Title: Lecture - QFT II, PHYS 603

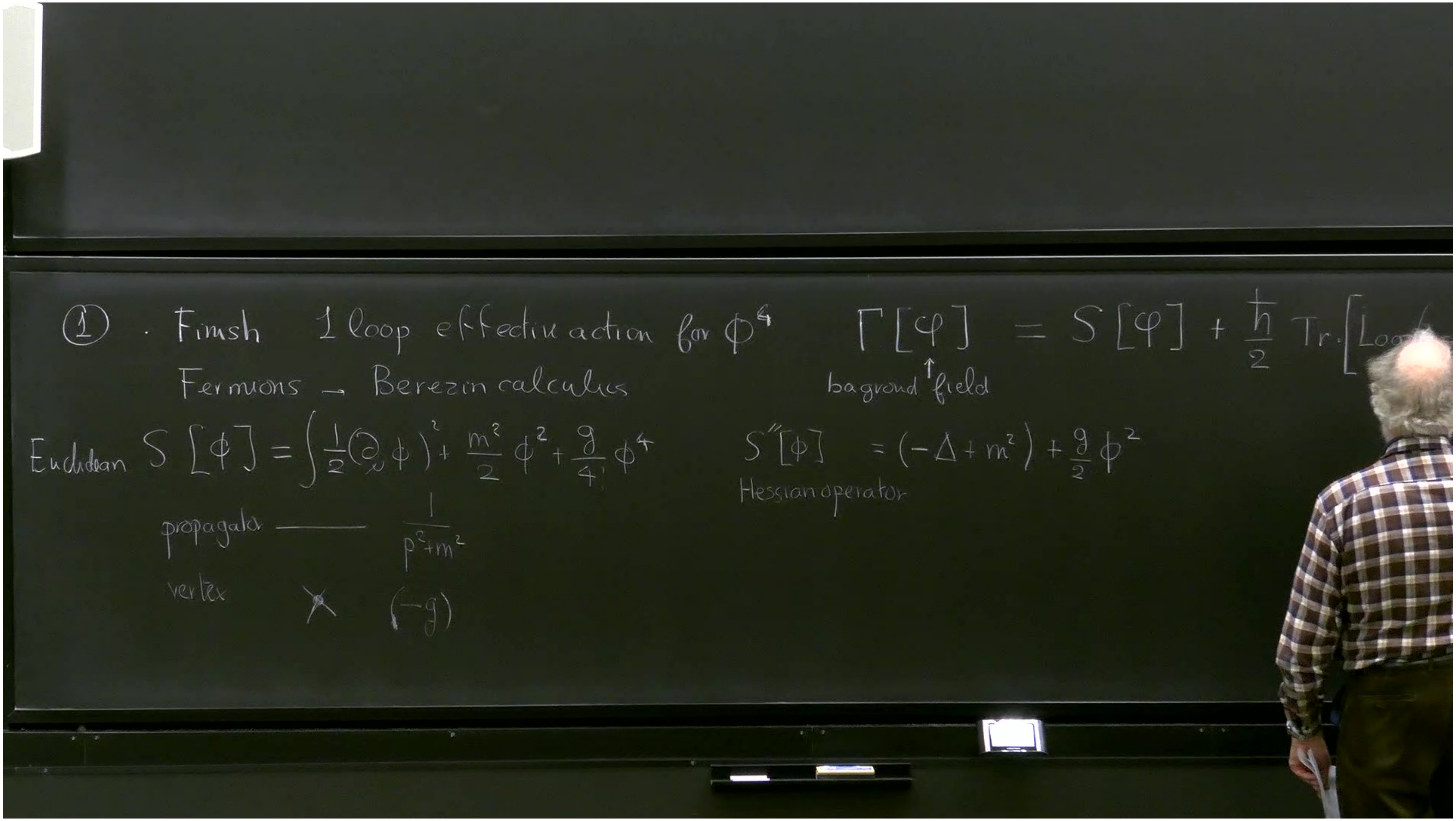
Speakers: Francois David

Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

Date: November 21, 2024 - 9:00 AM

URL: https://pirsa.org/24110013


① · Finish 1 loop effective action for ϕ^4
$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr.}[\mathrm{Log}$
Fermions → Berezin calculus
bagroud field
Euclidean $S[\phi] = \int \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$
$S''[\phi] = (-\Delta + m^2) + \frac{g}{2}\phi^2$
Hessian operator
propagator $\frac{1}{p^2+m^2}$
vertex $(-g)$

on for ϕ^4 $\qquad \Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\,\mathrm{Tr.}\left[\mathrm{Log}\left(-\Delta + m^2 + \frac{g}{2}\phi^4\right)\right] + O(\hbar^2)$

↑ baground field

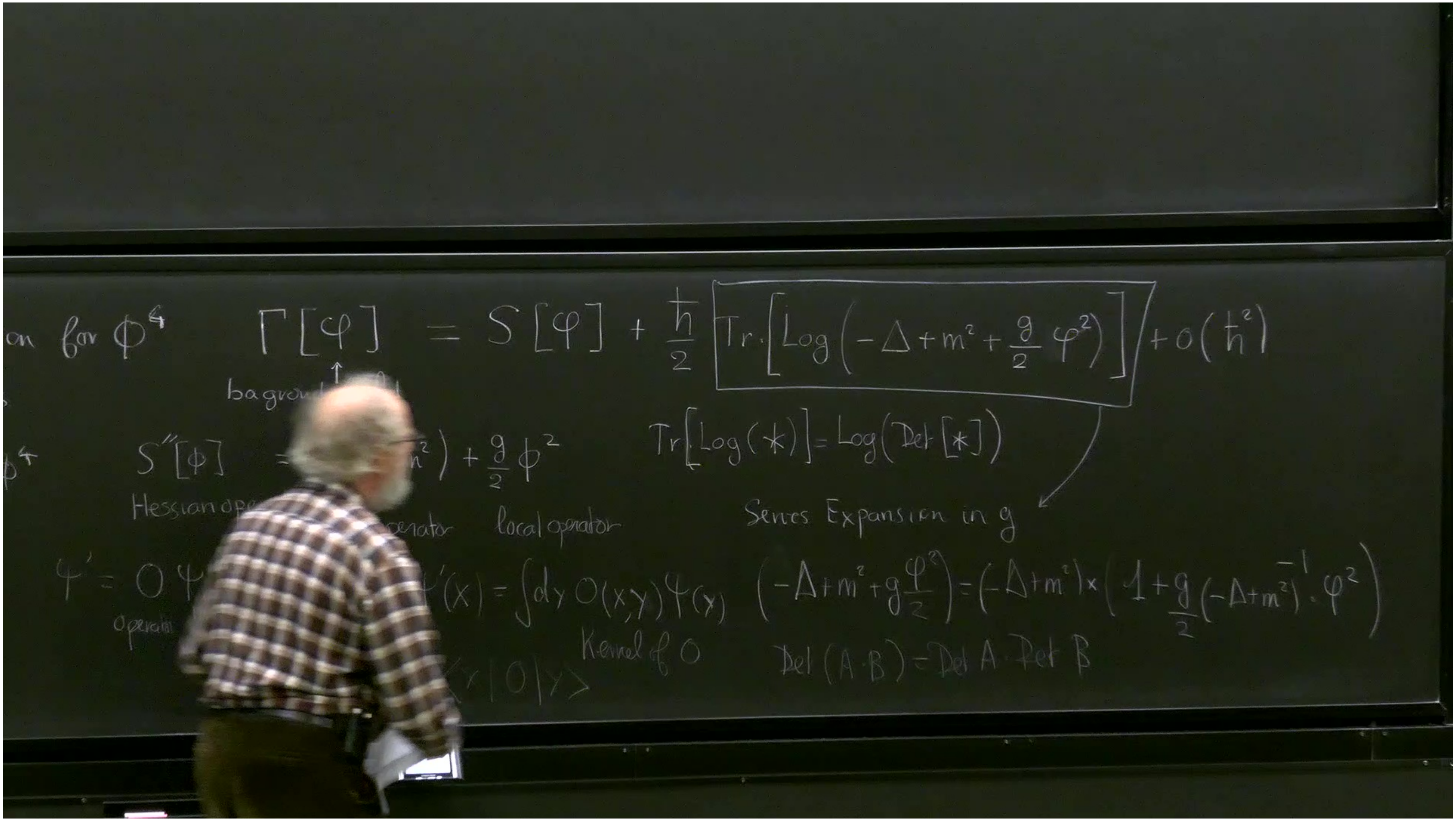
ϕ^4 $\qquad S''[\phi] = (-\Delta + m^2) + \frac{g}{2}\phi^2$

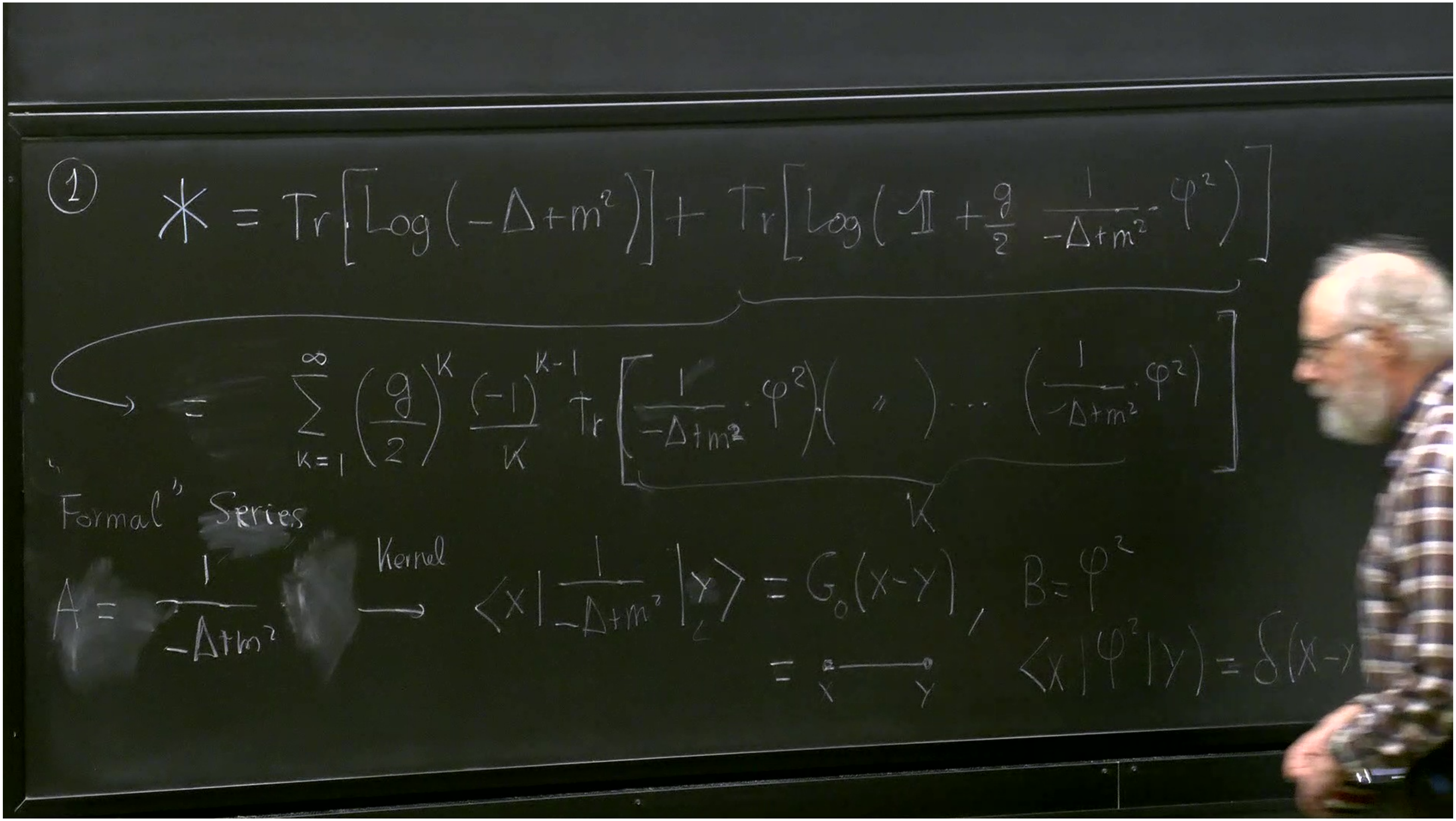
Hessian operator $\quad$ ↑ diff operator $\quad$ local operator

$\psi' = O\,\psi$ $\qquad \psi(x) \longrightarrow \psi'(x) = \int dy\, O(x,y)\,\psi(y)$

operator $\quad$ ↖ function $\qquad$ Kernel of O

$O(x,y) = \langle x | O | y \rangle$


for ϕ^4
$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr.}\left[\mathrm{Log}\left(-\Delta + m^2 + \frac{g}{2}\varphi^2\right)\right] + O(\hbar^2)$
$\mathrm{Tr}\left[\mathrm{Log}(*)\right] = \mathrm{Log}\left(\mathrm{Det}[*]\right)$
$S''[\phi]$
$+ \frac{g}{2}\phi^2$
local operator
Series Expansion in g
$\left(-\Delta + m^2 + g\frac{\varphi^2}{2}\right) = \left(-\Delta + m^2\right) \times \left(1 + \frac{g}{2}\left(-\Delta + m^2\right)^{-1} \cdot \varphi^2\right)$
$(x) = \int dy\, O(x,y)\, \varphi(y)$
Kernel of O
$\mathrm{Det}(A \cdot B) = \mathrm{Det}\, A \cdot \mathrm{Det}\, B$


① $* = \mathrm{Tr}\left[\mathrm{Log}(-\Delta + m^2)\right] + \mathrm{Tr}\left[\mathrm{Log}\left(\mathbb{1} + \frac{g}{2}\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]$
$= \sum_{K=1}^{\infty}\left(\frac{g}{2}\right)^K \frac{(-1)^{K-1}}{K}\mathrm{Tr}\left[\left(\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\left(\ \text{"}\ \right)\cdots\left(\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]$
K
"Formal" Series
Kernel
$A = \frac{1}{-\Delta+m^2} \longrightarrow \langle x|\frac{1}{-\Delta+m^2}|y\rangle = G_0(x-y)$, $B=\varphi^2$
$= \underset{x}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{y}{\circ}$
$\langle x|\varphi^2|y) = \delta(x-y$

(1) $$* = \mathrm{Tr}\left[\mathrm{Log}\left(-\Delta+m^2\right)\right] + \mathrm{Tr}\left[\mathrm{Log}\left(\mathbb{1} + \frac{g}{2}\,\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]$$

Truncated propagator o—|—o

$$= \sum_{k=1}^{\infty}\left(\frac{g}{2}\right)^{k}\frac{(-1)^{k-1}}{k}\,\mathrm{Tr}\Big[\underbrace{\left(\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\left(\;''\;\right)\cdots\left(\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)}_{k}\Big]$$

"Formal" Series

$$A = \frac{1}{-\Delta+m^2} \xrightarrow{\text{Kernel}} \left\langle x\left|\frac{1}{-\Delta+m^2}\right|y\right\rangle = G_0(x-y),$$

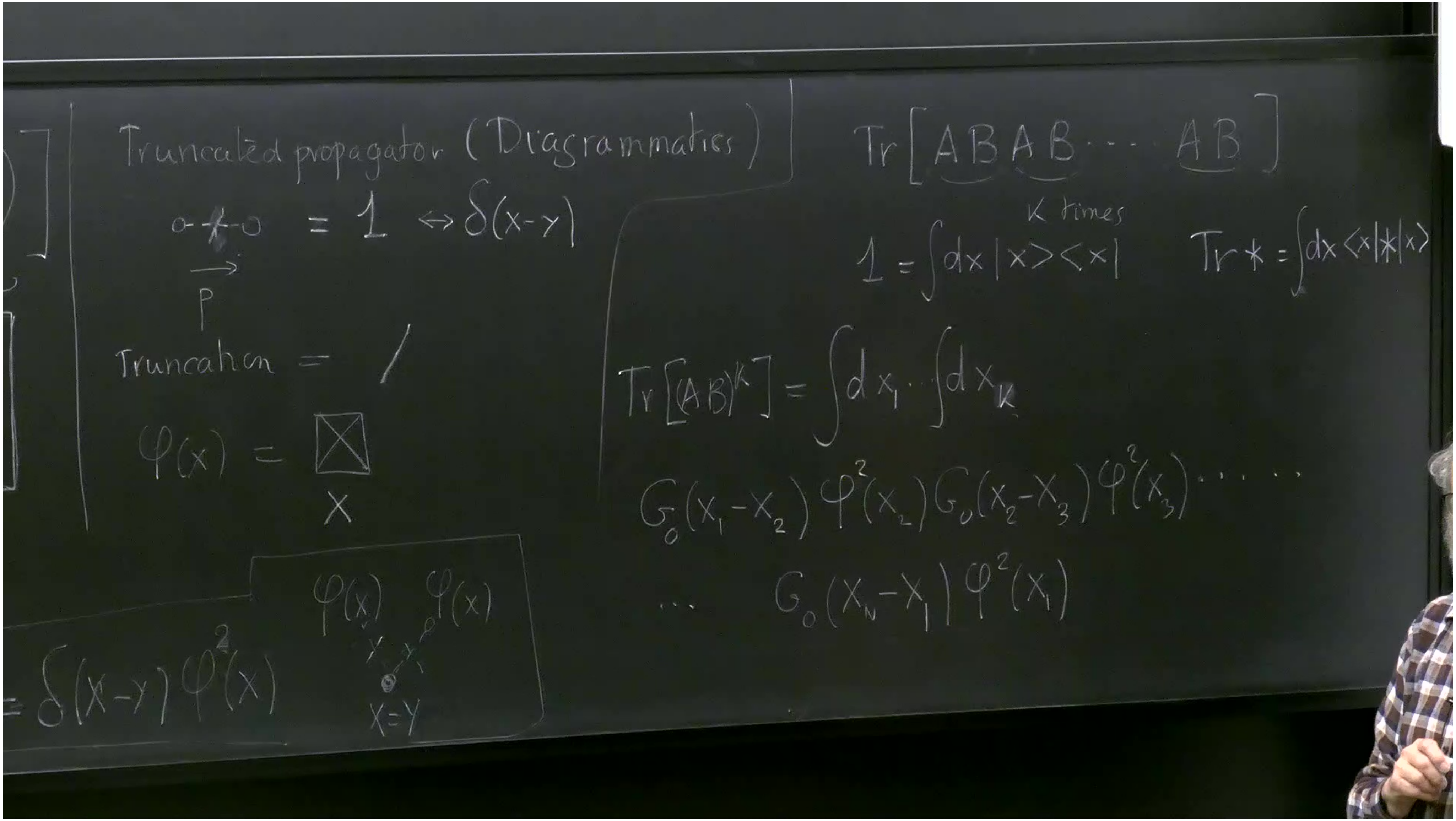
$= x$ o———o y

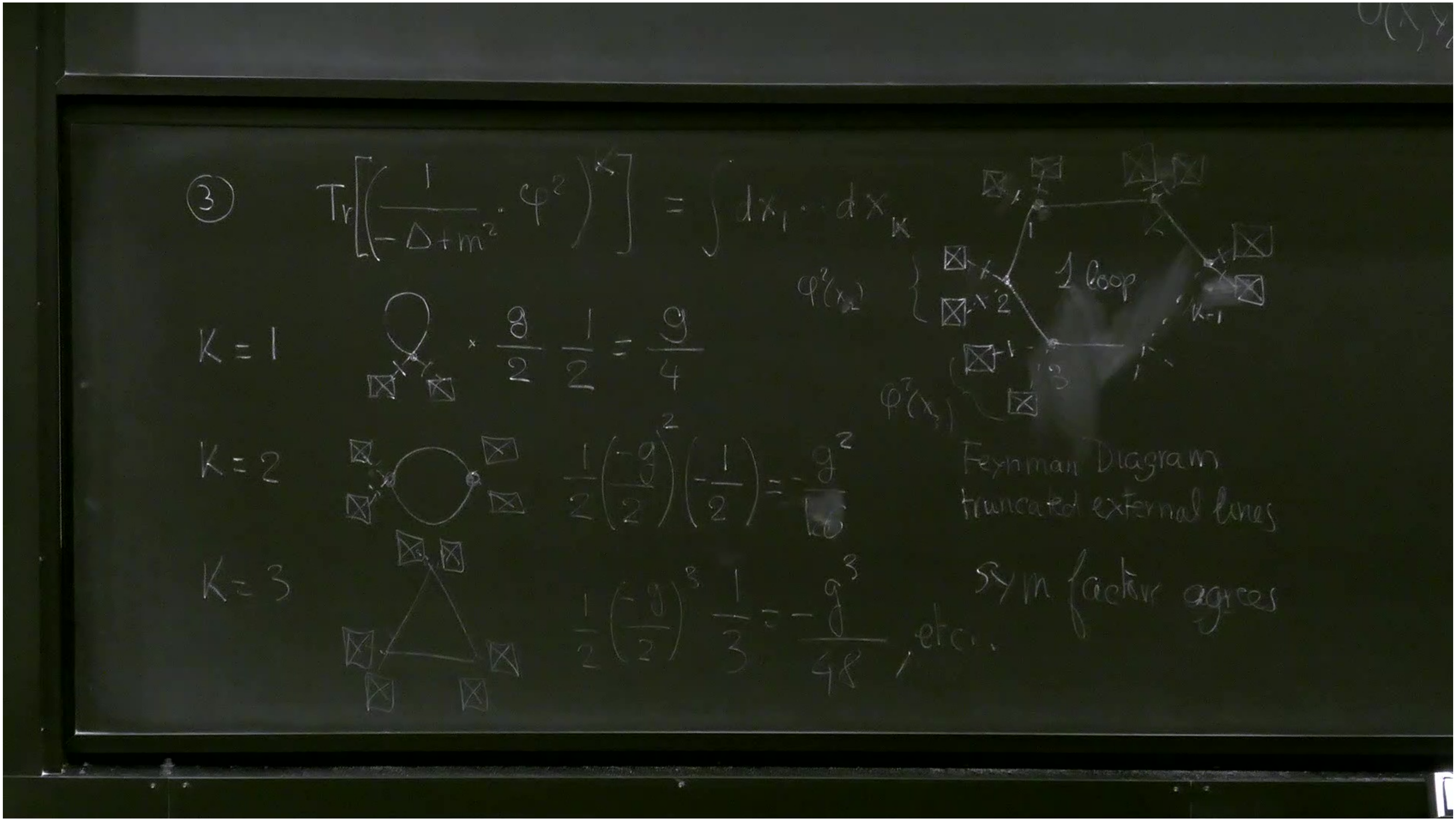
$$B = \varphi^2$$

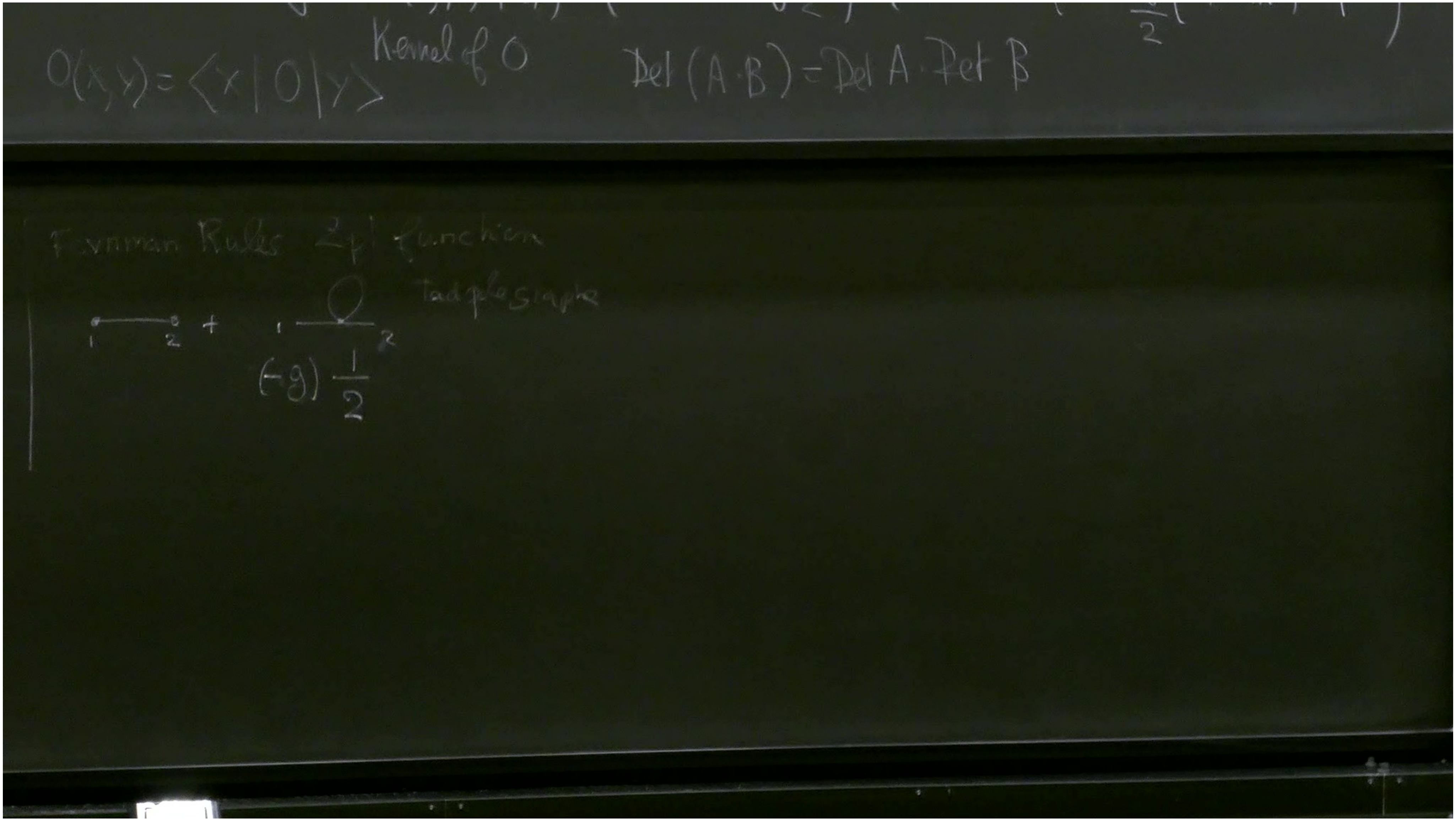
$$\langle x|\varphi^2|y\rangle = \delta(x-y)\,\varphi^2(x)$$

$\varphi(x)$ $\varphi(x)$

$x=y$


Truncated propagator (Diagrammatics)
= 1 ⇔ δ(x-y)
p
Truncation = /
φ(x) =
x
φ(x) φ(x)
x = y
= δ(x-y) φ²(x)
Tr [ABAB ··· AB]
K times
1 = ∫dx |x><x|
Tr * = ∫dx <x|*|x>
Tr [(AB)^K] = ∫dx₁ ··· ∫dx_K
G₀(x₁-x₂) φ²(x₂) G₀(x₂-x₃) φ²(x₃) ······
··· G₀(x_N-x₁) φ²(x₁)


③ Tr[(1/(−Δ+m²) · φ²)^K] = ∫dx₁ ⋯ dx_K
K = 1
g/2 · 1/2 = g/4
K = 2
1/2 (−g/2)² (−1/2) = −g²/16
K = 3
1/2 (−g/2)³ 1/3 = −g³/48, etc.
φ²(x₂)
φ²(x₁)
1 loop
1
2
3
K−1
K
Feynman Diagram
truncated external lines
sym factor agrees


$O(x,y) = \langle x | O | y \rangle$
Kernel of O
Det (A·B) = Det A · Det B
Feynman Rules 2pt function
$(-g)\frac{1}{2}$

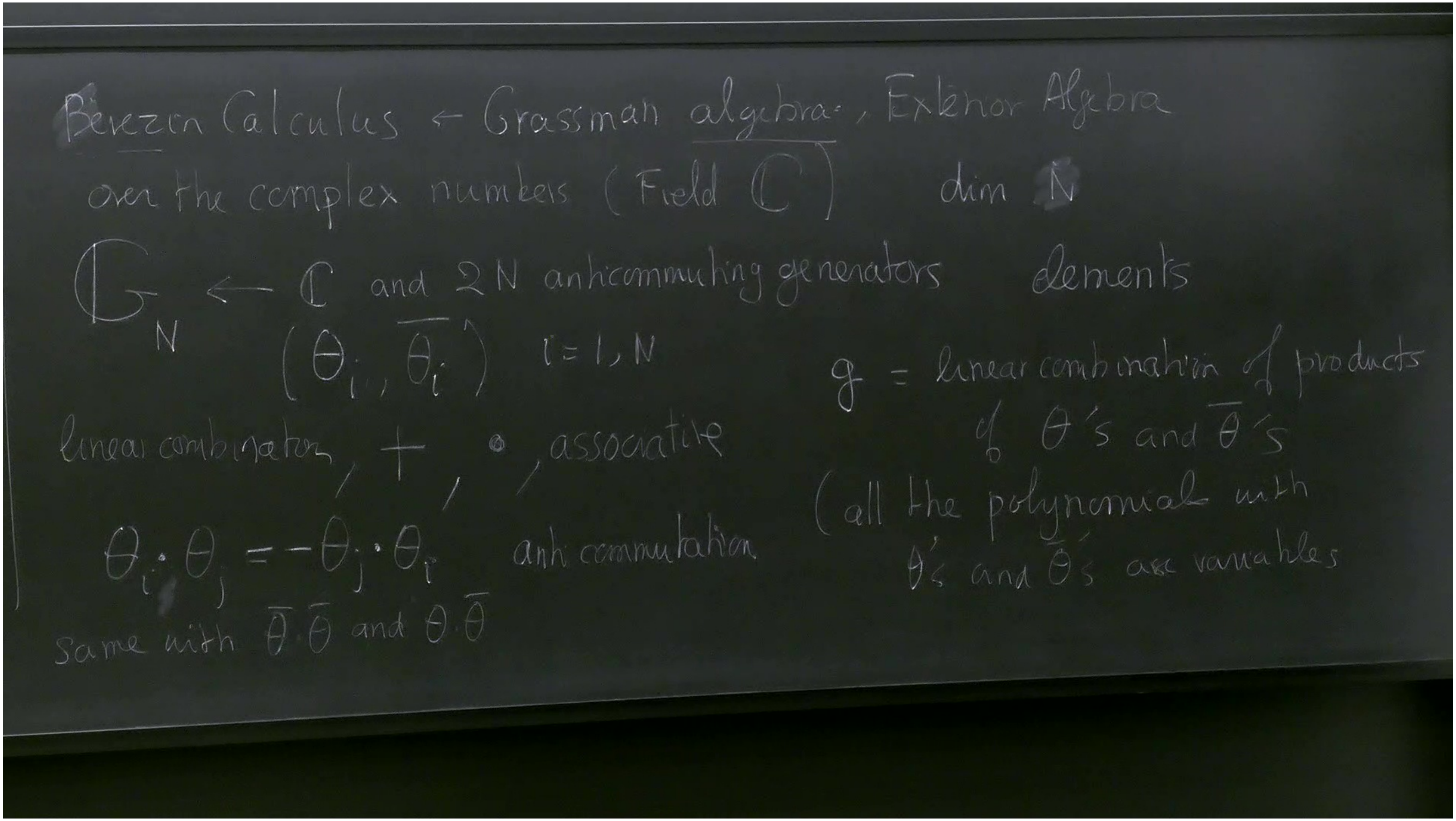
Path/Functional Integrals for Fermions : Chapter 5

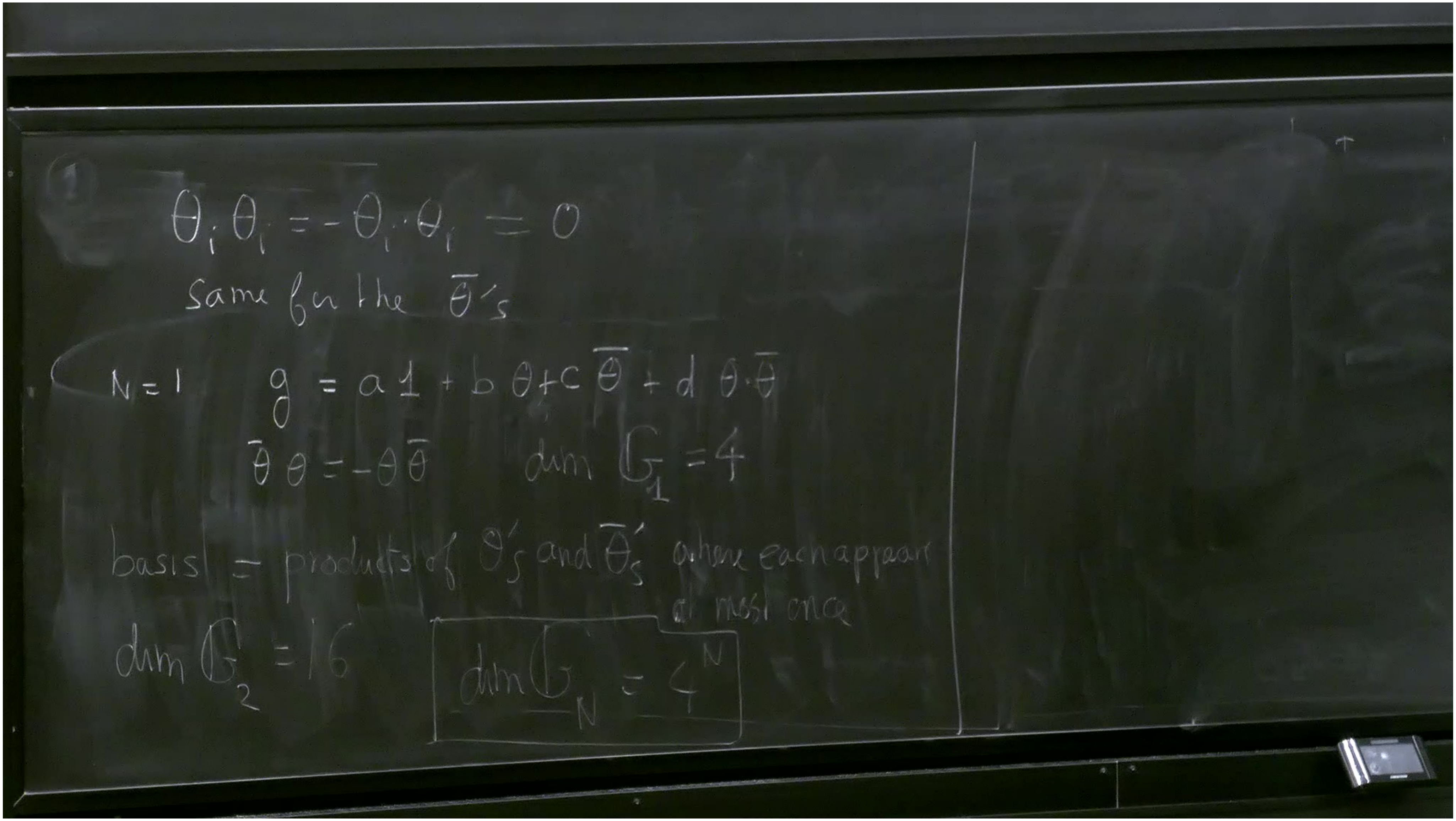
Dirac Equation $(i\not{\partial}+m)\psi=0$

Causality + locality $\Rightarrow$ creation and annihilation operators must anti-commute

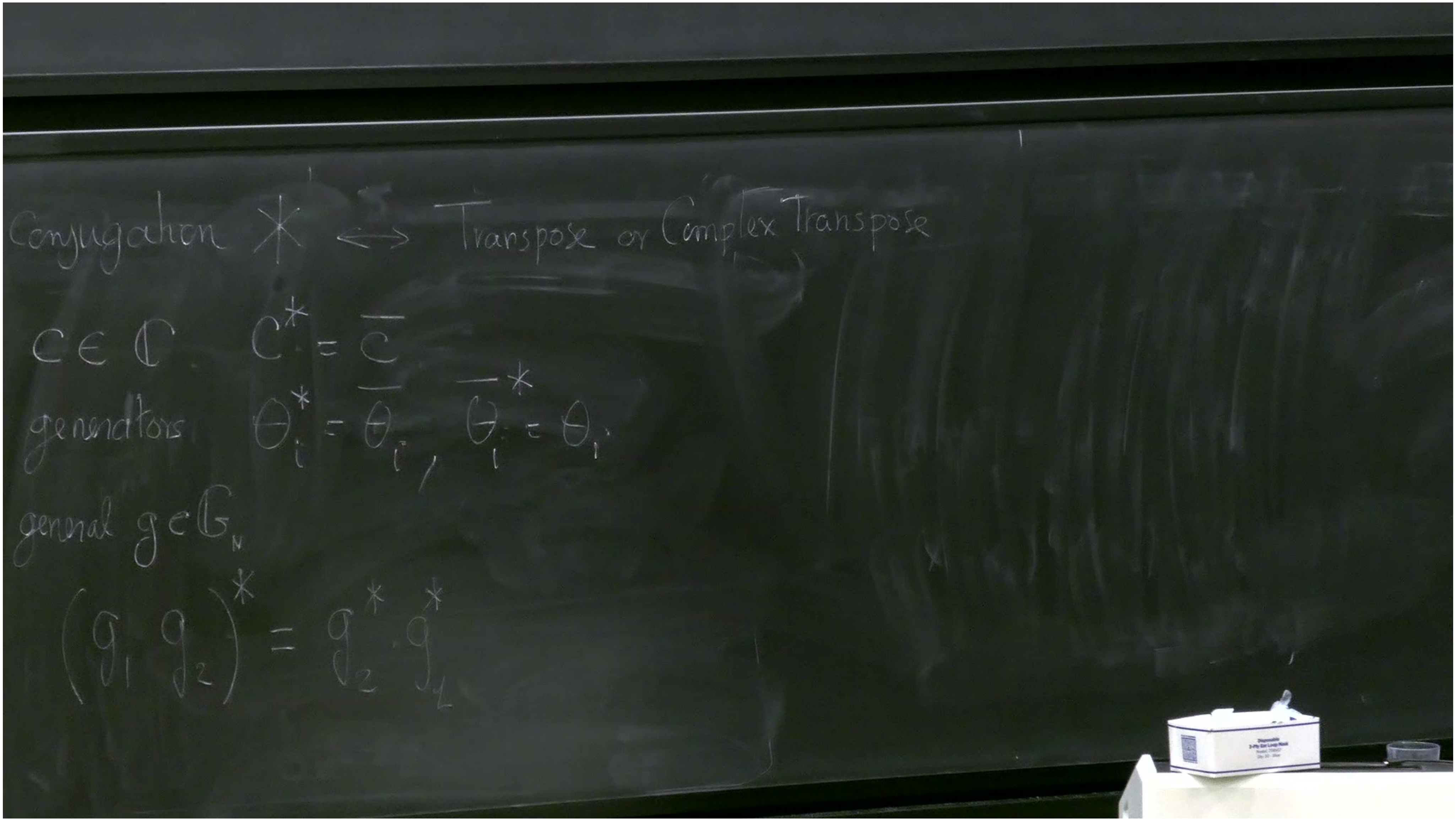
$$S[\psi,\bar{\psi}]=\int d^4x\ \bar{\psi}(i\not{\partial}-m)\psi$$

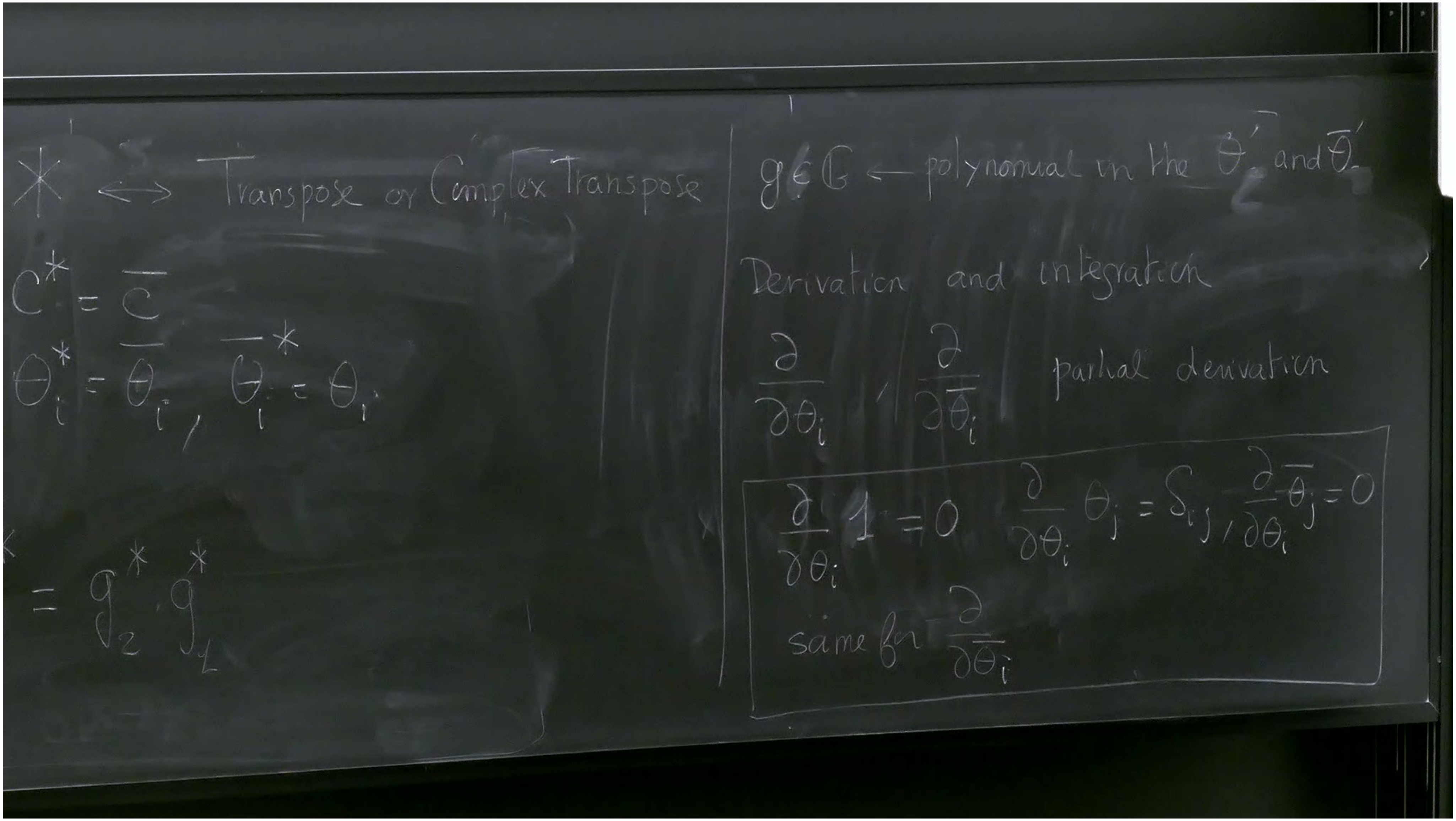
$$\int D[\bar{\psi},\bar{\psi}]\exp\left(\frac{i}{\hbar}S[\psi,\bar{\psi}]\right)$$ make sense of this ?

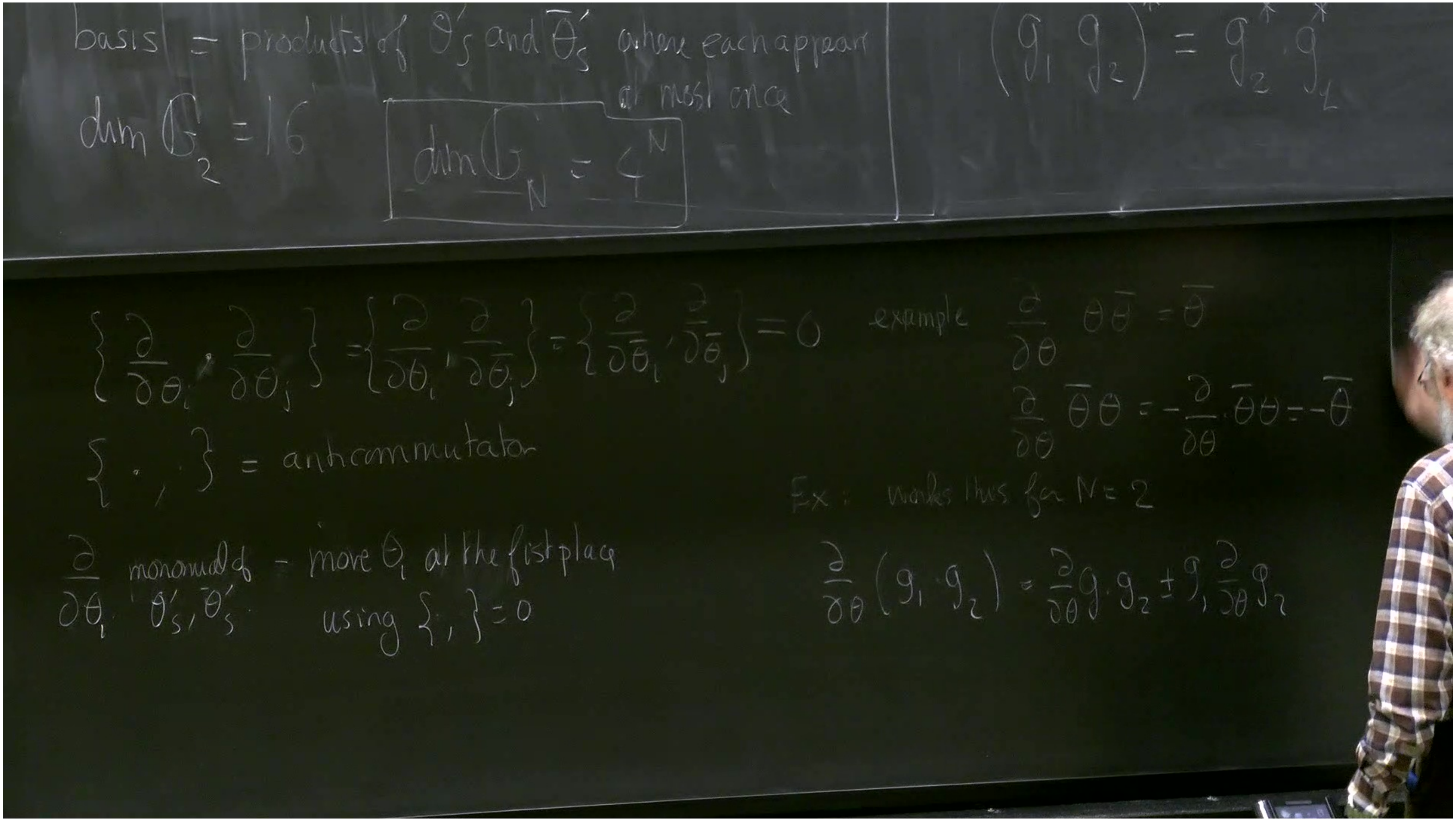

Berezin Calculus ← Grassman algebra, Exterior Algebra
over the complex numbers (Field ℂ)   dim N
$\mathbb{G}_N$ ← ℂ and 2N anticommuting generators   elements
$(\theta_i, \bar{\theta}_i)$ i = 1, N
g = linear combination of products
of θ's and $\bar{\theta}$'s
linear combination, +, ·, associative
(all the polynomials with
θ's and $\bar{\theta}$'s are variables
$\theta_i \cdot \theta_j = -\theta_j \cdot \theta_i$ anti commutation
same with $\bar{\theta} \cdot \bar{\theta}$ and $\theta \cdot \bar{\theta}$



$\theta_i \theta_i = -\theta_i \cdot \theta_i = 0$
same for the $\bar\theta$'s
$N=1$ $g = a\,1 + b\,\theta + c\,\bar\theta + d\,\theta\cdot\bar\theta$
$\bar\theta\,\theta = -\theta\,\bar\theta$ dim $G_1 = 4$
basis = products of θ's and $\bar\theta$'s where each appears at most once
dim $G_2 = 16$
dim $G_N = 4^N$


Conjugation $*$ $\longleftrightarrow$ Transpose or Complex Transpose
$c \in \mathbb{C}$ $c^* = \bar{c}$
generators $\theta_i^* = \bar{\theta}_i$, $\bar{\theta}_i^* = \theta_i$
general $g \in \mathbb{G}_N$
$(g_1 g_2)^* = g_2^* \cdot g_1^*$


* ⟷ Transpose or Complex Transpose
g ∈ G ← polynomial in the θ and θ̄
Derivation and integration
partial derivation
same for


basis = products of θ's and θ̄'s where each appear at most once
dim G₂ = 16
dim G_N = 4^N
(g₁ g₂)* = g₂* g₁*
{∂/∂θ_i, ∂/∂θ_j} = {∂/∂θ_i, ∂/∂θ̄_j} = {∂/∂θ̄_i, ∂/∂θ̄_j} = 0
example ∂/∂θ θθ̄ = θ̄
∂/∂θ θ̄θ = −∂/∂θ θθ̄ = −θ̄
{ , } = anticommutator
Ex: works this for N = 2
∂/∂θ_i monomial of θ's, θ̄'s - move θ_i at the first place using { , } = 0
∂/∂θ (g₁ · g₂) = ∂/∂θ g₁ · g₂ ± g₁ ∂/∂θ g₂

$= -\bar{\theta}$

$-\frac{\partial}{\partial\theta}\bar{\theta}\theta = -\bar{\theta}$

$\pm g_1 \frac{\partial}{\partial\theta} g_2$

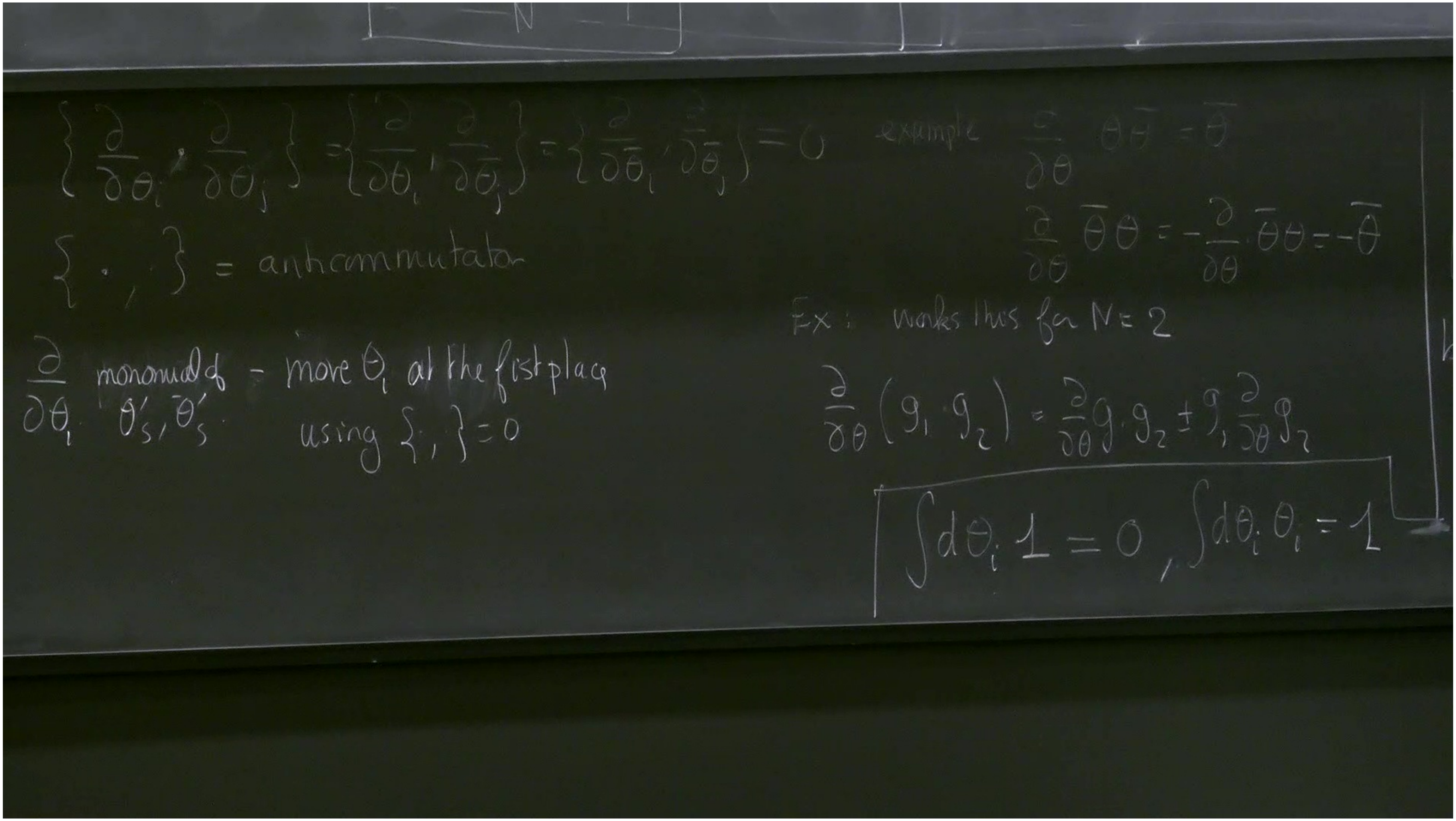
Integration $\int d\theta_i$ and $\int d\bar{\theta}_i$, $\int \frac{\partial}{\partial\theta} = 1$, boundary term

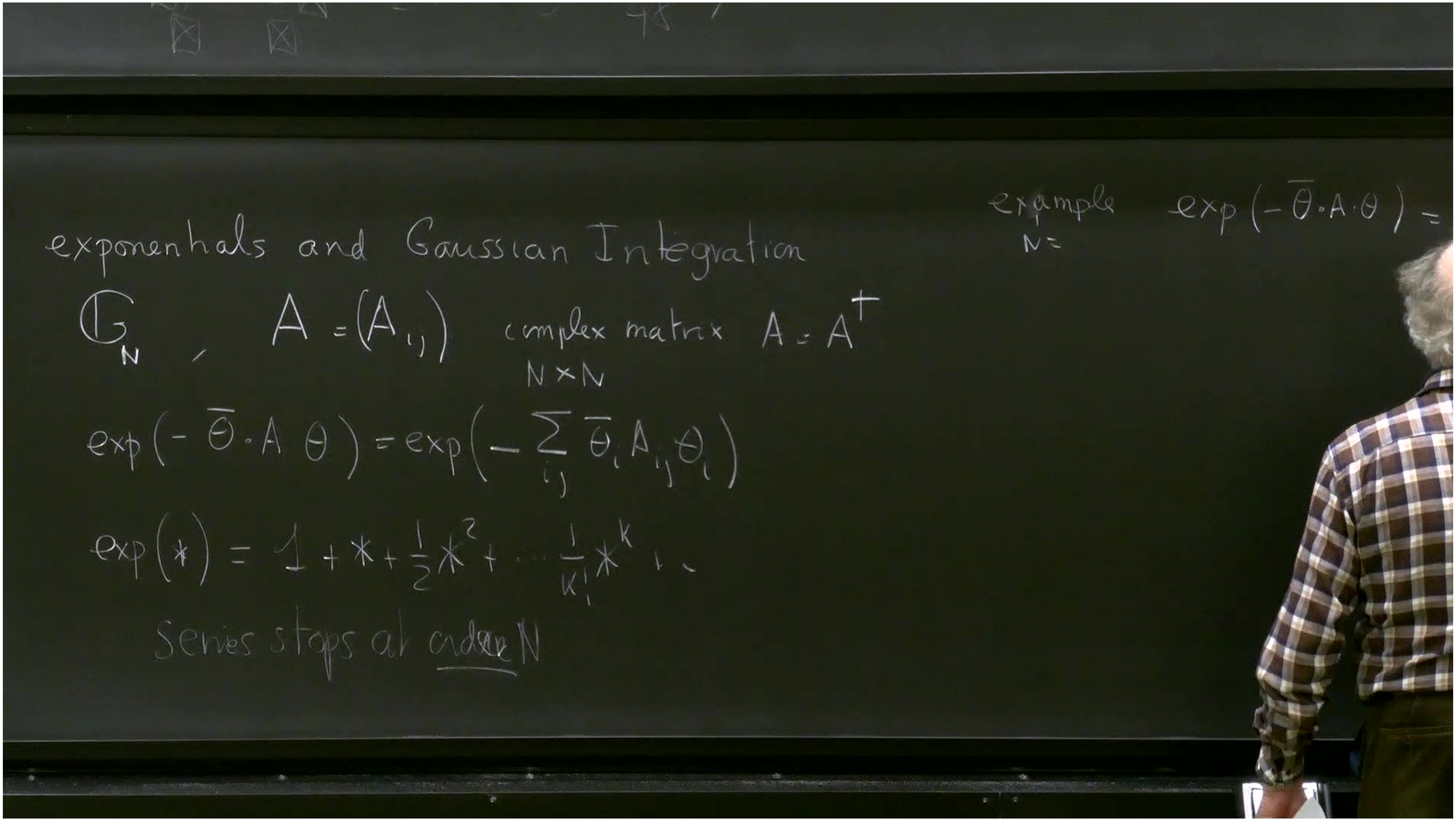
$$\boxed{\int d\theta_i \frac{\partial}{\partial\theta_i} g = 0}$$, same for $\int d\bar{\theta}_i$

but $\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_i} = 0$ so $\boxed{\text{take } \int d\theta_i := \frac{\partial}{\partial\theta_i}}$

Integration = Derivation $\neq$ Integration = $(\text{Derivation})^{-1}$

commuting case


$\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\} = \{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\bar\theta_j}\} = \{\frac{\partial}{\partial\bar\theta_i}, \frac{\partial}{\partial\bar\theta_j}\} = 0$
example $\frac{\partial}{\partial\theta}\theta\bar\theta = \bar\theta$
$\frac{\partial}{\partial\theta}\bar\theta\theta = -\frac{\partial}{\partial\theta}\theta\bar\theta = -\bar\theta$
$\{\cdot,\cdot\}$ = anticommutator
$\frac{\partial}{\partial\theta_i}$ monomial of $\theta_s', \bar\theta_s'$ - move θ_i at the first place using $\{,\} = 0$
Ex: works this for N = 2
$\frac{\partial}{\partial\theta}(g_1 \cdot g_2) = \frac{\partial}{\partial\theta}g_1 \cdot g_2 \pm g_1 \frac{\partial}{\partial\theta} g_2$
$\int d\theta_i\, 1 = 0, \int d\theta_i\, \theta_i = 1$


exponentials and Gaussian Integration
$\mathbb{G}_N$, $A = (A_{ij})$ complex matrix $A = A^\dagger$
$N \times N$
$\exp(-\bar{\theta} \cdot A\, \theta) = \exp\left(-\sum_{ij} \bar{\theta}_i A_{ij} \theta_j\right)$
$\exp(*) = 1 + * + \frac{1}{2} *^2 + \cdots \frac{1}{k!} *^k + \cdots$
Series stops at order N
example $\exp(-\bar{\theta} \cdot A \cdot \theta) =$
N=

...le $\exp(-\bar{\theta}\cdot A\cdot\theta) = 1 - \bar{\theta} A \theta$

$$\int d\bar{\theta}\cdot d\theta \, \exp(-\bar{\theta} A \theta) = -A\int d\bar{\theta}\cdot d\theta \, \bar{\theta}\,\theta = A\int d\bar{\theta}\, d\theta \, \theta\bar{\theta}$$

$$= A\int d\bar{\theta}\,\bar{\theta} = A$$

$$\iint \prod_{i=1}^{N} d\bar{\theta}_i \, d\theta_i \, \exp(-\bar{\theta}\cdot A\cdot\theta) = \text{Nth term of the expansion} = \det(A)$$

Integral $\Longrightarrow$ $\mathrm{Det}(A)$ instead of $[\mathrm{Det}(A)]^{-1}$ for complex gaussian integrals

$$\exp(-\bar{\theta}\cdot A\cdot\theta) = 1 - \bar{\theta} A \theta$$

$$\int d\bar{\theta}\cdot d\theta \exp(-\bar{\theta} A \theta) = -A\int d\bar{\theta}\cdot d\theta\ \bar{\theta}\ \theta = A\int d\bar{\theta}\, d\theta\ \theta\,\bar{\theta} = A\int d\bar{\theta}\ \bar{\theta} = A$$

$$\iint \prod_{i=1}^{N} d\bar{\theta}_i\, d\theta_i \exp(-\bar{\theta}\cdot A\cdot\theta) = \text{Nth term of the expansion} = \det(A)$$

Gaussian Integral $\Rightarrow$ $\mathrm{Det}(A)$ instead of $[\mathrm{Det}(A)]^{-1}$ for complex gaussian integrals ple $\exp(-\bar{\theta}\cdot A\cdot\theta) = 1 - \bar{\theta} A \theta$

$$\int d\bar{\theta}\cdot d\theta \exp(-\bar{\theta} A \theta) = -A\int d\bar{\theta}\cdot d\theta\ \bar{\theta}\ \theta = A\int d\bar{\theta}\, d\theta\ \theta\bar{\theta}$$

$$= A\int d\bar{\theta}\ \bar{\theta} = A$$

$$\iint \prod_{i=1}^{N} d\bar{\theta}_i\, d\theta_i \exp(-\bar{\theta}\cdot A\cdot\theta) =$$ Nth term of the expansion $= \det(A)$

Gaussian Integral $\Rightarrow$ Det(A) instead of $[\text{Det}(A)]^{-1}$ for complex gaussian integrals

$\Rightarrow$ Wick Theorem $\Rightarrow$ Dirac QFT