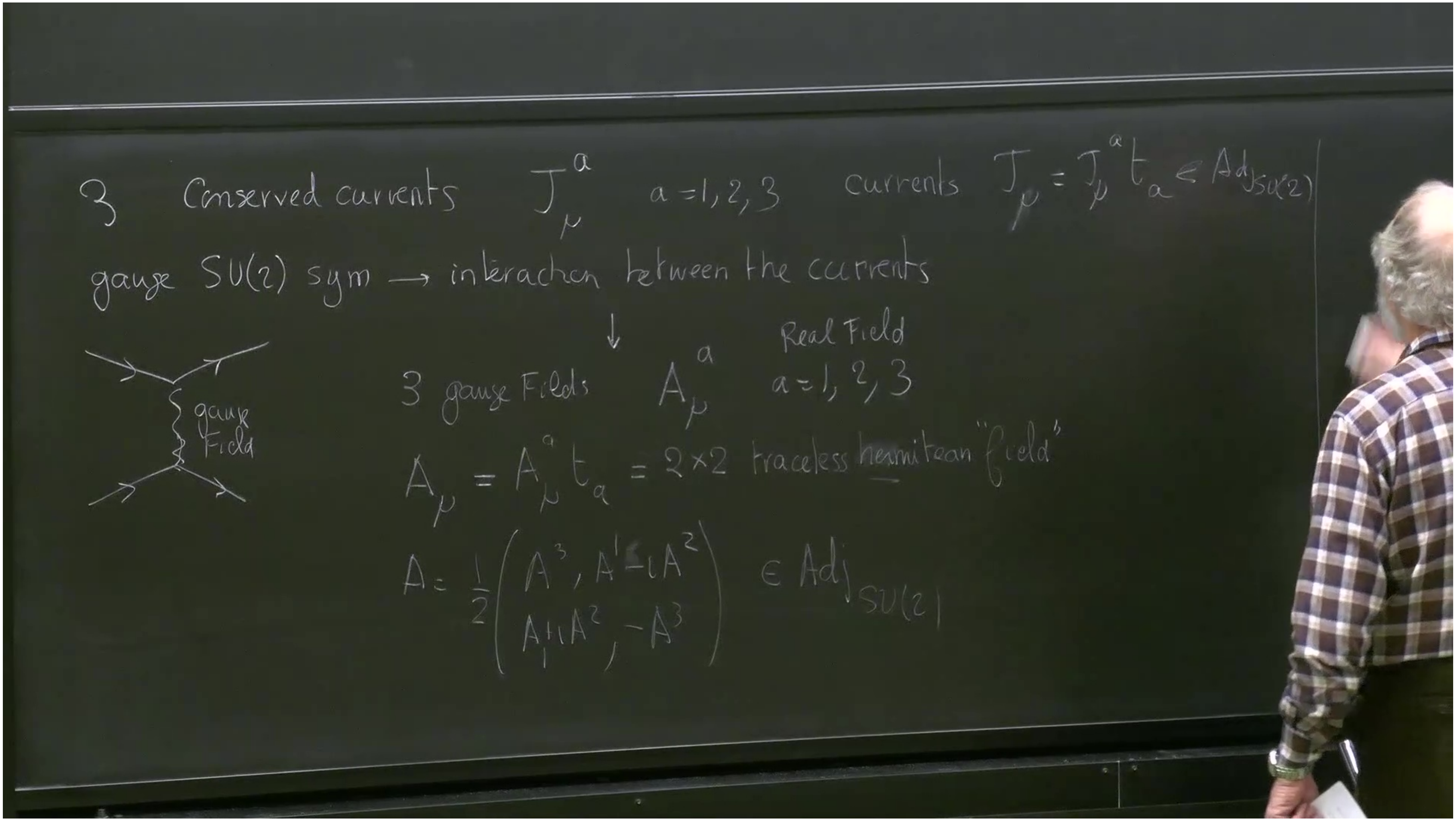


3 Conserved currents J_μ^a $a = 1, 2, 3$ currents $J_\mu = J_\mu^a t_a \in \mathrm{Adj}_{SU(2)}$

gauge SU(2) sym → interaction between the currents

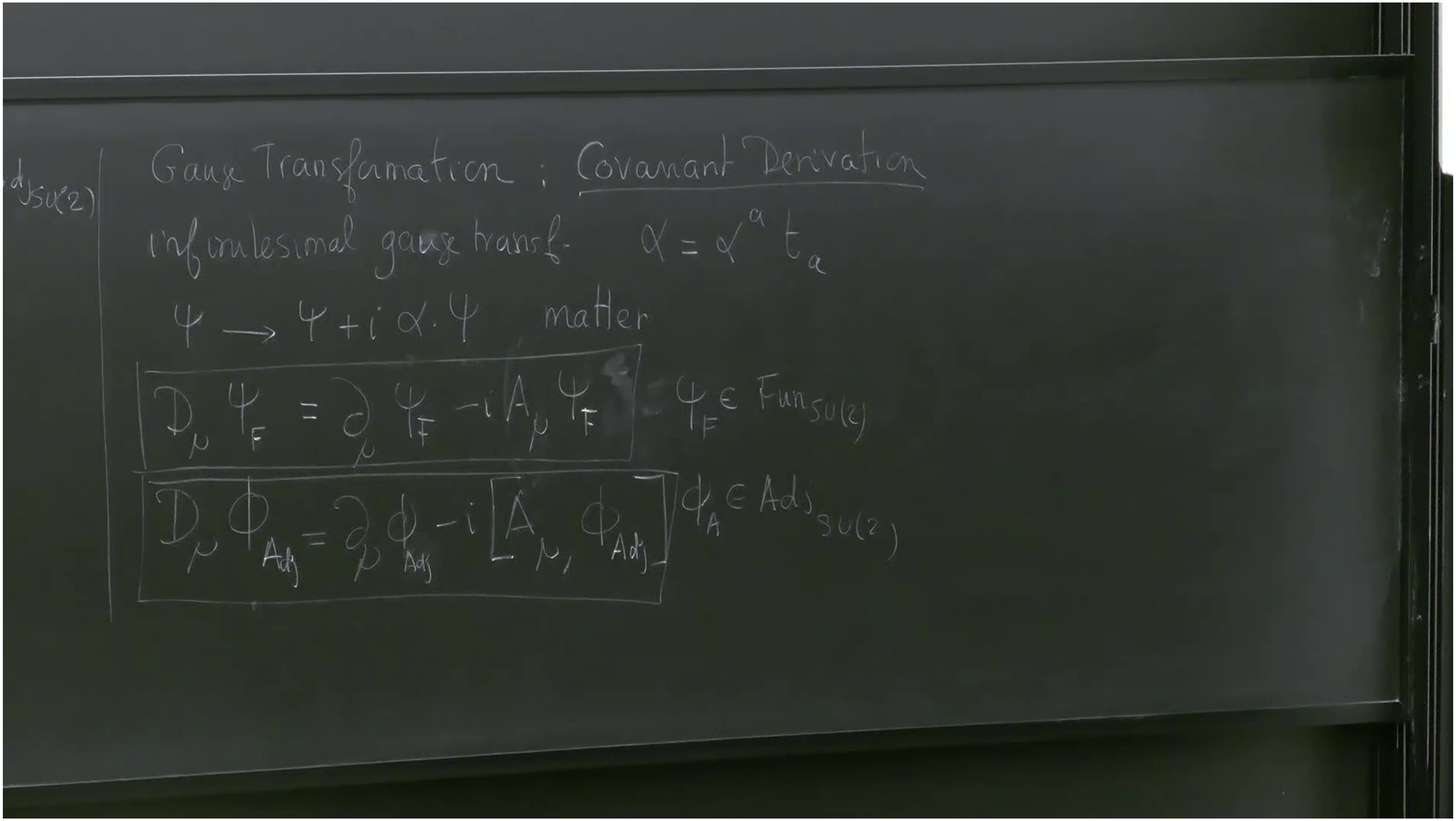
$\downarrow$

Real Field

3 gauge Fields A_μ^a $a = 1, 2, 3$

$A_\mu = A_\mu^a t_a = 2\times 2$ traceless hermitean "field"

$$A = \frac{1}{2}\begin{pmatrix} A^3, & A^1 - iA^2 \\ A^1 + iA^2, & -A^3 \end{pmatrix} \in \mathrm{Adj}_{SU(2)}$$



dj SU(2)

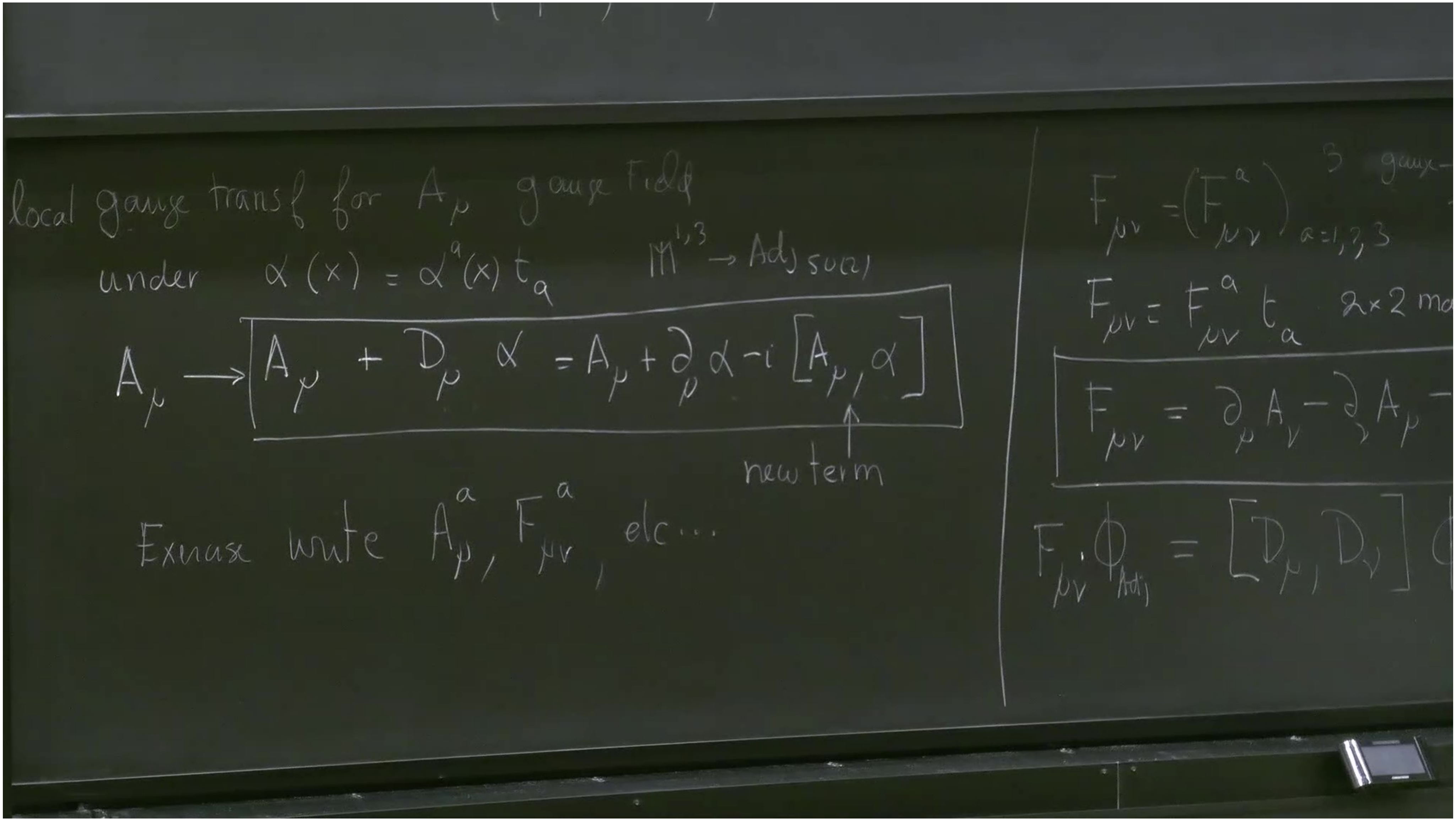
Gauge Transformation ; <u>Covariant Derivation</u>

infinitesimal gauge transf. $\alpha = \alpha^a t_a$

$\psi \longrightarrow \psi + i\,\alpha\cdot\psi$ matter

$$\boxed{D_\mu \psi_F = \partial_\mu \psi_F - i A_\mu \psi_F} \quad \psi_F \in \mathrm{Fun}_{SU(2)}$$

$$\boxed{D_\mu \phi_{Adj} = \partial_\mu \phi_{Adj} - i\left[A_\mu , \phi_{Adj}\right]} \quad \phi_A \in \mathrm{Adj}_{SU(2)}$$



local gauge transf for A_μ gauge Field

under $\alpha(x) = \alpha^a(x)\, t_a$ $\quad \mathbb{M}^{1,3} \to \mathrm{Adj}\, su(2)$

$$A_\mu \longrightarrow \boxed{A_\mu + D_\mu \alpha = A_\mu + \partial_\mu \alpha - i\,[A_\mu, \alpha]}$$

new term

Exercise write A^a_μ, $F^a_{\mu\nu}$, etc...

$$F_{\mu\nu} = \left(F^a_{\mu\nu}\right)_{a=1,2,3}$$

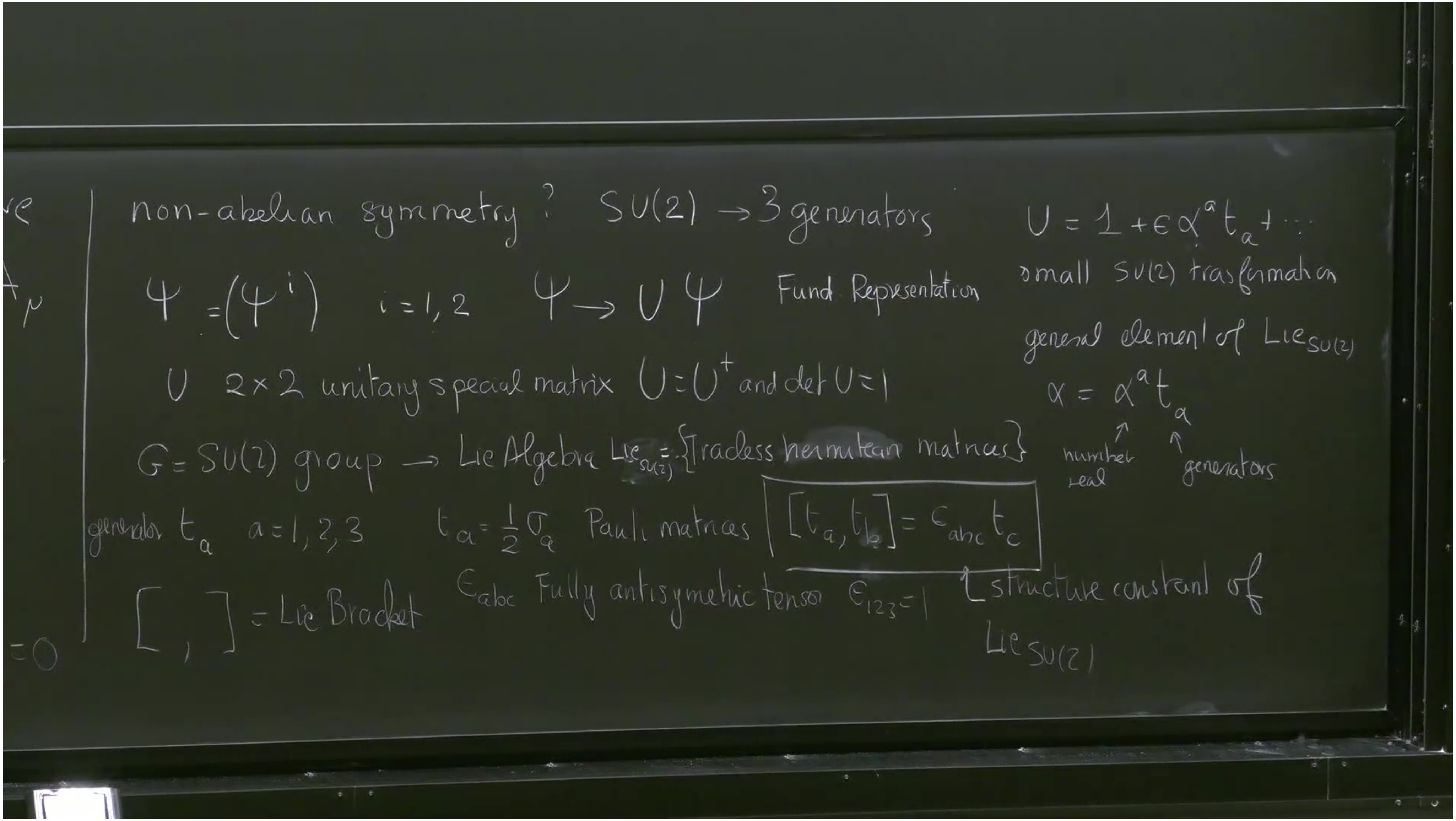
3 gauge-

$$F_{\mu\nu} = F^a_{\mu\nu}\, t_a$$

2×2 ma

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu -}$$

$$F_{\mu\nu} \cdot \phi_{\mathrm{Adj}} = [D_\mu, D_\nu]\, \phi$$



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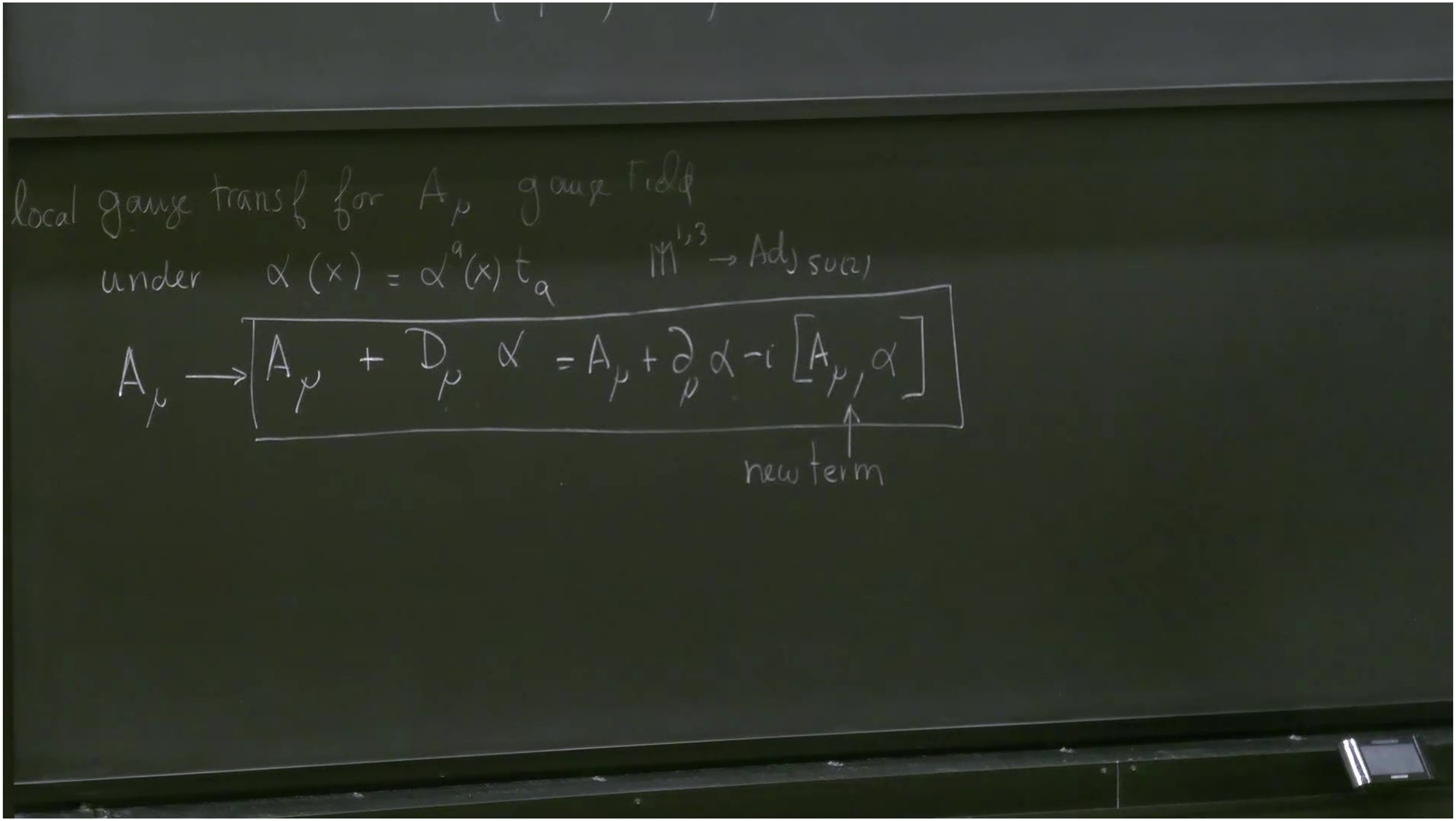
$\alpha = \underset{\substack{\uparrow \\ \text{real number}}}{\alpha^a} \underset{\substack{\uparrow \\ \text{generators}}}{t_a}$

$G = SU(2)$ group $\longrightarrow$ Lie Algebra $Lie_{SU(2)} = \{$Traceless hermitean matrices$\}$

generator t_a $a = 1, 2, 3$ $t_a = \frac{1}{2} \sigma_a$ Pauli matrices $\boxed{[t_a, t_b] = \epsilon_{abc} t_c}$

$[\ ,\ ]$ = Lie Bracket ϵ_{abc} Fully antisymmetric tensor $\epsilon_{123} = 1$

$\uparrow$ structure constant of $Lie_{SU(2)}$

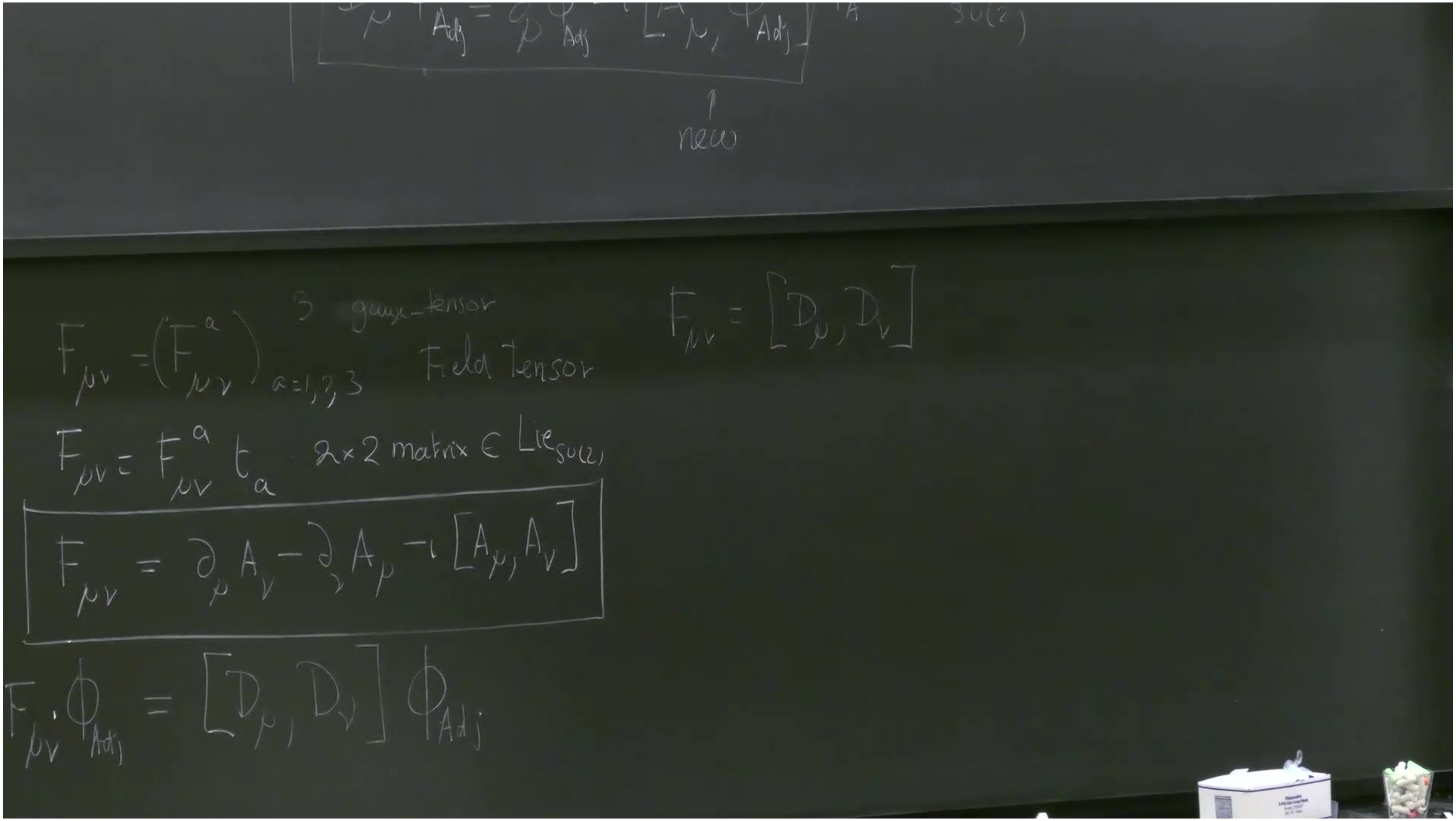


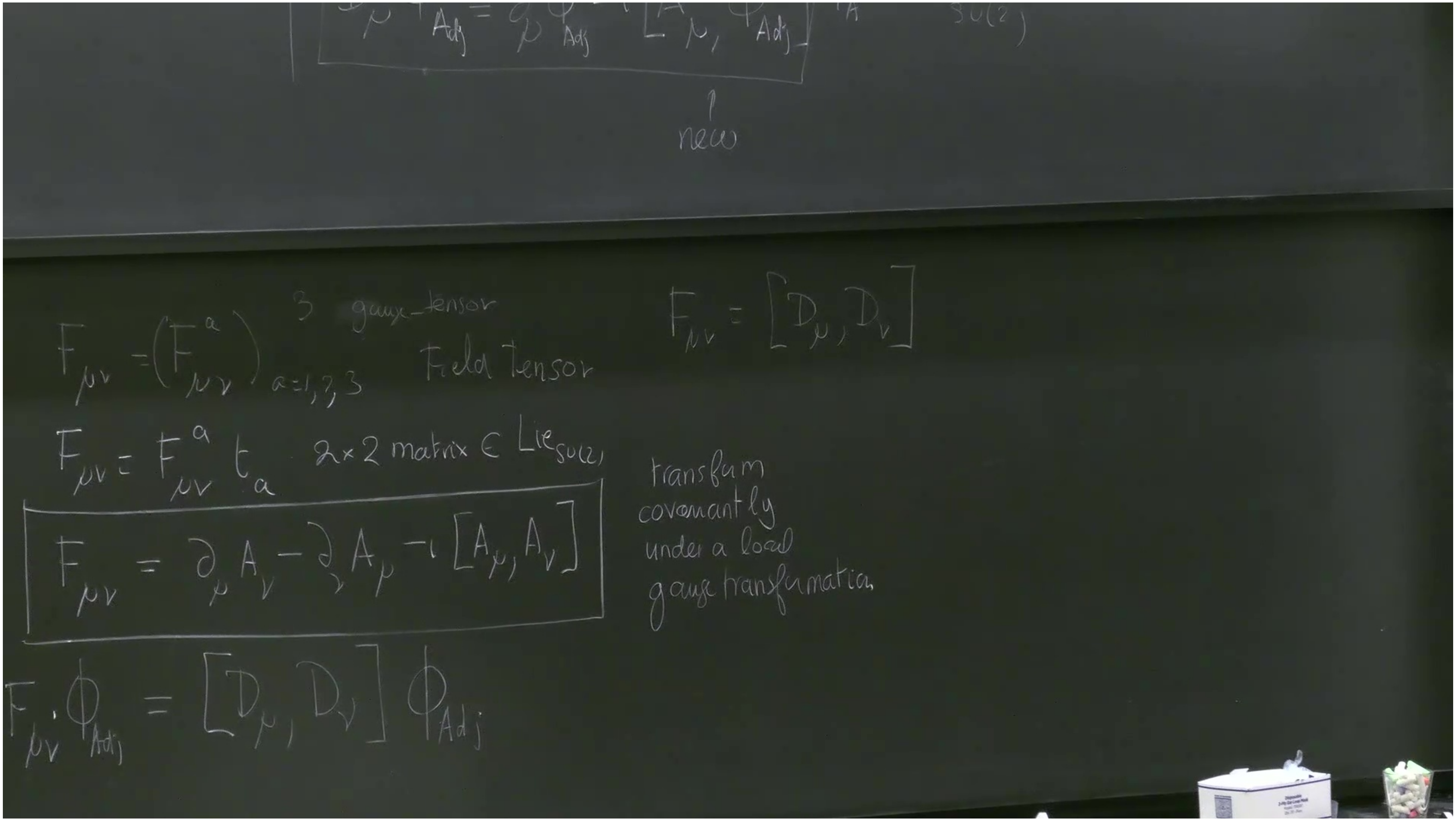
local gauge transf for A_μ gauge field

under $\alpha(x) = \alpha^a(x)\, t_a$ $\quad \mathbb{M}^{1,3} \rightarrow \text{Adj}\ su(2)$

$$A_\mu \longrightarrow \boxed{A_\mu + D_\mu \alpha = A_\mu + \partial_\mu \alpha - i\,[A_\mu, \alpha]}$$

$\uparrow$ new term





$$\boxed{D_\mu \phi_{Adj} = \partial_\mu \phi_{Adj} - i\left[A_\mu, \phi_{Adj}\right]}$$ SU(2)

new

3 gauge-tensor

$$F_{\mu\nu} = \left(F^a_{\mu\nu}\right)_{a=1,2,3}$$ Field Tensor

$$F_{\mu\nu} = F^a_{\mu\nu} t_a$$ 2×2 matrix $\in \mathrm{Lie}_{SU(2)}$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]}$$

$$F_{\mu\nu}\phi_{Adj} = \left[D_\mu, D_\nu\right]\phi_{Adj}$$

$$F_{\mu\nu} = \left[D_\mu, D_\nu\right]$$

transform
covariantly
under a local
gauge transformation