Title: Lecture - QFT II, PHYS 603

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Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

Date: November 15, 2024 - 9:00 AM

URL: https://pirsa.org/24110011

Correlation functions: Generating functionals

N point function

Euclidean metric (simplicity)

no boundary terms

$$S_E[\phi] = \frac{1}{2}\,\phi\cdot(-\Delta+m^2)\cdot\phi = \int d^dx\,\frac{1}{2}\left((\partial_\mu\phi)^2+m^2\phi^2\right)$$

$$Z = \int D[\phi]\,\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)$$

$$\langle\phi(x_1)\phi(x_2)\rangle = G(x_1,x_2) = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1x_2}$$

Gaussian integration $\Rightarrow$ Wick Theorem functionals

no boundary terms

$\frac{1}{2}\left((\partial_\nu \phi)^2 + m^2 \phi^2\right)$

$)_{x_1 x_2}$

N point function $G_N(x_1 \cdots x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle$

correlation, Green Functions, Wightman F

$$\boxed{Z[j] = \int D[\phi]\, \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)} \leftarrow \text{correlation functions}$$

Functional of j

random variable "quantum field"

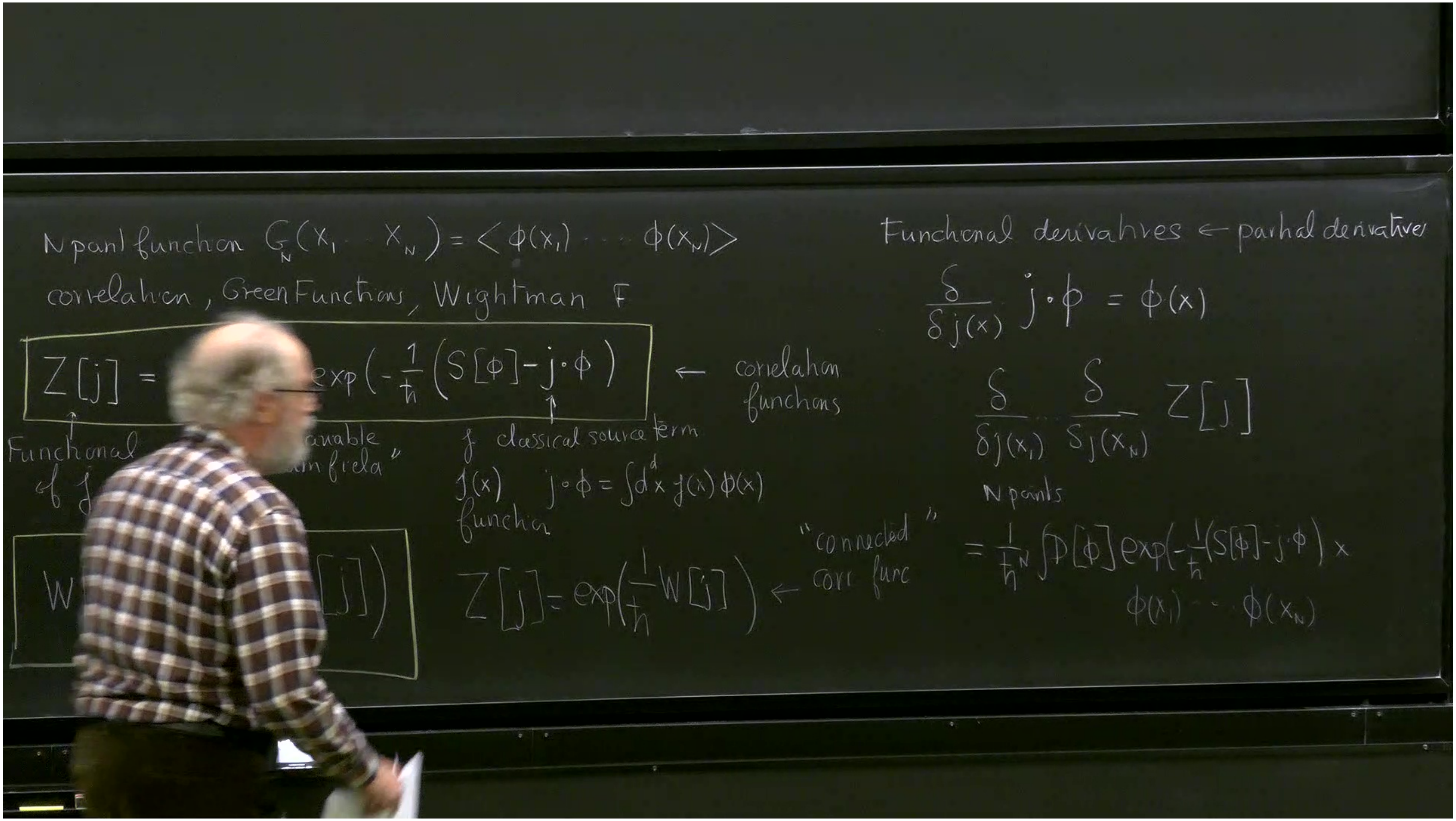
j classical source term

$j(x)$ function

$j\circ\phi = \int d^dx\, j(x)\,\phi(x)$

$$\boxed{W[j] = \hbar\, \mathrm{Log}(Z[j])}$$

$$Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right) \leftarrow \text{"connected" corr func}$$


N point function $G_N(x_1 \dots x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle$
correlation, Green Functions, Wightman F
$Z[j] = \dots \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$
← correlation functions
Functional of j
variable "field"
j classical source term
j(x) function
$j \cdot \phi = \int d^d x \, j(x)\phi(x)$
$Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right)$ ← "connected" corr func
Functional derivatives ← partial derivatives
$\frac{\delta}{\delta j(x)} j\cdot\phi = \phi(x)$
$\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j]$
N points
$= \frac{1}{\hbar^N} \int D[\phi] \exp\left(-\frac{1}{\hbar}(S[\phi] - j\cdot\phi)\right) \times \phi(x_1) \cdots \phi(x_N)$

$$\left.\frac{\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_N)} Z[j]}{Z[0]}\right|_{j=0} = \langle \phi(x_1)\cdots\phi(x_N)\rangle \cdot \hbar^{-N}$$

N point function

Free Field : Saddle point approx (exact)

extremum of $\frac{1}{2}\phi\cdot(-\Delta+m^2)\cdot\phi - j\phi$ $\quad \phi_c[j]$ $\quad (-\Delta+m^2)\phi_c = j$

critical point

$$\phi_c[j] = \frac{1}{-\Delta+m^2}\cdot j$$

$$\phi_c[j](x) = \int d^d y\, G_2(x-y)\, j(y) =$$

x

Diagrammatics (Feynman)

$G_2(x,y) =$ $\quad x \quad y$ $\qquad j(y)$ $\quad y$

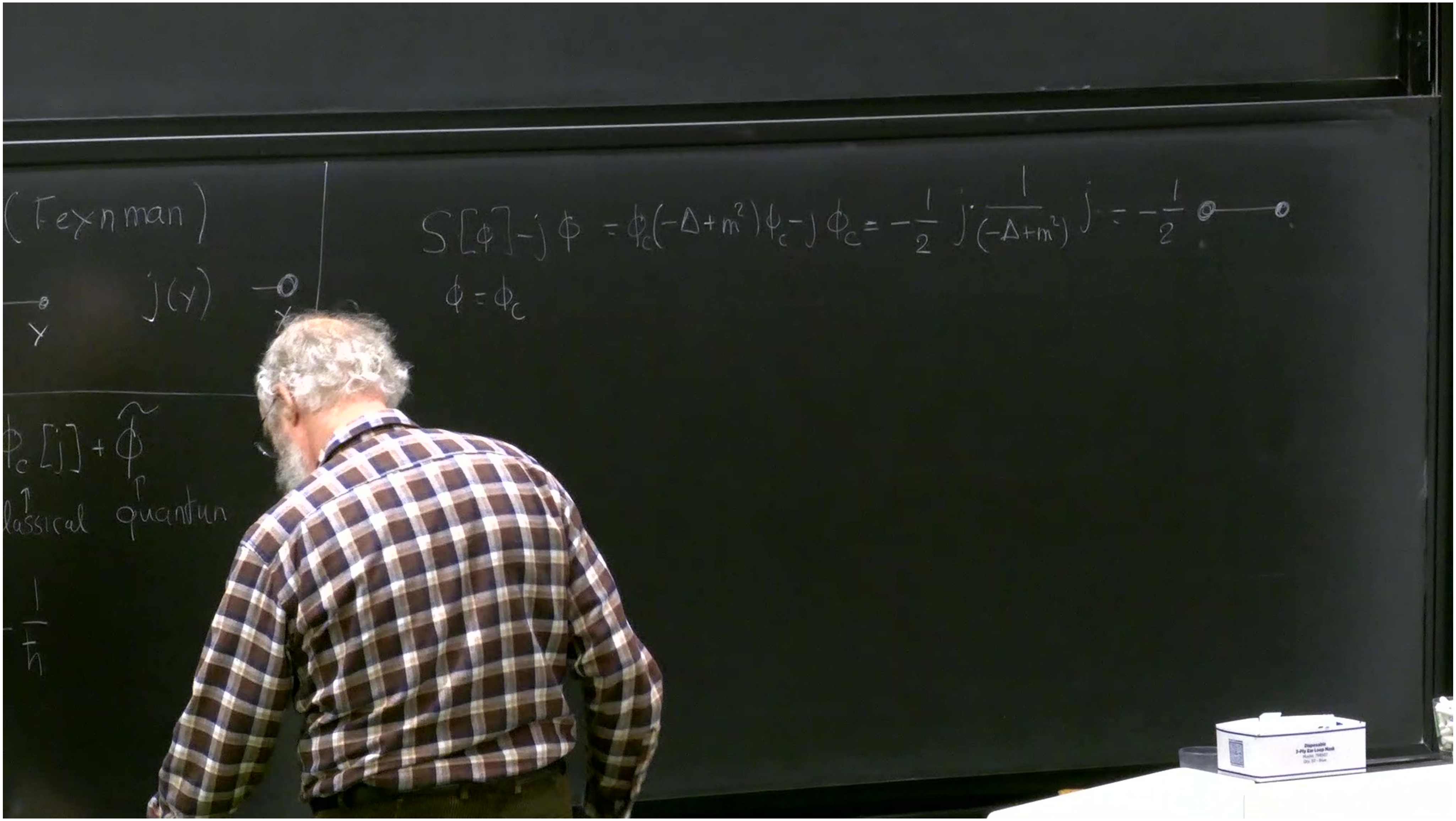
replace $\phi = \phi_c[j] + \tilde{\phi}$

classical quantum

$$Z[j] = \exp\left(-\frac{1}{\hbar}\right.$$

$S[\phi]$

$\phi = \phi_c$


(Feynman)
$S[\phi] - j\phi = \phi_c(-\Delta + m^2)\phi_c - j\phi_c = -\frac{1}{2} j \cdot \frac{1}{(-\Delta+m^2)} j = -\frac{1}{2}$
$\phi = \phi_c$
$j(y)$
$\phi_c[j] + \tilde{\phi}$
classical quantum

function

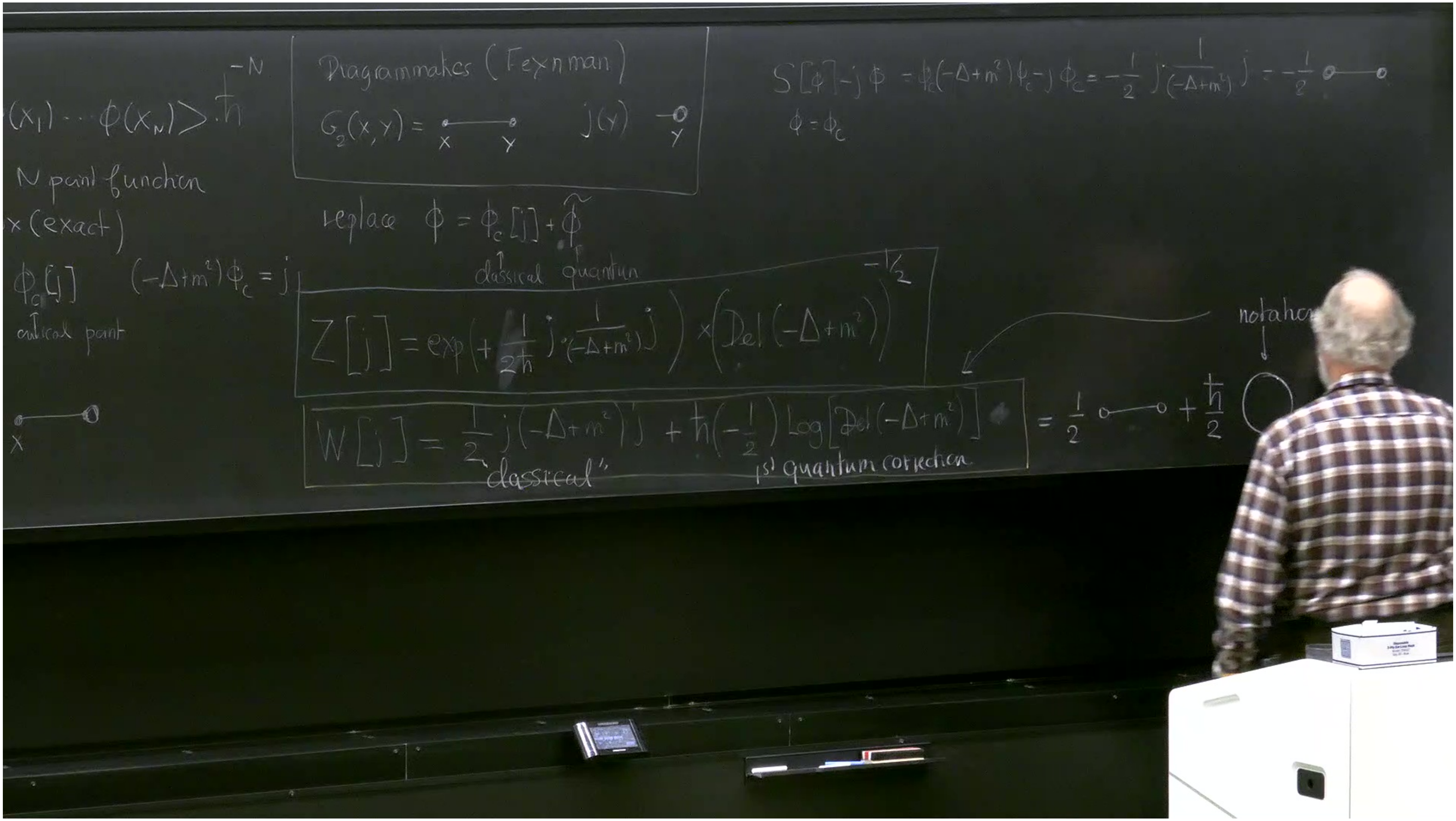
$(-\Delta + m^2)\phi_c = j$

point

replace $\phi = \underset{\uparrow \text{ classical}}{\phi_c[j]} + \underset{\uparrow \text{ quantum}}{\tilde{\phi}}$

$$\boxed{Z[j] = \exp\left(+\frac{1}{2\hbar}\, j \circ \frac{1}{(-\Delta+m^2)}\, j\right) \times \left(\mathrm{Det}\,(-\Delta+m^2)\right)^{-1/2}}$$

$W[j] =$


Diagrammatics (Feynman)
N point function
(exact)
critical point
replace
classical
quantum
"classical"
1st quantum correction
notation

$Y \qquad Y \qquad \phi = \phi_c$

$\phi_c[j] + \tilde{\phi}$

classical quantum

$$\frac{1}{2\hbar}\, j \cdot \frac{1}{(-\Delta+m^2)}\, j \Big) \times \Big(\mathrm{Det}\,(-\Delta+m^2)\Big)^{-1/2}$$

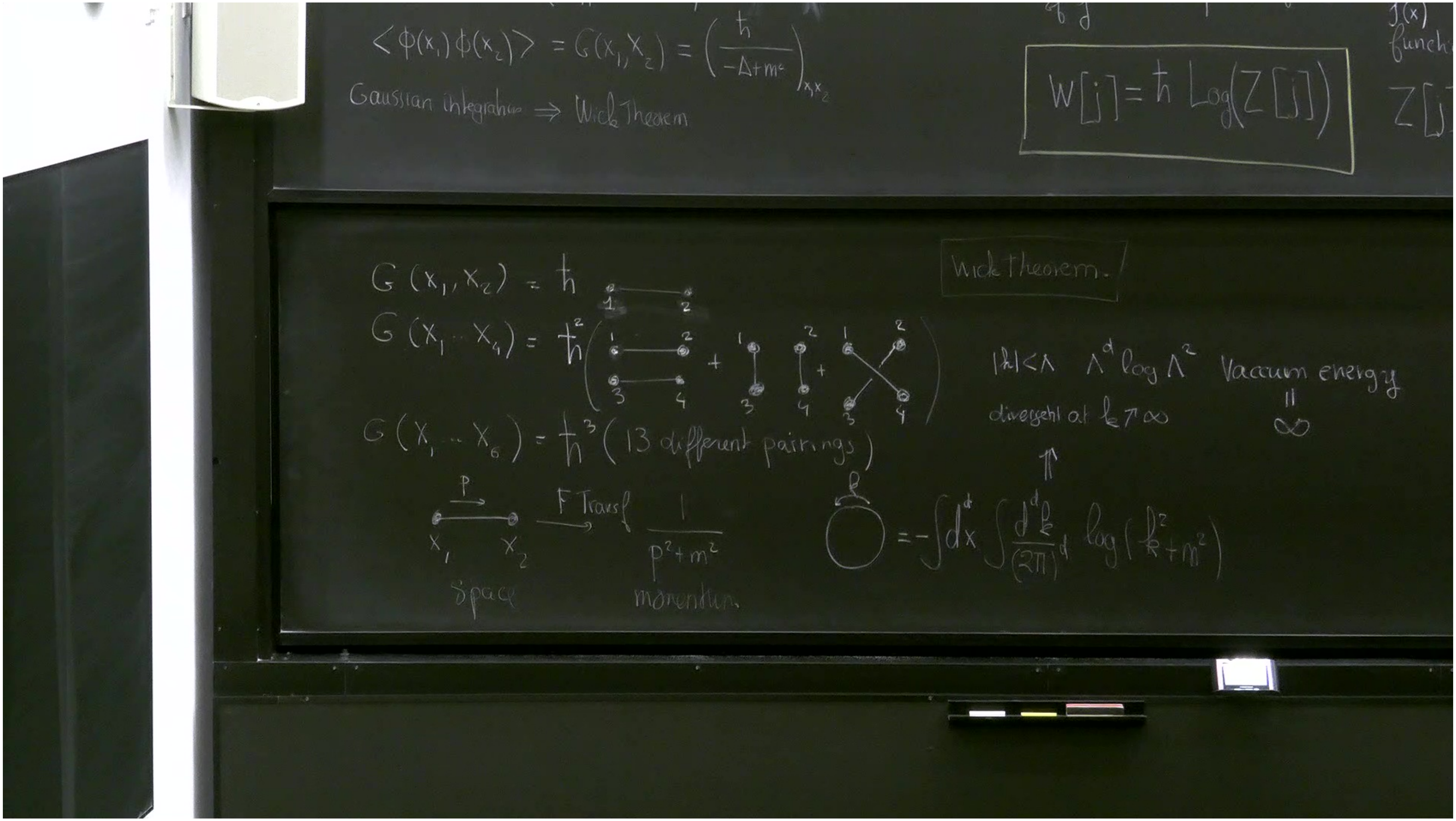
$$j(-\Delta+m^2)^{-1} j + \hbar\left(-\frac{1}{2}\right)\mathrm{Log}\left[\mathrm{Det}\,(-\Delta+m^2)\right] = \frac{1}{2}\ \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\ + \frac{\hbar}{2}\ \bigcirc$$

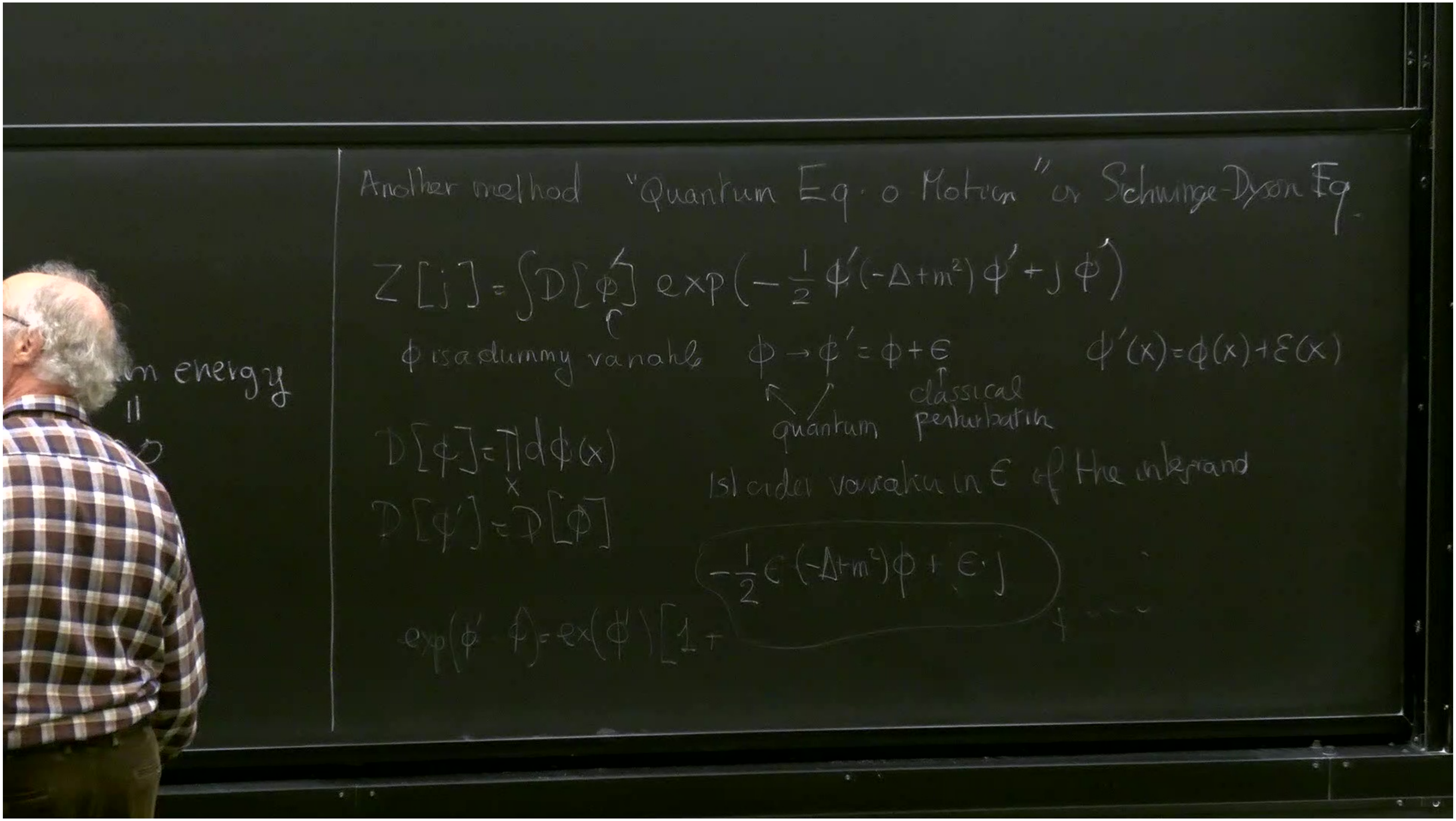
"classical" 1st quantum correction

notation

closed loop with no end point

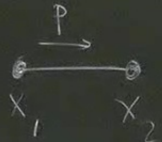
"Ouroboros" diagram.


$\langle \phi(x_1)\phi(x_2)\rangle = G(x_1,x_2) = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1x_2}$
Gaussian integration ⇒ Wick Theorem
$W[j] = \hbar \,\mathrm{Log}(Z[j])$
Wick Theorem.
$G(x_1,x_2) = \hbar$
$G(x_1 \ldots x_4) = \hbar^2(\;+\;+\;)$
$G(x_1 \ldots x_6) = \hbar^3$ (13 different pairings)
$|k|<\Lambda$ $\Lambda^d \log \Lambda^2$ Vacuum energy $= \infty$
divergent at $k \to \infty$
F Transf $\frac{1}{p^2+m^2}$
space
momentum
$= -\int d^dx \int \frac{d^dk}{(2\pi)^d} \log(k^2+m^2)$



m energy
Another method "Quantum Eq. o Motion" or Schwinger-Dyson Eq.
$Z[j] = \int D[\phi] \exp\left(-\frac{1}{2}\phi'(-\Delta+m^2)\phi' + j\phi'\right)$
ϕ is a dummy variable $\phi \to \phi' = \phi + \epsilon$
$\phi'(x) = \phi(x) + \epsilon(x)$
classical perturbation
quantum
$D[\phi] = \prod_x d\phi(x)$
$D[\phi'] = D[\phi]$
1st order variation in ϵ of the integrand
$-\frac{1}{2}\epsilon(-\Delta+m^2)\phi + \epsilon\cdot j$
$\exp(\phi'\ \phi) = \exp(\phi)[1 +$
$+ \ldots]$

$G(x_1 \ldots x_6) = \hbar^3$ (13 different pairings)



$\xrightarrow{\text{F Transf}} \frac{1}{p^2+m^2}$

space   momentum

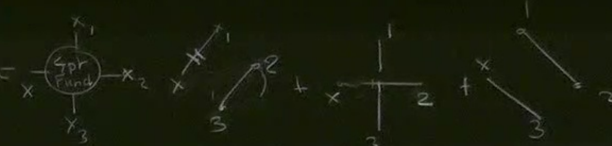
$= -\int d^d x \int \frac{d^d k}{(2\pi)^d} \log(k^2+m^2)$

$D[\phi] = \prod_x d\phi(x)$

$D[\phi'] = D[\phi]$

For any $\epsilon(x)$ $\epsilon(x) \boxed{\langle (-\Delta+m^2)\phi + j \rangle_{j\neq 0}} = 0$

$(-\Delta_x + m^2) =$

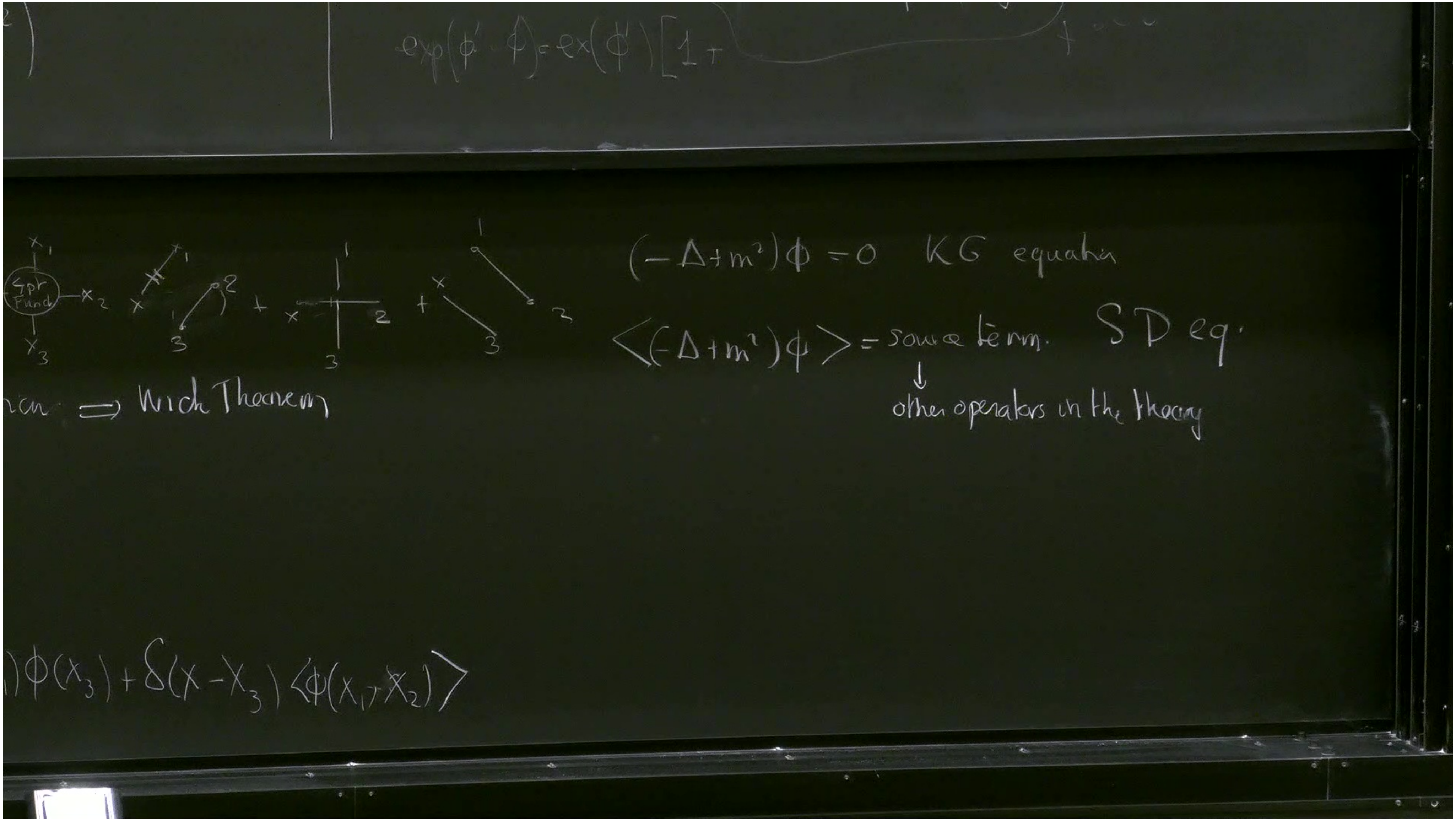


$(-\Delta_x+m^2)\int D[\phi]\, e^{-S+j\phi}\,\phi(x) = \int D[\phi]\, e^{-S+j\phi}\, j(x)$ → SD equation $\Rightarrow$ Wick Theorem

Application $N=2$

$(-\Delta_x+m^2)\langle\phi(x)\phi(x_1)\rangle = \delta(x-x_1) \Rightarrow$ propagator

$(-\Delta_x+m^2)\langle\phi(x)\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \delta(x-x_1)\langle\phi(x_2)\phi(x_3)\rangle + \delta(x-x_2)\langle\phi(x_1)\phi(x_3)\rangle + \delta(x-x_3)\langle\phi(x_1)\phi(x_2)\rangle$



$(-\Delta+m^2)\phi=0$ KG equation
$\langle(-\Delta+m^2)\phi\rangle=$ source term. SD eq.
other operators in the theory
$\Rightarrow$ Wick Theorem
$\phi(x_3)+\delta(x-x_3)\langle\phi(x_1,x_2)\rangle$

From Free Scalar Field to Interactions

Euclidean

$$S_E[\phi] = \int d^d x \, \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4(x) \quad \text{term}$$

symmetry reason

coupling constant

Interaction term

ϕ^4 theory

free term

$$+ \frac{h}{3!}\phi^3(x)$$

perturbative expansion in powers of g c.c

apply Wick theorem

1 particle mass m

contact repulsive interacting

strength g

$2 \to 2$ process

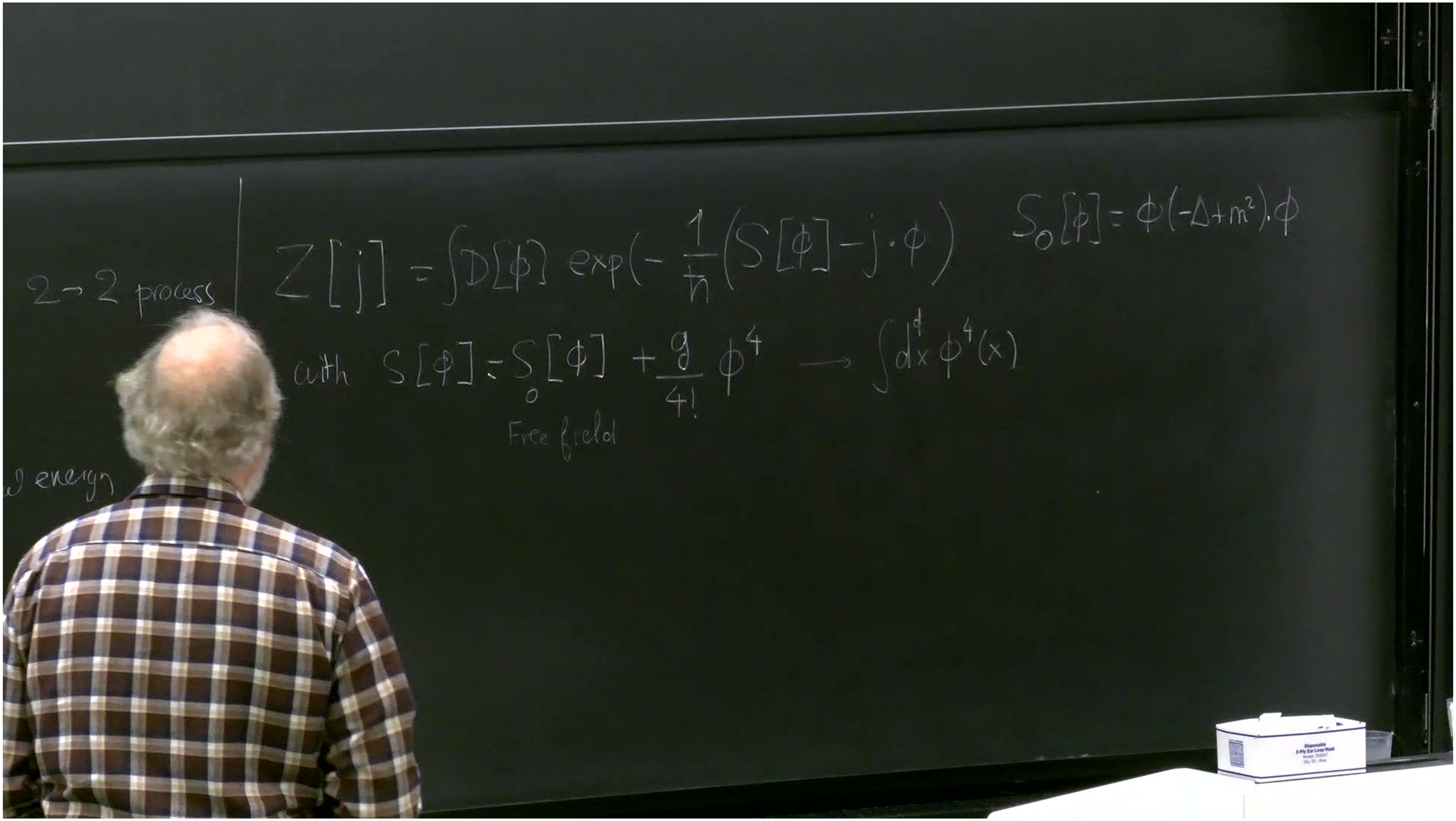
attractive $\to$ Bose Einstein Cond

no $|0\rangle$ with finite classical energy

$2 \to 1$ process

statistical mechanics

dynamical processes


2 → 2 process
$Z[j] = \int D[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$
$S_0[\phi] = \phi(-\Delta + m^2).\phi$
with $S[\phi] = S_0[\phi] + \frac{g}{4!}\phi^4 \longrightarrow \int d^dx\, \phi^4(x)$
Free field

...l energy

with $S[\phi] = \underset{\text{Free field}}{S_0[\phi]} + \frac{g}{4!}\phi^4 \longrightarrow \int d^dx\, \phi^4(x)$

$$\text{PT} = Z[j] = \int D[\phi]\, \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \exp\left(-\frac{1}{\hbar}\frac{g}{4!}\phi^4\right) \exp\left(+\frac{1}{\hbar} j\cdot\phi\right)$$

expand them in g

$$\exp\left(-\frac{1}{\hbar}\frac{g}{4!}\phi^4\right) = \sum_{K=0}^{\infty} \frac{1}{K!}\left(-\frac{1}{\hbar}\right)^K \left(\frac{g}{4!}\right)^K \underbrace{\phi^4 \cdots \phi^4}_{K}$$

K integer ≥ 0

$$\underbrace{\phi^4 \cdots \phi^4}_{K} = \int d^dx_1 \cdots d^dx_K\, \phi^4(x_1) \cdots \phi^4(x_K)$$

$\frac{\ \ }{3!}\phi(x)$

perturbative expansion in powers of g i.e.
apply Wick theorem

statistical mechanics
dynamical processes

$$\exp\left(-\frac{1}{\hbar}\frac{g}{4!}\phi^4\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{\hbar}\right)^k\left(\frac{g}{4!}\right)^k$$

$$\underbrace{\phi^4 \cdots \phi^4}_{k} = \int d^d x_1 \; d^d x_k \; \phi^4(x_1) \cdots \phi$$

commute $\int D[\phi]$ with $\sum_k$

$$Z[j] = \sum_{k=0}^{\infty} \int D[\phi] \exp\left(-\frac{1}{\hbar}S_0[\phi]\right)\exp\left(+\frac{1}{\hbar}j\cdot\phi\right)\cdot\frac{1}{k!}\left(-\frac{1}{\hbar}\right)^k\left(\frac{g}{4!}\right)^k \underbrace{\phi^4 \quad \phi^4}_{k}$$

by taking $\frac{\delta}{\delta j}$

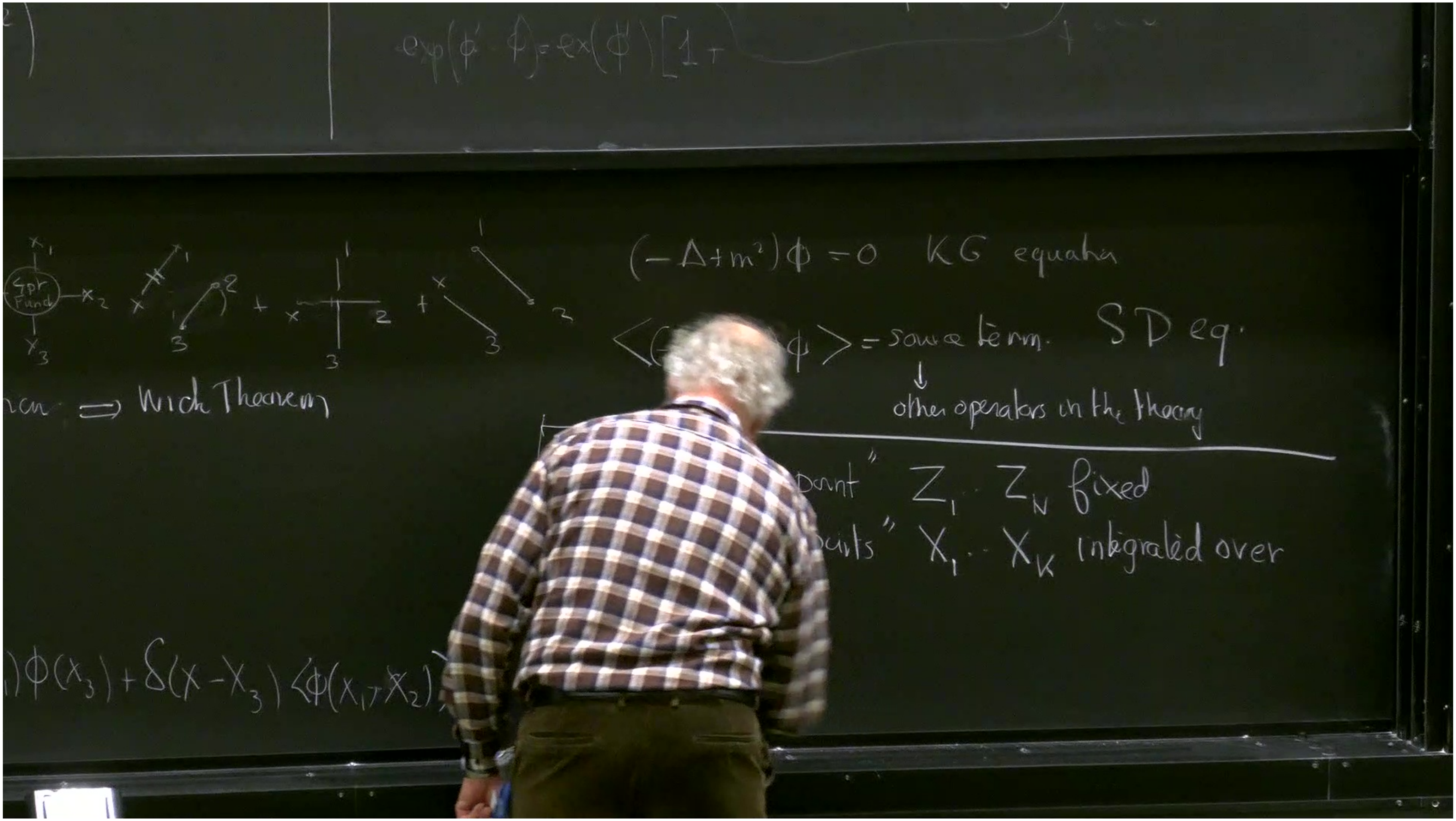
$$\prod_{i=1}^{N}\frac{\delta}{\delta j(z_i)} Z[j]\Big|_{j=0} / Z[0] = \boxed{\hbar^{-N}\langle\phi(z_1)\cdots\phi(z_N)\rangle_{\text{interacting theory}}}$$

as for Free theory expand $\sum_k$ both

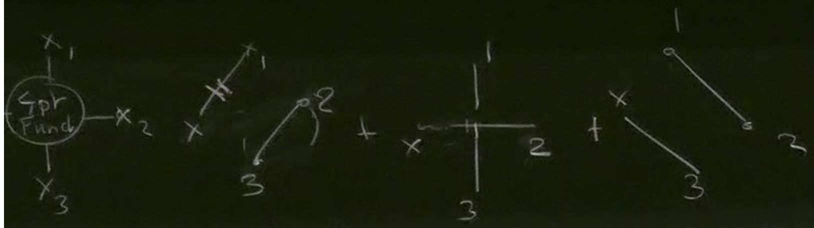
$$\boxed{= \frac{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-g}{\hbar 4!}\right)^k\int d^d x_1 \; d^d x_k \langle\phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_N)\rangle_{\text{Free theory}}}{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-g}{\hbar 4!}\right)^k\int d^d x_1 \cdots d^d x_k \langle\phi^4(x_1)\cdots\phi^4(x_N)\rangle}}$$

as series in g^k

$$\langle\phi\cdots\phi\,\phi^4\cdots\phi^4\rangle_{\text{Free theory}}$$


(−Δ+m²)φ = 0 KG equation
> = source term. SD eq.
other operators in the theory
point" Z1 .. ZN fixed
points" X1 .. XK integrated over
⟹ Wick Theorem

$\exp(\phi' \cdot \phi) = \exp(\phi')\,[1 + \ldots$



...ion $\Longrightarrow$ Wick Theorem

$\ldots)\,\phi(x_3) + \delta(x - x_3)\,\langle \phi(x_1, x_2) \rangle$

$(-\Delta + m^2)\phi = 0$ KG equation

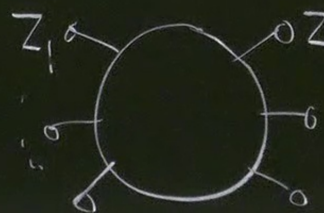
$\langle (-\Delta + m^2)\phi \rangle$ = source term. SD eq.

$\downarrow$

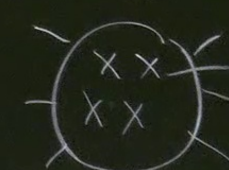
other operators in the theory

N "external point" $Z_1 \dots Z_N$ fixed

K "internal points" $X_1 \dots X_K$ integrated over



$= \sum_H$



$\leftarrow$ K internal ϕ^4 vertices