Title: Lecture - QFT II, PHYS 603

Speakers: Francois David

Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

Date: November 18, 2024 - 9:00 AM

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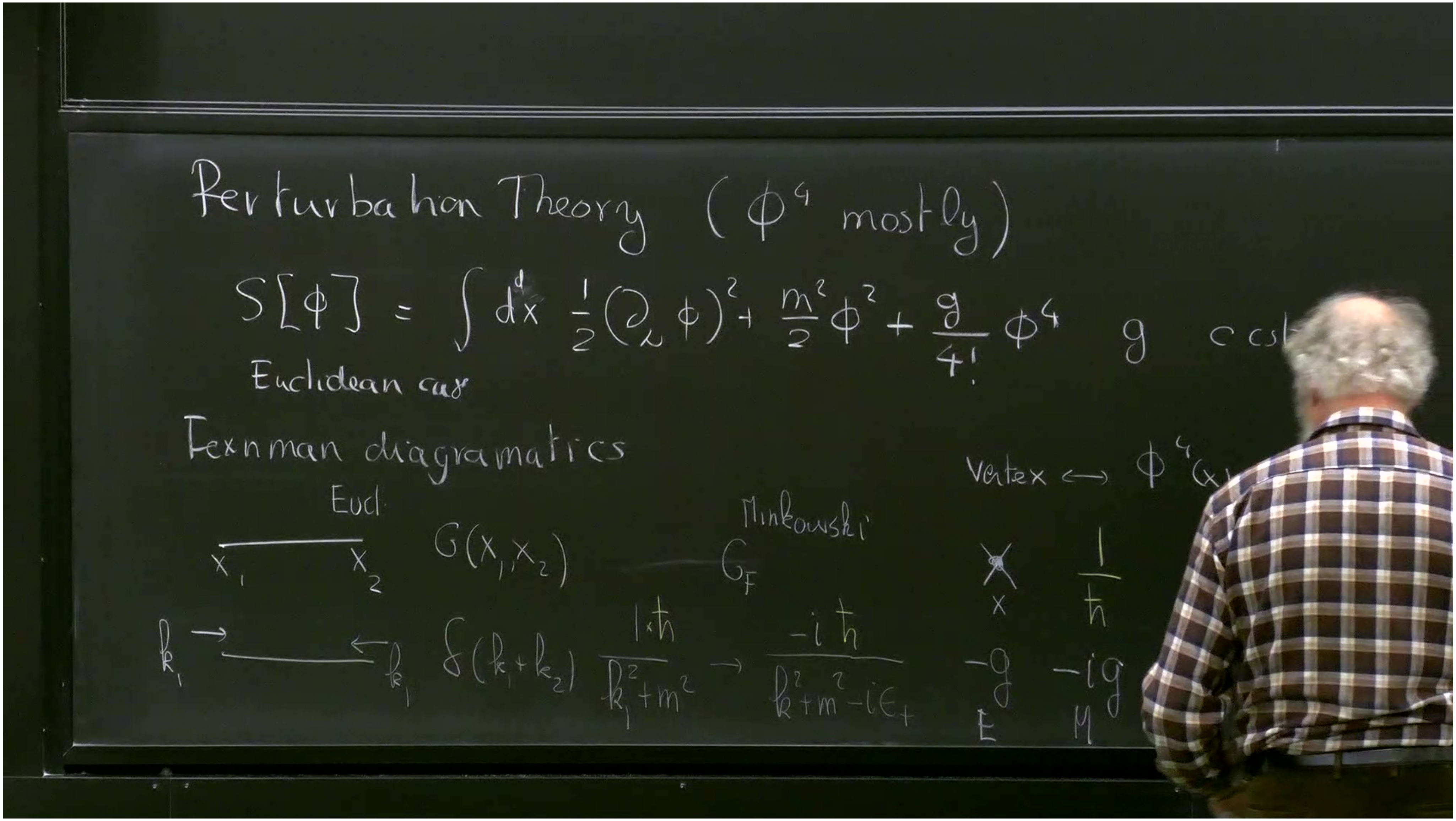

Perturbation Theory (ϕ^4 mostly)
$S[\phi] = \int d^dx \, \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$
g
Euclidean case
Feynman diagramatics
Vertex $\longleftrightarrow$ ϕ^4
Eucl.
Minkowski
x_1 x_2
$G(x_1,x_2)$
G_F
k_1 k_1
$\delta(k_1+k_2)\,\frac{1\cdot\hbar}{k_1^2+m^2} \rightarrow \frac{-i\hbar}{k^2+m^2-i\epsilon_+}$
x
$\frac{1}{\hbar}$
$-g$ E
$-ig$ M

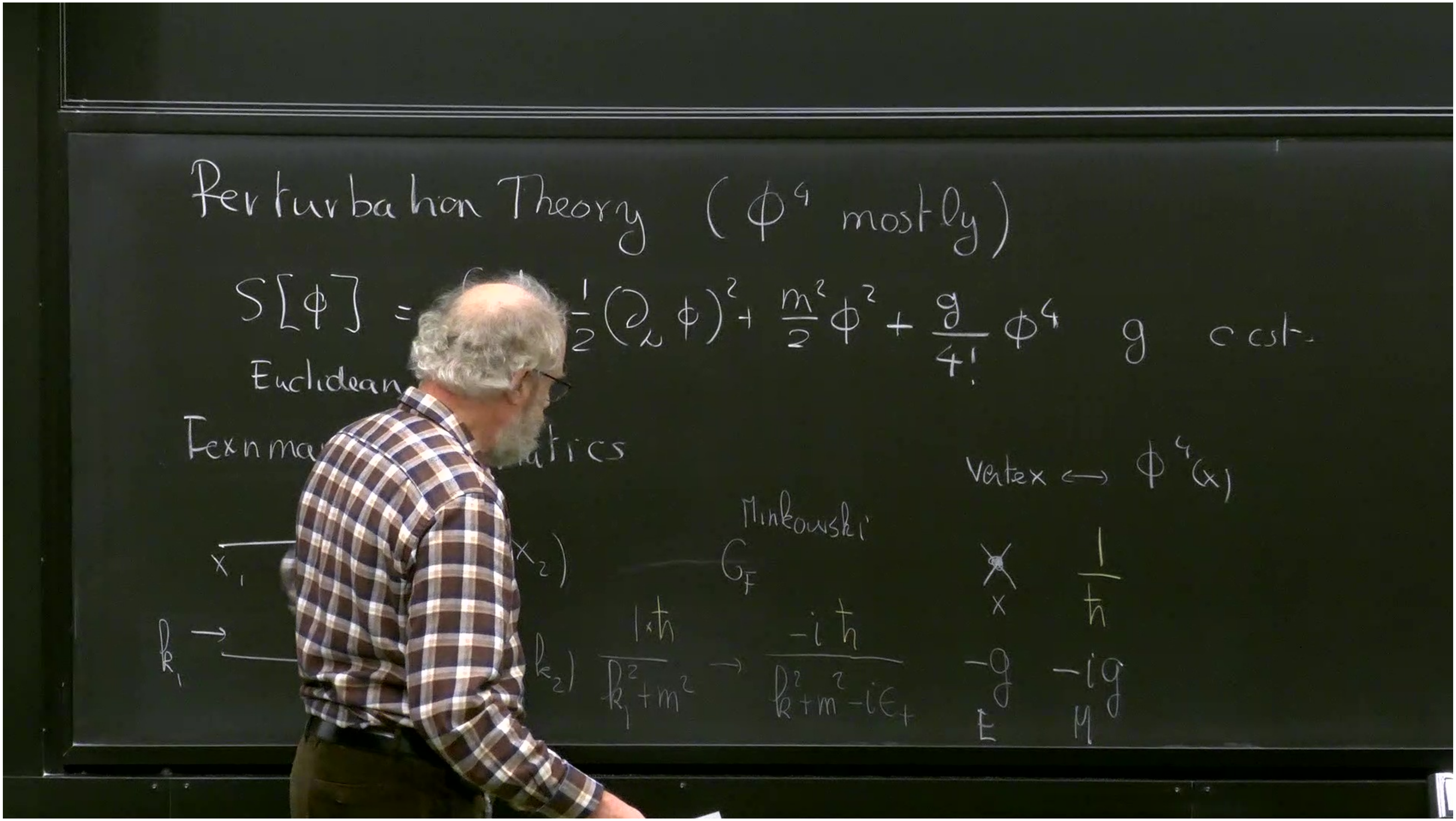

Perturbation Theory (ϕ^4 mostly)
$S[\phi] = \ldots \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$ g cst.
Euclidean
Vertex $\longleftrightarrow$ $\phi^4(x)$
Minkowski
G_F
$\frac{1}{\hbar}$
$\frac{-i\hbar}{k^2+m^2-i\epsilon_+}$
$-g$
E
$-ig$
M

Diagram $\langle \phi(x_1) \cdots \phi(x_4) \rangle$

cst-

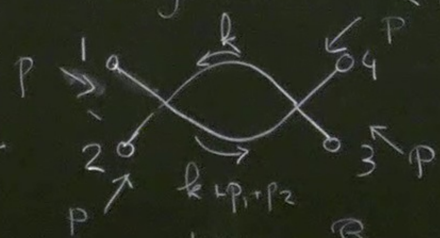


diagram $\mathcal{B}$

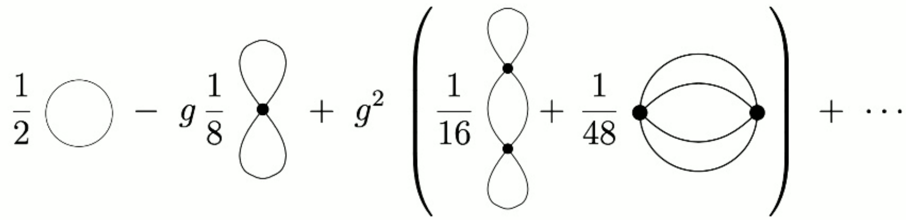
$\mathcal{B} \longrightarrow$ Feynman Integral

complicated in general
(a lot of physics)

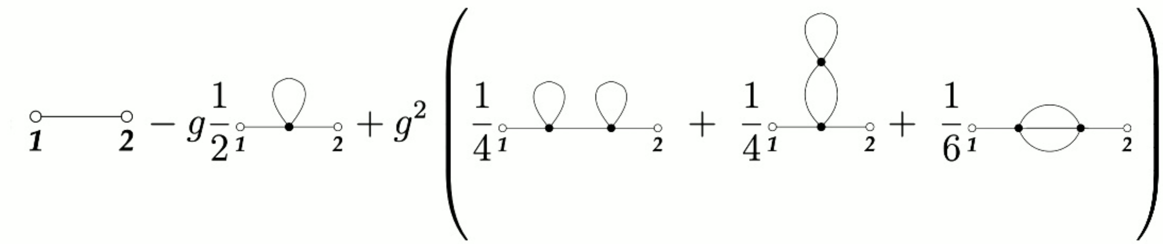
$$I_{\mathcal{B}}(p_1 \cdots p_4) = \delta(p_1 + \cdots + p_4)\, \frac{1}{p_1^2+m^2} \cdots \frac{1}{p_4^2+m^2}$$

$$\int_{\text{internal}} dk\, \frac{1}{k^2+m^2}\, \frac{1}{(k+p_1+p_2)^2+m^2}$$

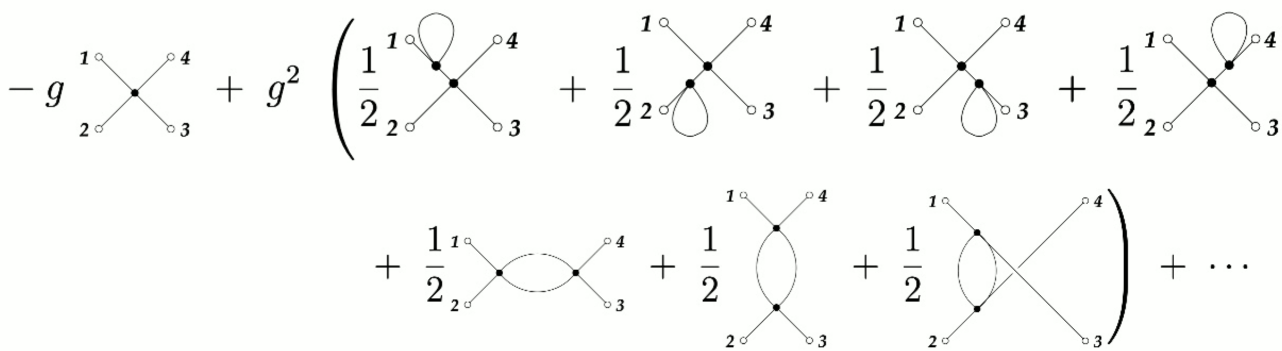
0 pt connected function



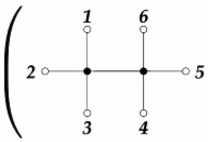
2 pt connected function

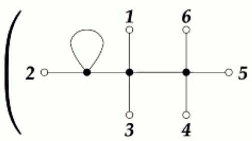
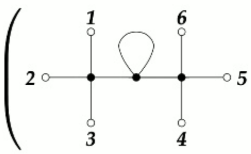


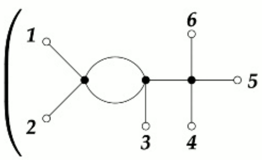
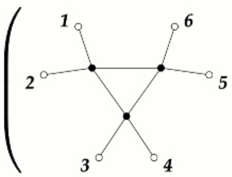
4 pt connected function



6 pt connected function

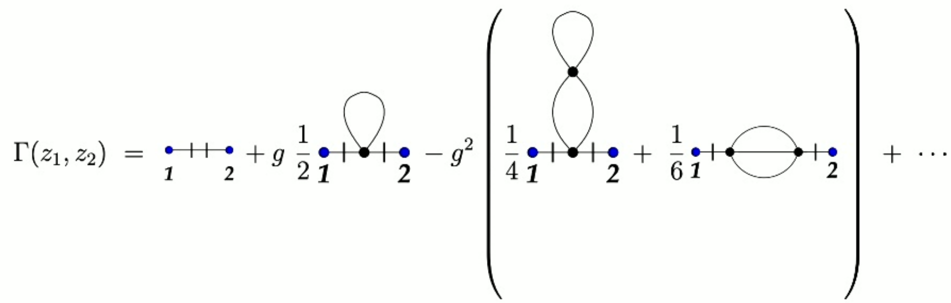
$g^2 \Bigg($  $+ \cdots$ (20 terms) $\Bigg)$

$- g^3 \Bigg(\frac{1}{2} \Bigg($  $+ \cdots$ (120 terms) $\Bigg) + \frac{1}{2} \Bigg($  $+ \cdots$ (20 terms) $\Bigg)$

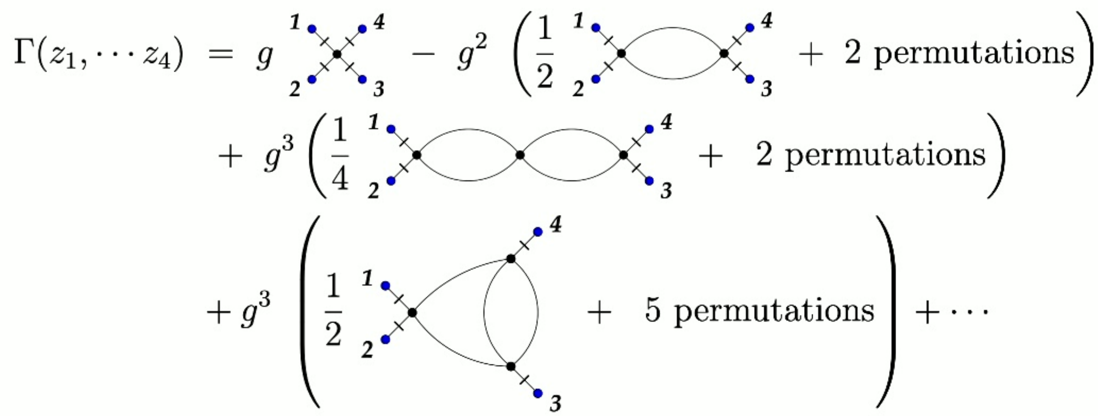
$+ \frac{1}{2} \Bigg($  $+ \cdots$ (60 terms) $\Bigg) + \frac{1}{2} \Bigg($  $+ \cdots$ (15 terms) $\Bigg)\Bigg)$

$$+ \cdots$$

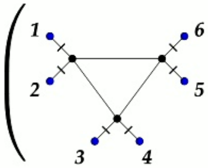
2 pt irreducible function

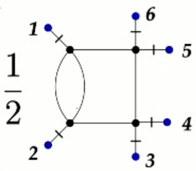


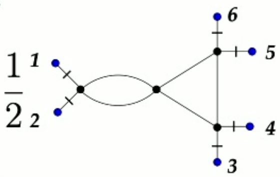
4 pt irreducible function

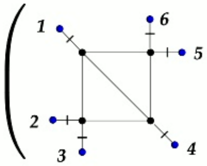


6 pt irreducible function

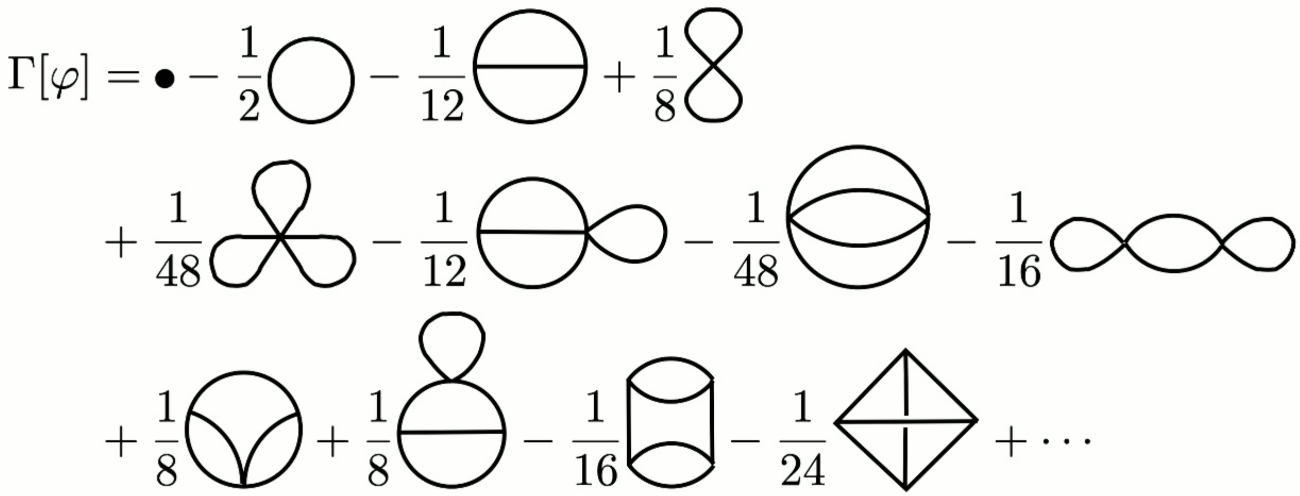
$\Gamma(z_1, \cdots z_6) \ = \ g^3 \ \Bigg($  $+ \ 14$ permutations $\Bigg)$

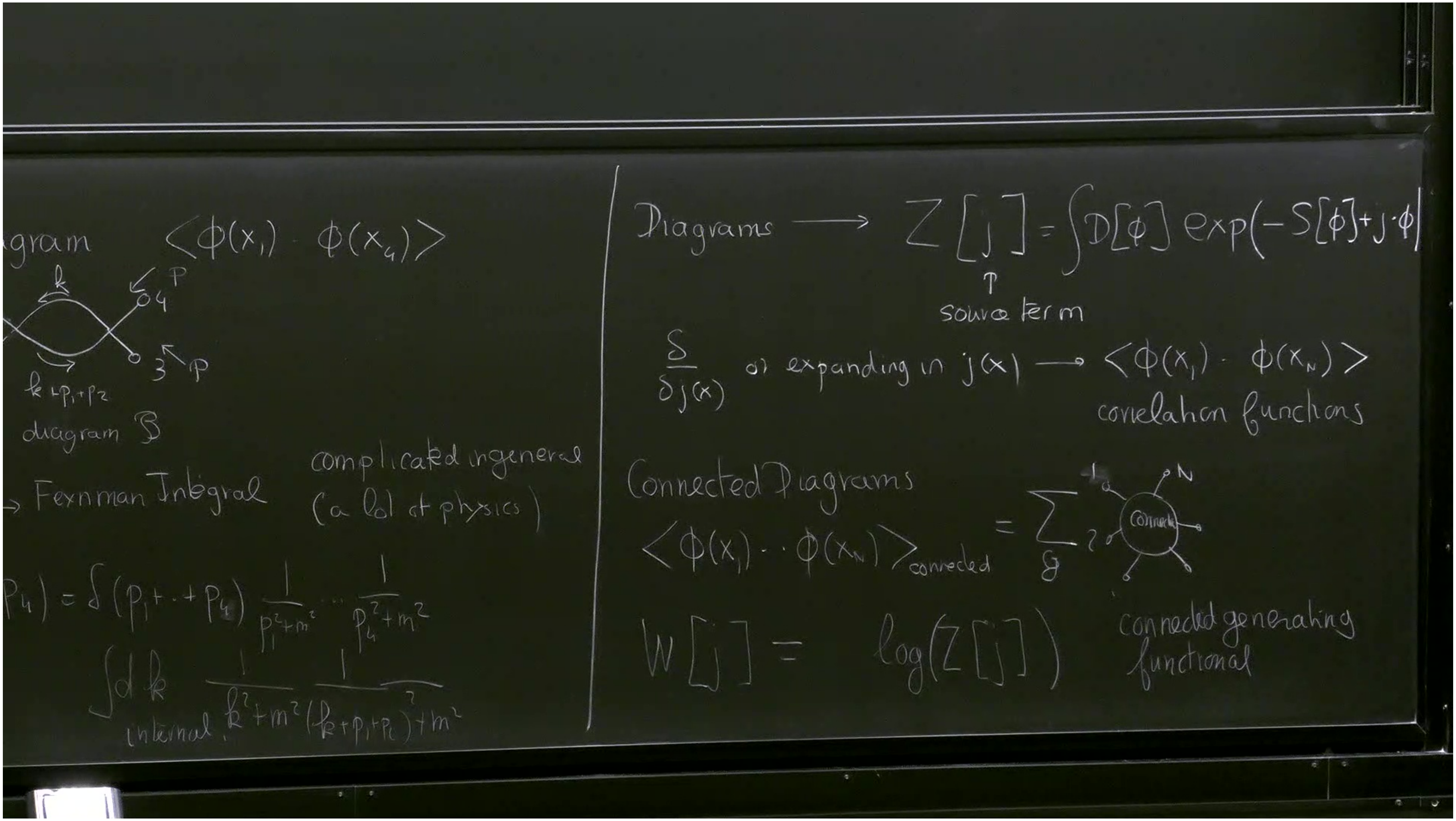
$- \, g^4 \Bigg(\dfrac{1}{2}$  $+ \ 89$ permutations $\Bigg)$

$- \, g^4 \Bigg(\dfrac{1}{2}$  $+ \ 44$ permutations $\Bigg)$

$- \, g^4 \Bigg($  $+ \ 44$ permutations $\Bigg) \ + \ \cdots$

The « generic » irreducible graphs up to 3 loops





gram $\langle \phi(x_1) \cdot \phi(x_4) \rangle$
k p 4
3 p
$k+p_1+p_2$
diagram B
→ Feynman Integral
complicated in general
(a lot of physics)
$p_4) = \delta(p_1 + \cdot + p_4) \frac{1}{p_1^2+m^2} \cdots \frac{1}{p_4^2+m^2}$
$\int dk \frac{1}{k^2+m^2} \frac{1}{(k+p_1+p_2)^2+m^2}$
internal
Diagrams $\longrightarrow$ $Z[j] = \int D[\phi] \exp(-S[\phi] + j \cdot \phi)$
source term
$\frac{\delta}{\delta j(x)}$ or expanding in $j(x) \longrightarrow \langle \phi(x_1) \cdot \phi(x_N) \rangle$
correlation functions
Connected Diagrams
$\langle \phi(x_1) \cdots \phi(x_N) \rangle_{\text{connected}} = \sum_{g}$
1 N
Connected
$W[j] = \log(Z[j])$
connected generating functional

Z as a formal power series

$$\langle 1 \rangle = 1$$

$$\log(1 + *) = * - \frac{1}{2} *^2 + \frac{1}{3} *^3 \dots$$

$$W[J] = \sum c \Rightarrow \exp(W[J]) = 1 + \sum c + \frac{1}{2} \sum \underbrace{c \;\; c}_{G} + \dots$$

$$= \sum_{\substack{\text{all} \\ G}} (-g)^k S_G \, I_G$$

sym factors

integral

$$S_{G_1 \cup G_2} = S_{G_1} \cdot S_{G_2}$$

factorization

Irreducible graphs

$$G \to G_1 \cup G_2$$

...gram $\langle \phi(x_1) \cdots \phi(x_4) \rangle$

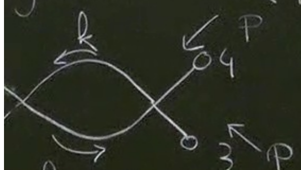


diagram B

→ Feynman Integral

complicated in general (a lot of physics)

$$(\ldots p_4) = \delta(p_1 + \cdots + p_4)\, \frac{1}{p_1^2+m^2} \cdots \frac{1}{p_4^2+m^2} \int_{\text{internal}} dk\, \frac{1}{k^2+m^2}\, \frac{1}{(k+p_1+p_2)^2+m^2}$$

Diagrams $\longrightarrow$ $\boxed{Z[j] = \int D[\phi]\, \exp(-S[\phi] + j\cdot\phi)}$

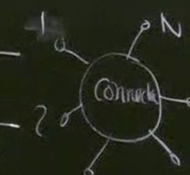
source term

$\frac{\delta}{\delta j(x)}$ or expanding in $j(x) \longrightarrow \langle \phi(x_1) \cdots \phi(x_N) \rangle$

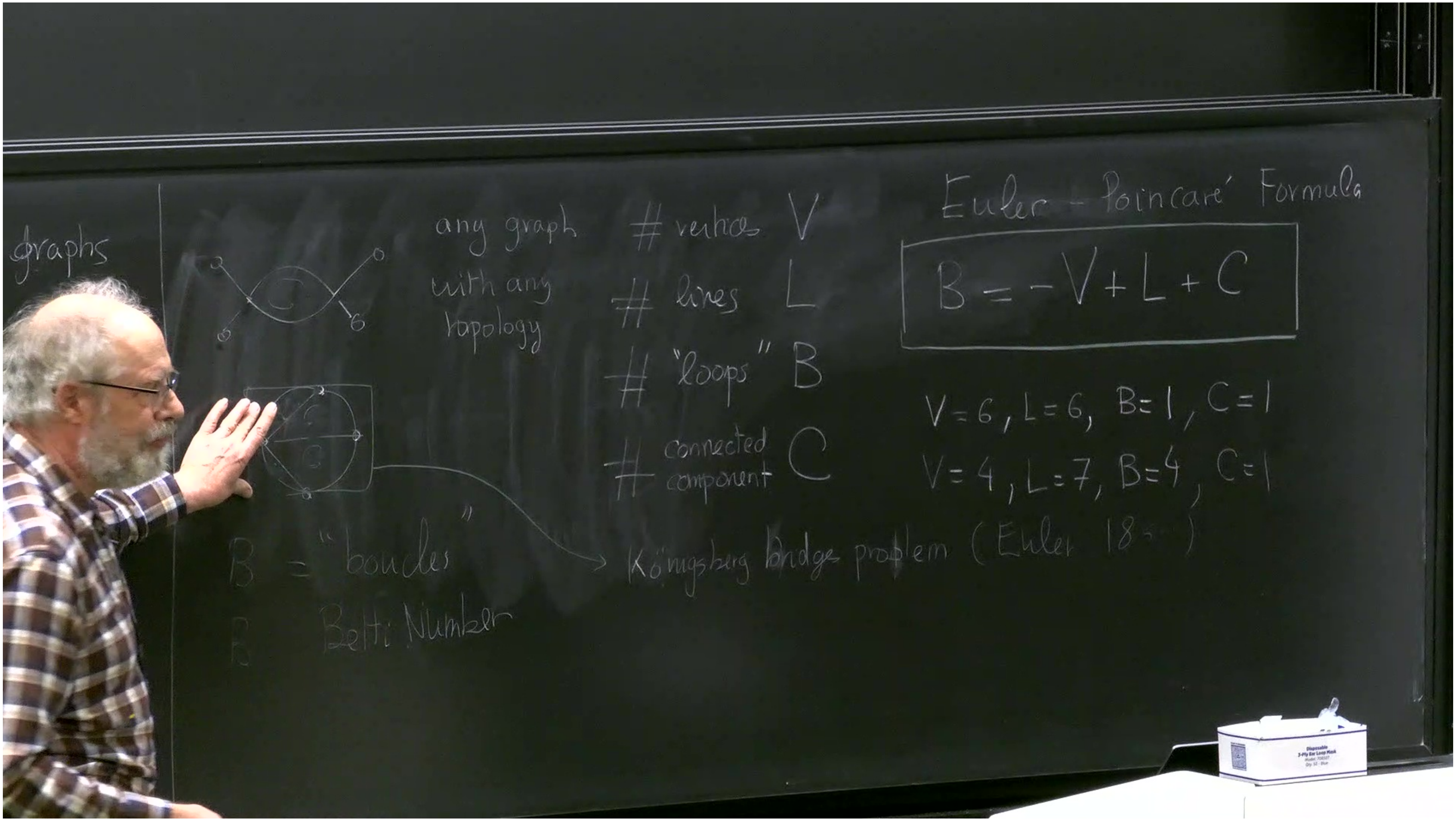
correlation functions

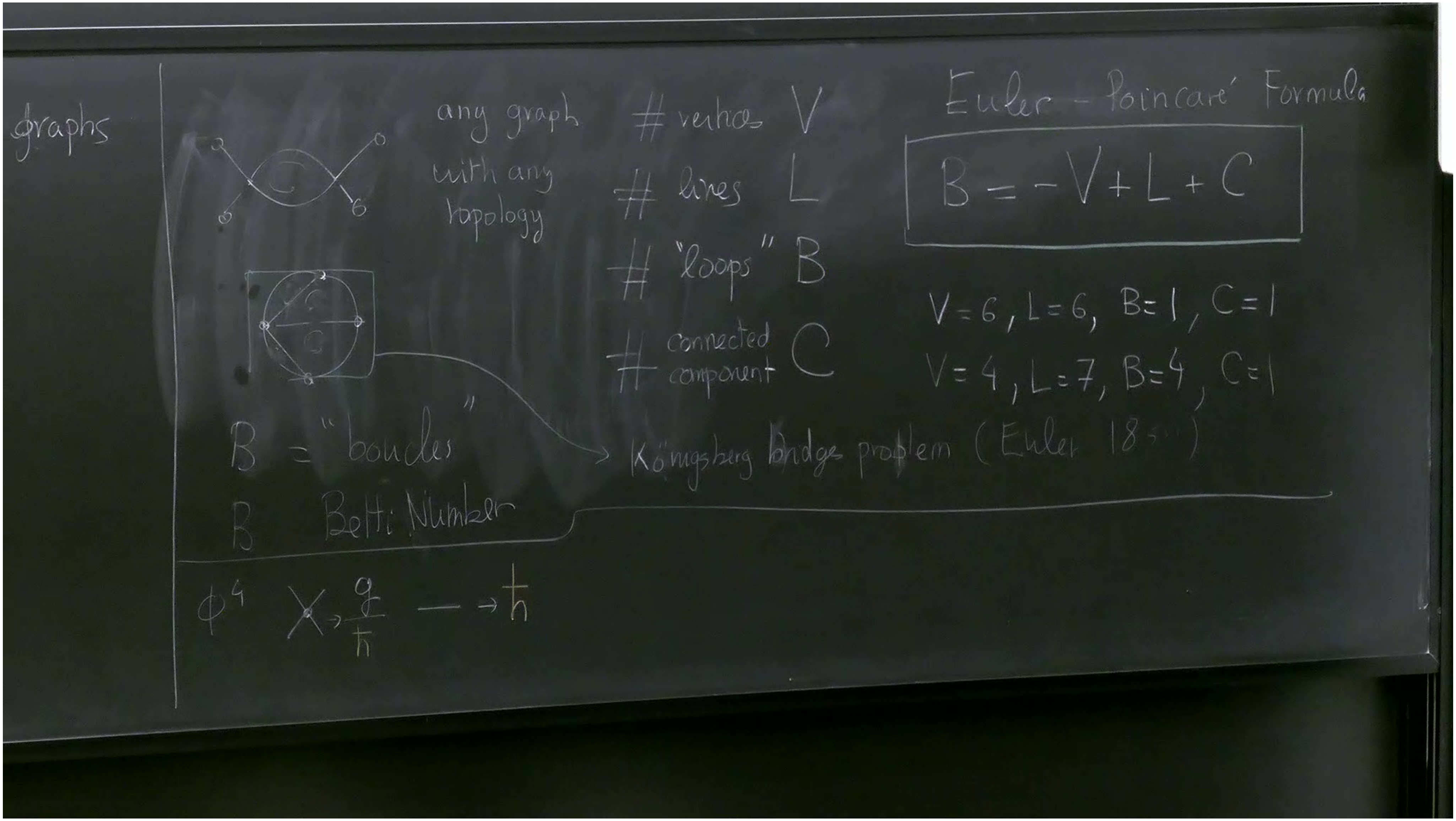
Connected Diagrams

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle_{\text{connected}} = \sum_{g}$$



$\boxed{W[j] = \log(Z[j])}$ connected generating functional


graphs
any graph
with any
topology
vertices V
lines L
"loops" B
connected component C
Euler - Poincaré Formula
B = - V + L + C
V = 6, L = 6, B = 1, C = 1
V = 4, L = 7, B = 4, C = 1
B = "boucles"
B Betti Number
Königsberg bridge problem (Euler 18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...gram $\langle \phi(x_1) \cdots \phi(x_4) \rangle$

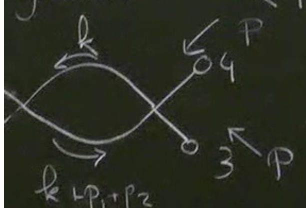


diagram ℬ

→ Feynman Integral   complicated in general (a lot of physics)

$$(\ldots p_4) = \delta(p_1+\cdots+p_4)\,\frac{1}{p_1^2+m^2}\cdots\frac{1}{p_4^2+m^2}$$

$$\int dk_{\text{internal}}\;\frac{1}{k^2+m^2}\,\frac{1}{(k+p_1+p_2)^2+m^2}$$

Diagrams $\longrightarrow$ $\boxed{Z[j] = \int D[\phi]\,\exp\left(-\frac{1}{\hbar}S[\phi]+j\cdot\phi\right)}$

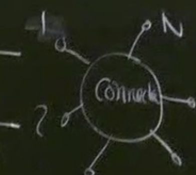
source term

$\frac{\delta}{\delta j(x)}$ or expanding in $j(x)$ $\longrightarrow$ $\langle \phi(x_1)\cdots\phi(x_N)\rangle$

correlation functions

Connected Diagrams

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle_{\text{connected}} = \sum_{g}$$

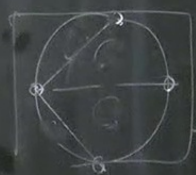


$$\boxed{W[j] = \hbar\,\log(Z[j])}$$

connected generating functional

graphs

any graph with any topology



\# vertices V

\# lines L

\# "loops" B

\# connected component C

Euler-Poincaré Formula

$$\boxed{B = -V + L + C}$$

$V = 6,\ L = 6,\ B = 1,\ C = 1$

$V = 4,\ L = 7,\ B = 4,\ C = 1$

B = "boucles" → Königsberg bridge problem (Euler 18..)

B Betti Number

$\phi^4 \quad \times \to \frac{g}{\hbar} \qquad — \to \hbar \qquad$ Conn comp $\to \hbar$

Graph Feynman of ϕ^4 $\quad \hbar^{L-V+C} = \hbar^{B}\, g^{V_{internal}}$

Z as a formal power series

$\langle 1 \rangle = 1$

$\log(1+*) = * - \frac{1}{2} *^2 + \frac{1}{3} *^3 \dots$

$W[j] = \sum G \Rightarrow \exp\left(\frac{1}{\hbar} W[j]\right) = 1 + \sum G + \frac{1}{2} \sum \underbrace{G \quad G}_{G} + \dots$

$= \sum_{\text{all } G} (-g)^k S_G I_G$

sym factor

integral

$S_{G_1 \cup G_2} = S_{G_1} \cdot S_{G_2}$

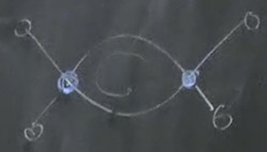
factorization

Irreducible graphs

$G \to G_1 \cup G_2$

ϕ^4

graphs



any graph
with any
topology

vertices V

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Euler – Poincaré Formula

$$\boxed{B = -V + L + C}$$

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ϕ^4 $\quad \times \to \frac{g}{\hbar} \quad — \to \hbar \quad$ Conn. Comp $\to \hbar$

Graph Feynman of ϕ^4

$$\boxed{\hbar^{L-V+C} = \hbar^{B}}\ g^{V_{internal}}$$

Z as a formal power series

Irreducible graphs

$$\langle 1 \rangle = 1$$

$$G \to G_1 \cup G_2$$

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$$W[j] = \sum c \Rightarrow \exp\left(\frac{1}{\hbar} W[j]\right) = 1 + \sum G + \frac{1}{2} \sum \underbrace{G \; G}_{G} + \dots$$

$$\sum_{\substack{\text{all}\\ G}} (-g)^k S_G \, I_G$$

sym factor

integrand

$$S_{G_1 \cup G_2} = S_{G_1} \cdot S_{G_2}$$

factorization

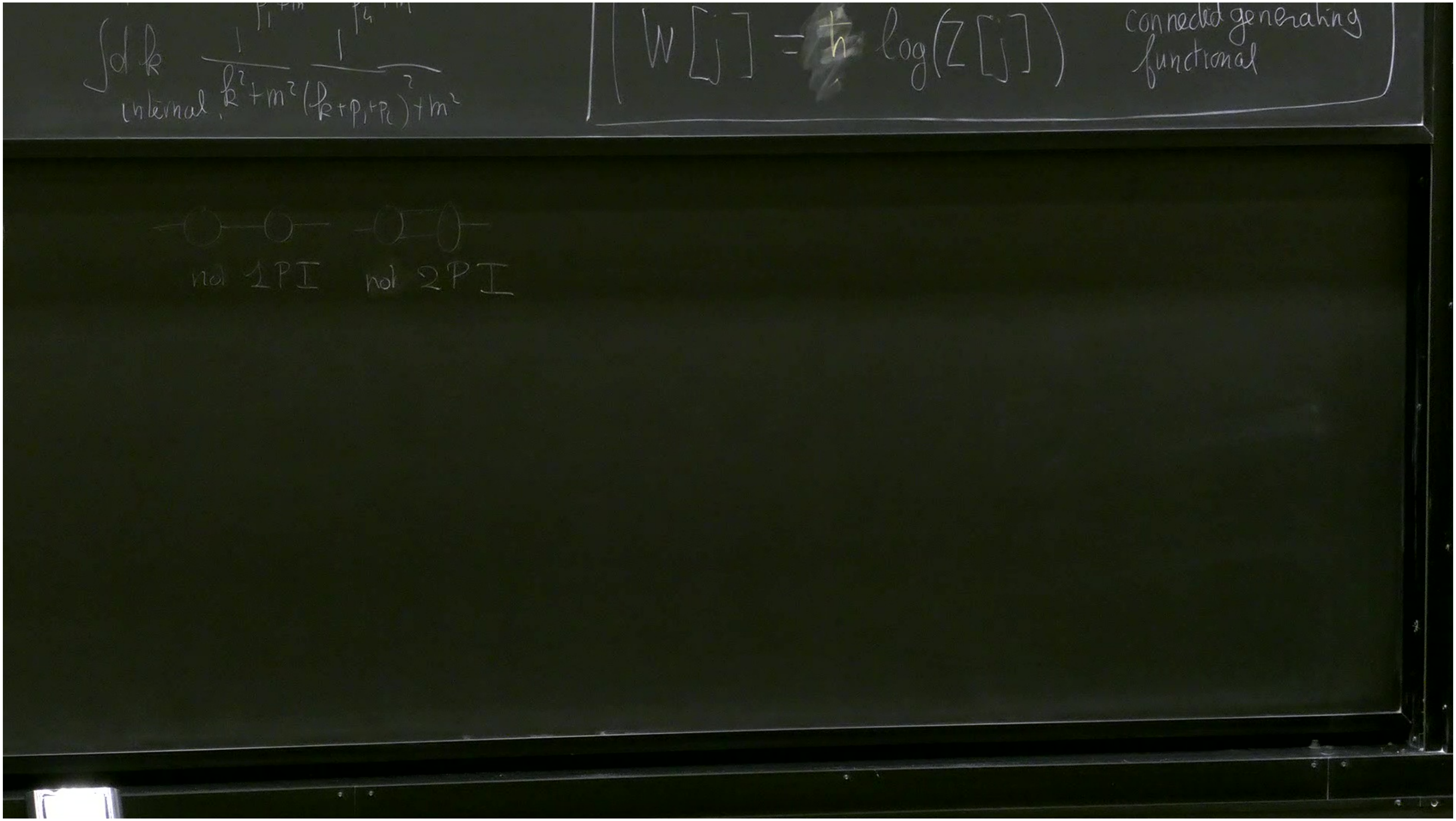
semi-classical

perturbation theory = Loop expansion = $\hbar$ expansion

$k_1 \longleftarrow k_1 \quad \delta(k_1+k_2) \quad \frac{\hbar}{k_1^2+m^2} \to \frac{-i\hbar}{k^2+m^2-i\epsilon_+} \quad \underset{E}{-g} \quad \underset{M}{-ig}$

Generating function for Irreducible Diagrams (one particle irreducible)

1PI


∫dk internal 1/(k²+m²) 1/((k+p₁+p₂)²+m²)
W[j] = ħ log(Z[j])
connected generating functional
not 1PI
not 2PI

$$\int_{\text{internal}} dk \; \frac{1}{k^2+m^2}\,\frac{1}{(k+p_1+p_2)^2+m^2}$$

$$\boxed{W[j] = \hbar \log(Z[j])}$$ connected generating functional

not 1PI not 2PI

HEP also in Cond Matter

$$\int_{\text{internal}} dk \; \frac{1}{k^2+m^2} \, \frac{1}{(k+p_1+p_2)^2+m^2}$$

$$\boxed{W[j] = \hbar \log(Z[j])}$$ connected generating functional

not 1PI not 2PI

HEP also in Cond Matter

$k_1 \rightarrow$ ——— $\leftarrow k_1$ $\quad \delta(k_1+k_2)$ $\quad \frac{1\cdot\hbar}{k_1^2+m^2} \to \frac{-i\hbar}{k^2+m^2-i\epsilon_+}$ $\quad \underset{E}{-g} \quad \underset{M}{-ig}$ $\quad \int dk$ internal k

Generating function for Irreducible Diagrams (one particle irreducible)

Effective action $\Gamma[\varphi]$ $\quad \varphi$ classical field "background field" $\quad$ 1PI

$\downarrow$ Legendre Transform

of $W[j]$ $\quad$ Conn $\rightarrow$ 1PI

ev of ϕ in the "j" vacuum

$$W[j] = \hbar \log Z[j]$$

quantum field

$$\frac{\delta}{\delta j} W[j] = \langle \phi \rangle_j = \frac{\int D[\phi]\, e^{-\frac{1}{\hbar}(S[\phi]+j\phi)}\, \phi}{\int D[\phi] \exp\left(-\frac{1}{\hbar}(S[\phi]-j\phi)\right)}$$

$x_1 \quad x_2 \quad G(x_1, x_2) \quad G_F$

$\vec{k}_1 \longrightarrow \longleftarrow k_1 \quad \delta(k_1+k_2) \dfrac{1 \cdot \hbar}{k_1^2+m^2} \to \dfrac{-i\hbar}{k^2+m^2-i\epsilon_+}$

$\times \; x \quad \dfrac{1}{\hbar} \quad -g \; E \quad -ig \; M$

$I_\beta(p_1 \ldots p_4) = \delta(p_1+\ldots+p_4) \dfrac{1}{p_1^2+m^2} \cdots \dfrac{1}{p_4^2+m^2}$

$\int dk_{\text{internal}} \dfrac{1}{k^2+m^2} \dfrac{1}{(k+p_1+p_2)^2+m^2}$

Generating function for Irreducible Diagrams (one particle irreducible)

not 1PI   not 2PI

Effective action $\Gamma[\varphi]$ φ classical field "background field"   1PI

↓ Legendre Transform

of $W[j]$

Conn → 1PI

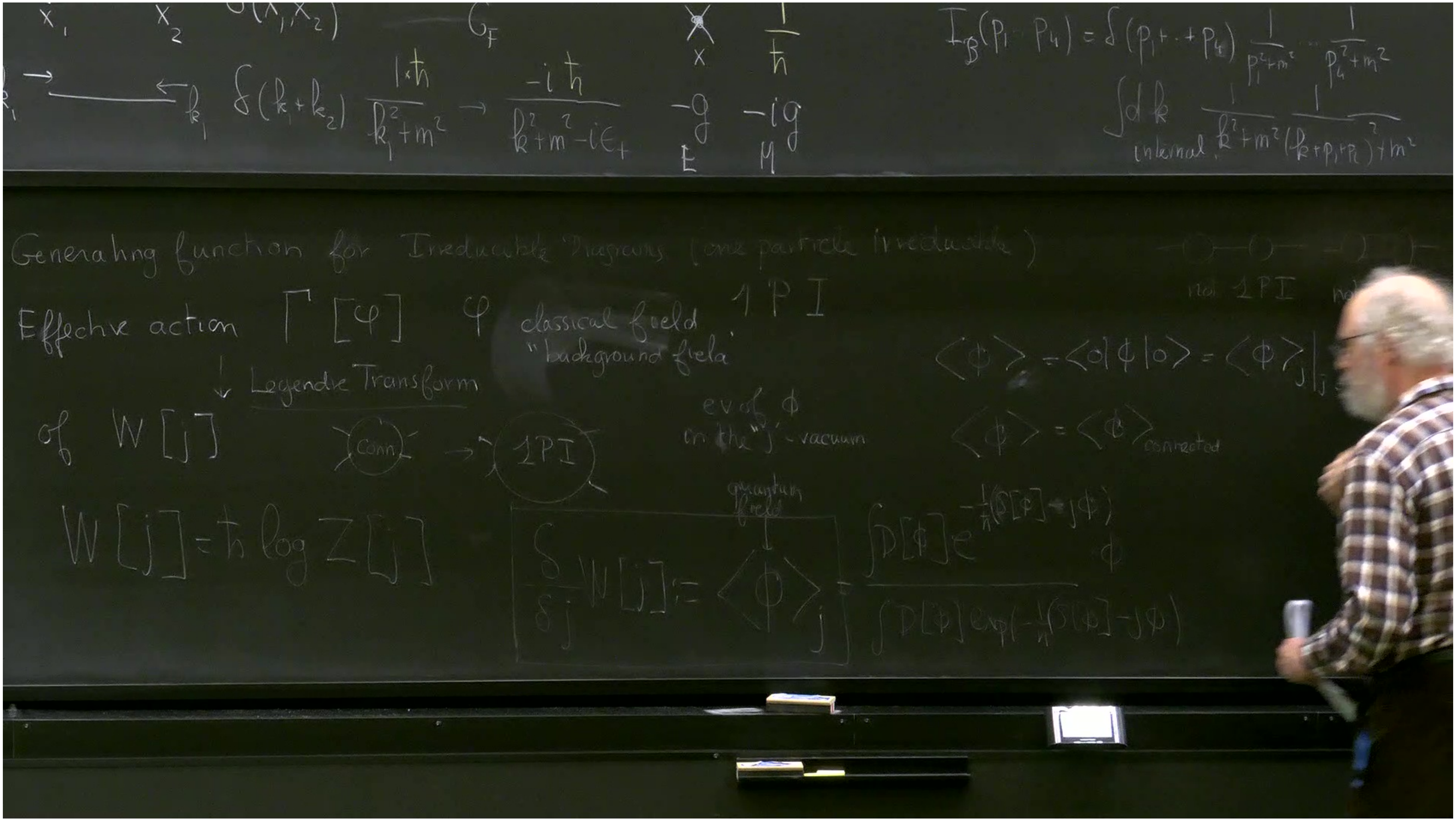
$\langle\phi\rangle = \langle 0|\phi|0\rangle = \langle\phi\rangle_j\Big|_{j=0}$

ev of ϕ in the "j"-vacuum

$W[j] = \hbar \log Z[j]$

quantum field

$$\frac{\delta}{\delta j} W[j] := \langle\phi\rangle_j = \frac{\int D[\phi]\, e^{-\frac{1}{\hbar}(S[\phi]-j\phi)}\,\phi}{\int D[\phi]\exp\left(-\frac{1}{\hbar}(S[\phi]-j\phi)\right)}$$


Generating function for Irreducible Diagrams (one particle irreducible)
not 1PI
Effective action $\Gamma[\varphi]$ φ classical field
"background field"
1PI
Legendre Transform
of $W[j]$
Conn
1PI
ev of ϕ in the "j"-vacuum
$\langle\phi\rangle = \langle 0|\phi|0\rangle = \langle\phi\rangle_j\big|_{j}$
$\langle\phi\rangle = \langle\phi\rangle_{connected}$
$W[j] = \hbar \log Z[j]$
quantum field
$\frac{\delta}{\delta j} W[j] := \langle\phi\rangle_j = \frac{\int D[\phi]\, e^{-\frac{1}{\hbar}(S[\phi] - j\phi)}\,\phi}{\int D[\phi] \exp(-\frac{1}{\hbar}(S[\phi] - j\phi))}$
$\delta(k_1+k_2)\frac{i\hbar}{k_1^2+m^2} \to \frac{-i\hbar}{k^2+m^2-i\epsilon_+}$
$I_B(p_1 \dots p_4) = \delta(p_1+\dots+p_4)\frac{1}{p_1^2+m^2}\cdots\frac{1}{p_4^2+m^2}$
$\int dk_{internal} \frac{1}{k^2+m^2}\frac{1}{(k+p_1+p_2)^2+m^2}$

$$\int_{\text{internal}} dk \, \frac{1}{k^2+m^2} \, \frac{1}{(k+p_1+p_2)^2+m^2}$$

$$\boxed{W[j] = \hbar \log(Z[j])}$$ connected generating functional

3

not 1PI   not 2PI

~~HEP also in Cond Matter~~

$= \langle 0|\phi|0\rangle = \langle \phi \rangle_j \Big|_{j=0}$

$= \langle \phi \rangle_{\text{connected}}$

$] - j\phi)$

ϕ

$\frac{1}{\hbar}(S[\phi] - j\phi)$

For a given j $\varphi(x) = \langle \phi(x) \rangle_j \longleftarrow \{ j(x) \}$

\varphi: classical function   \phi: operator

background field = ev of ϕ in j $\varphi \longleftarrow j$

$\longrightarrow$

⚠   invertible   (in perturbation theory)

$$\int d k \frac{1}{k^2+m^2} \frac{1}{(k+p_1+p_2)^2+m^2}$$

internal

$$\boxed{W[j] = \hbar \log(Z[j])}$$ connected generating functional

not 1PI not 2PI

$$= \langle 0|\phi|0\rangle = \langle \phi \rangle_j \Big|_{j=0}$$

$$= \langle \phi \rangle_{\text{connected}}$$

$$(S[\phi] - j\phi)$$

(3)

HEP also in Cond Matter

For a given j $\quad \varphi(x) = \langle \phi(x) \rangle_j \longleftarrow \{ j(x) \}$

\varphi $\quad$ \phi

classical function $\quad$ operator

background field = ev of ϕ in j $\quad \varphi \longleftarrow j$

$\triangle$ $\quad j \longrightarrow \varphi[j] \xrightarrow{\text{invertible}}$ (in perturbation theory)

$$j[\varphi] \longleftarrow \varphi$$

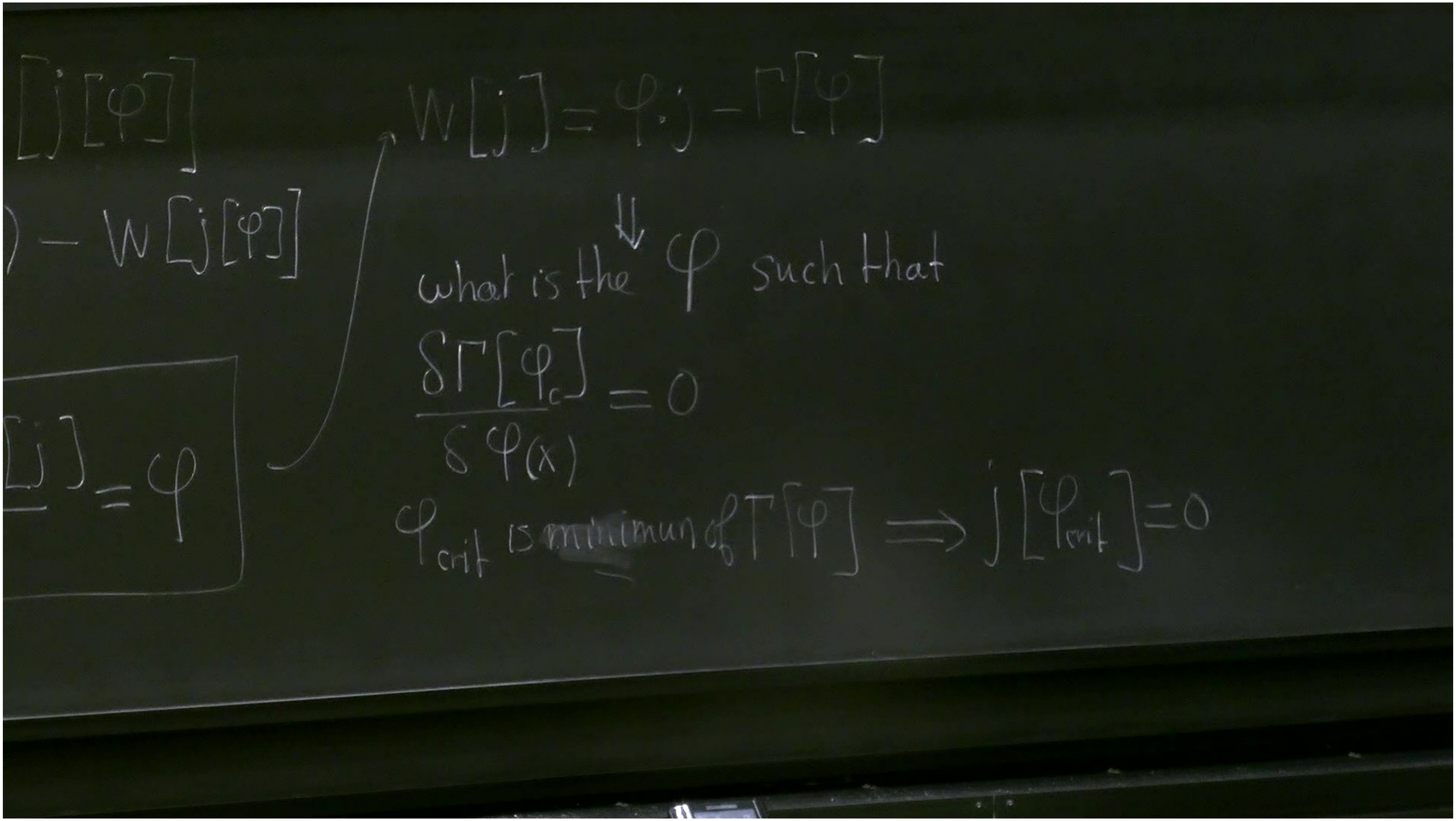
(4) $\Gamma[\varphi] := j[\varphi]\cdot\varphi - W[j[\varphi]]$

$$= \int dx\, j[\varphi](x)\cdot\varphi(x) - W[j[\varphi]]$$

$W[j] = \varphi\cdot j - \Gamma[\varphi]$

Properties: Leg. Trans is involutive

$$\boxed{\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = j[\varphi](x)} \Leftrightarrow \boxed{\frac{\delta W[j]}{\delta j} = \varphi}$$


$W[j] = \varphi \cdot j - \Gamma[\varphi]$
$\Downarrow$
what is the φ such that
$\frac{\delta \Gamma[\varphi_c]}{\delta \varphi(x)} = 0$
φ_{crit} is minimum of $\Gamma[\varphi] \Longrightarrow j[\varphi_{crit}] = 0$

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v.e.v. ϕ in $|0\rangle$

Vacuum of the original theory

$$\underline{\varphi_{crit} \text{ is minimum of } \Gamma[\varphi]} \Longrightarrow j[\varphi_{crit}] = 0 \Rightarrow \varphi_{crit} = \left.\frac{\delta W}{\delta j}\right|_{j=0} = \langle \phi \rangle_{j=0}$$

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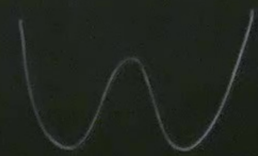
$\phi \to -\phi$ symmetry $\langle \phi \rangle = 0$ $\varphi_{crit} = 0$

But not true $m^2 < 0 \Rightarrow$ Spontaneous Sym. Breaking



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Vacuum of the original theory

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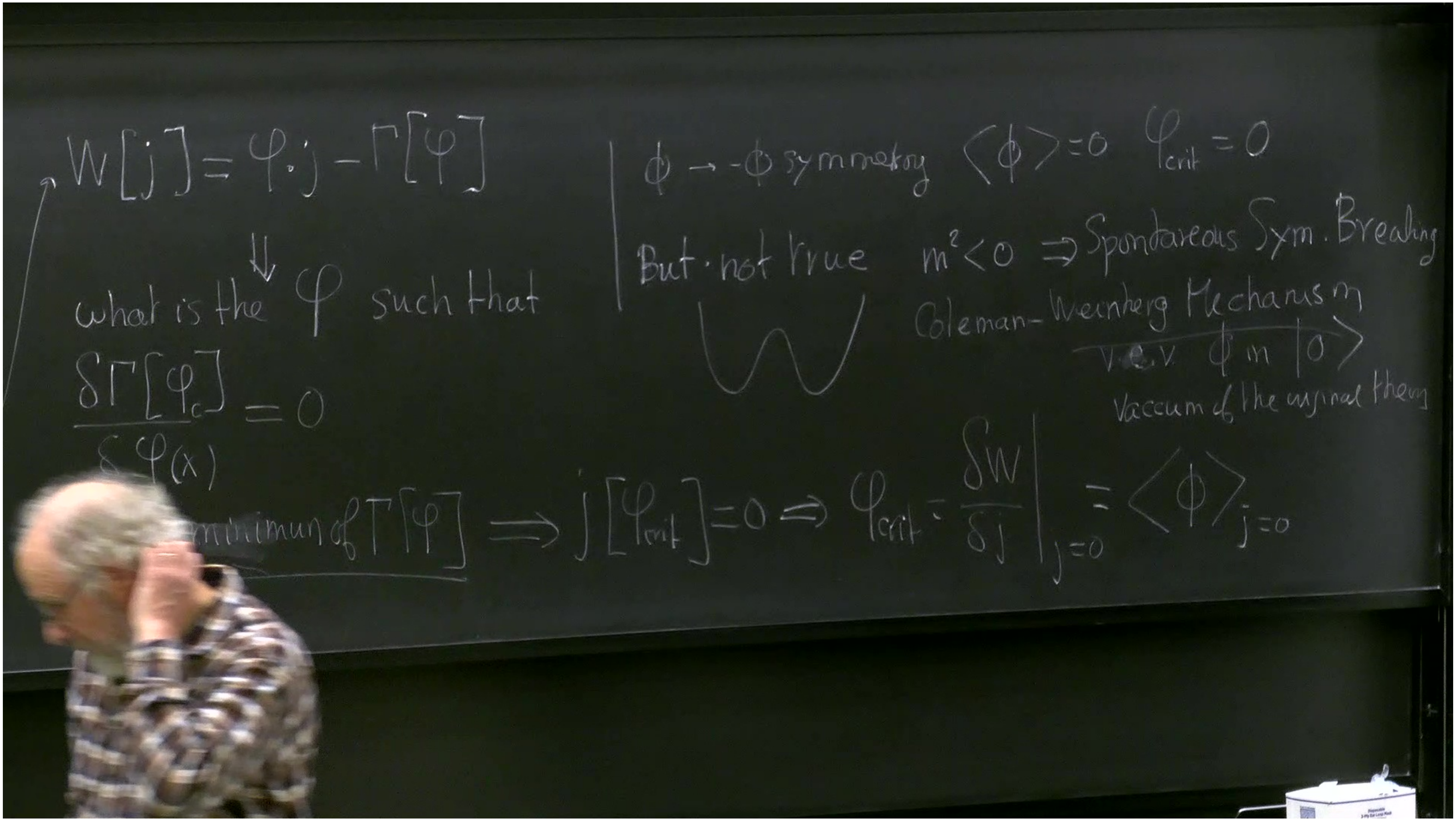
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Coleman-Weinberg Mechanism
v.e.v. ϕ in $|0\rangle$
Vaccum of the original theory

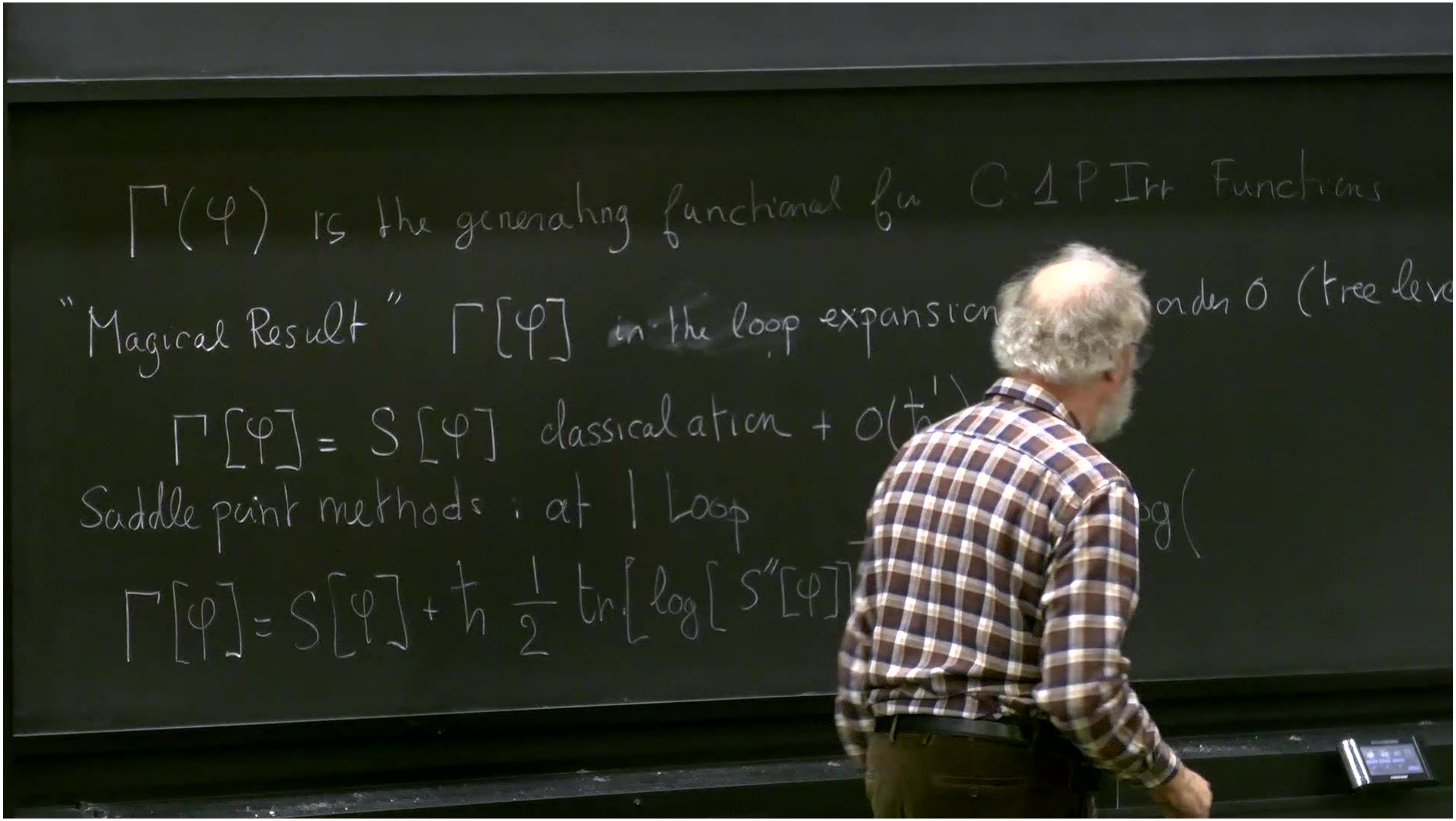
$\Gamma(\varphi)$ is the generating functional for C 1 P Irr Functions

"Magical Result" $\Gamma[\varphi]$ in the loop expansion : at order 0 (tree level)

$$\Gamma[\varphi] = S[\varphi] \text{ classical action} + O(\hbar^1) \text{ terms}$$

Saddle point methods : at 1 Loop

$$\Gamma[\varphi] = S[\varphi] + \hbar \frac{1}{2} \mathrm{tr}\left[\log\left[S''[\varphi]\right]\right] + O(\hbar^2)$$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unctions

0 (tree level)

$$S''[\varphi](x_1,x_2) = \frac{\delta^2 S[\varphi]}{\delta\varphi(x_1)\,\delta\varphi(x_1)}$$

Kernel of an operator acting on function of x

$$\frac{1}{4!}\,4\cdot 3 = \frac{1}{2}$$

$$\psi \longmapsto S''[\varphi]\cdot\psi \qquad \psi(x) \to \int dy\, \frac{\delta^2 S[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)}\,\psi(y)$$

now ϕ^4 what is $S''[\varphi]$?

$$\phi^4 \quad S''[\varphi] = (-\Delta + m^2) + \frac{g}{2}\varphi^2$$

$$\boxed{S''[\varphi]\psi\cdot(x) = (-\Delta_x + m^2)\psi(x) + \frac{g}{2}\,\varphi(x)^2\,\psi(x)}$$

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diff op   local operator

Saddle point methods : at 1 Loop

$$\longrightarrow \mathrm{tr}\cdot \mathrm{Log}\left(\overbrace{-\Delta + m^2} + \frac{g}{2}\varphi^2\right)'$$

$$\Gamma[\varphi] = S[\varphi] + \hbar\,\frac{1}{2}\,\mathrm{tr}\left[\log\left[S''[\varphi]\right]\right] + o(\hbar^2)$$