Title: Lecture - QFT II, PHYS 603

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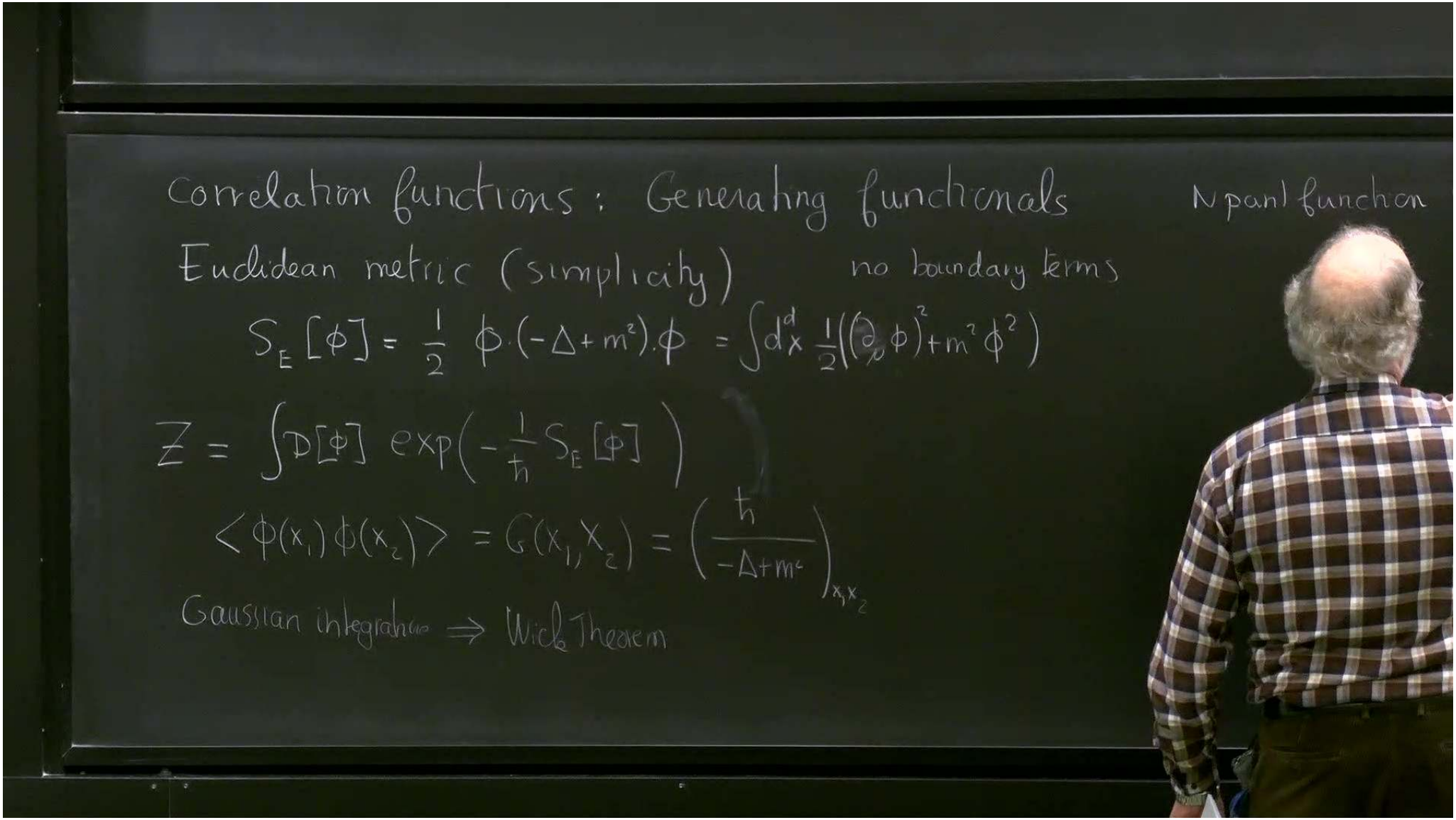
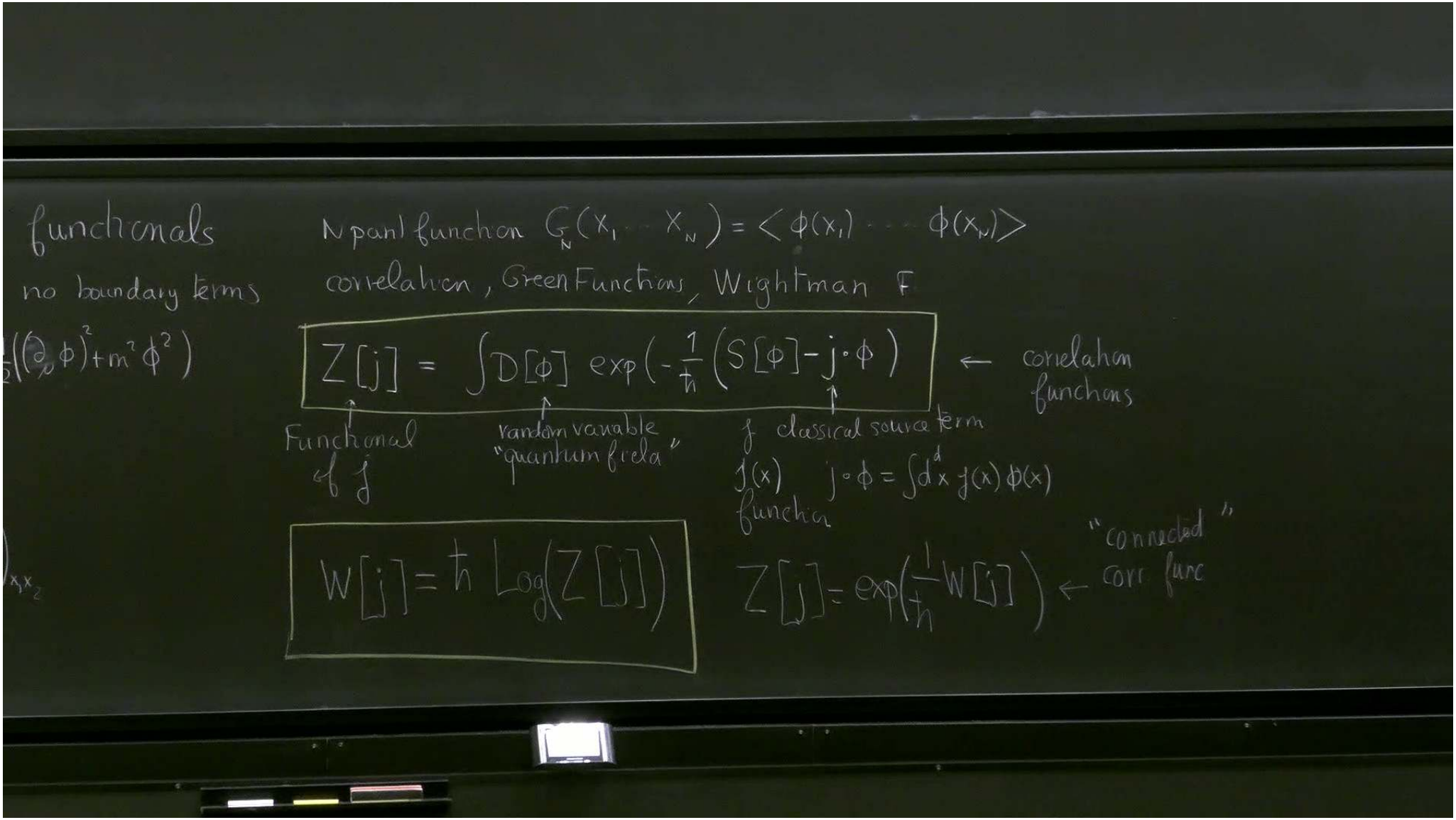
Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

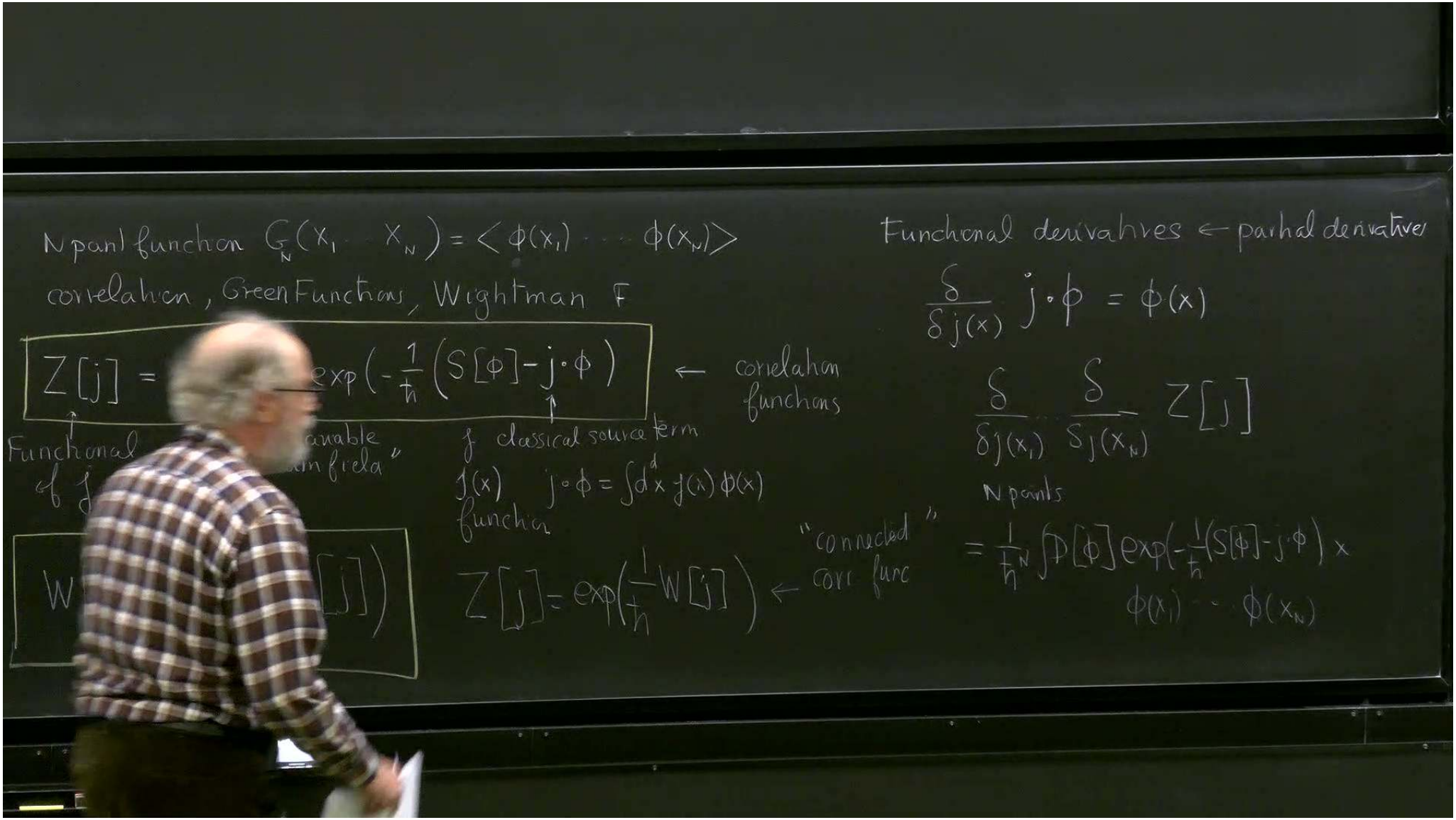
Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

Date: November 15, 2024 - 9:00 AM

URL: https://pirsa.org/24110011


Correlation functions : Generating functionals
N point function
Euclidean metric (simplicity)
no boundary terms
$S_E[\phi] = \frac{1}{2}\,\phi\cdot(-\Delta+m^2)\cdot\phi = \int d^dx\,\frac{1}{2}\left((\partial_\mu\phi)^2+m^2\phi^2\right)$
$Z = \int \mathcal{D}[\phi]\,\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)$
$\langle\phi(x_1)\phi(x_2)\rangle = G(x_1,x_2) = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1,x_2}$
Gaussian integration $\Rightarrow$ Wick Theorem


functionals
no boundary terms
$\frac{1}{2}\left((\partial_\mu\phi)^2+m^2\phi^2\right)$
N point function $G_N(x_1 \cdots x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle$
correlation, Green Functions, Wightman F
$Z[j] = \int D[\phi]\, \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$
← correlation functions
Functional of j
random variable "quantum field"
j classical source term
$j(x)$ function
$j\cdot\phi = \int d^d x\, j(x)\phi(x)$
$W[j] = \hbar\, \mathrm{Log}(Z[j])$
$Z[j] = \exp\left(\frac{1}{\hbar}W[j]\right)$
← "connected" corr func


N point function $G_N(x_1 \cdots x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle$
correlation, Green Functions, Wightman F
$Z[j] = \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$ ← correlation functions
Functional of j
j classical source term
$j(x)$ function
$j\cdot\phi = \int d^d x\, j(x)\phi(x)$
$Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right)$ ← "connected" corr func
Functional derivatives ← partial derivatives
$\frac{\delta}{\delta j(x)}\, j\cdot\phi = \phi(x)$
$\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j]$
N points
$= \frac{1}{\hbar^N} \int \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar}(S[\phi] - j\cdot\phi)\right) \times$
$\phi(x_1) \cdots \phi(x_N)$

$$\frac{\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j]}{Z[0]}\Bigg|_{j=0} = \langle \phi(x_1) \cdots \phi(x_N) \rangle \cdot \hbar^{-N}$$

N point function

Diagrammatics (Feynman)

$G_2(x,y) =$ x — y $j(y)$ — y

Free Field : Saddle point approx (exact)

extremum of $\frac{1}{2}\phi \cdot (-\Delta + m^2) \cdot \phi - j\phi$ $\quad \phi_c[j]$ $\quad (-\Delta + m^2)\phi_c = j$

critical point

$$\phi_c[j] = \frac{1}{-\Delta + m^2} \cdot j$$

$$\phi_c[j](x) = \int d^d y \, G_2(x-y) j(y) = \; x \text{ —— } 0$$

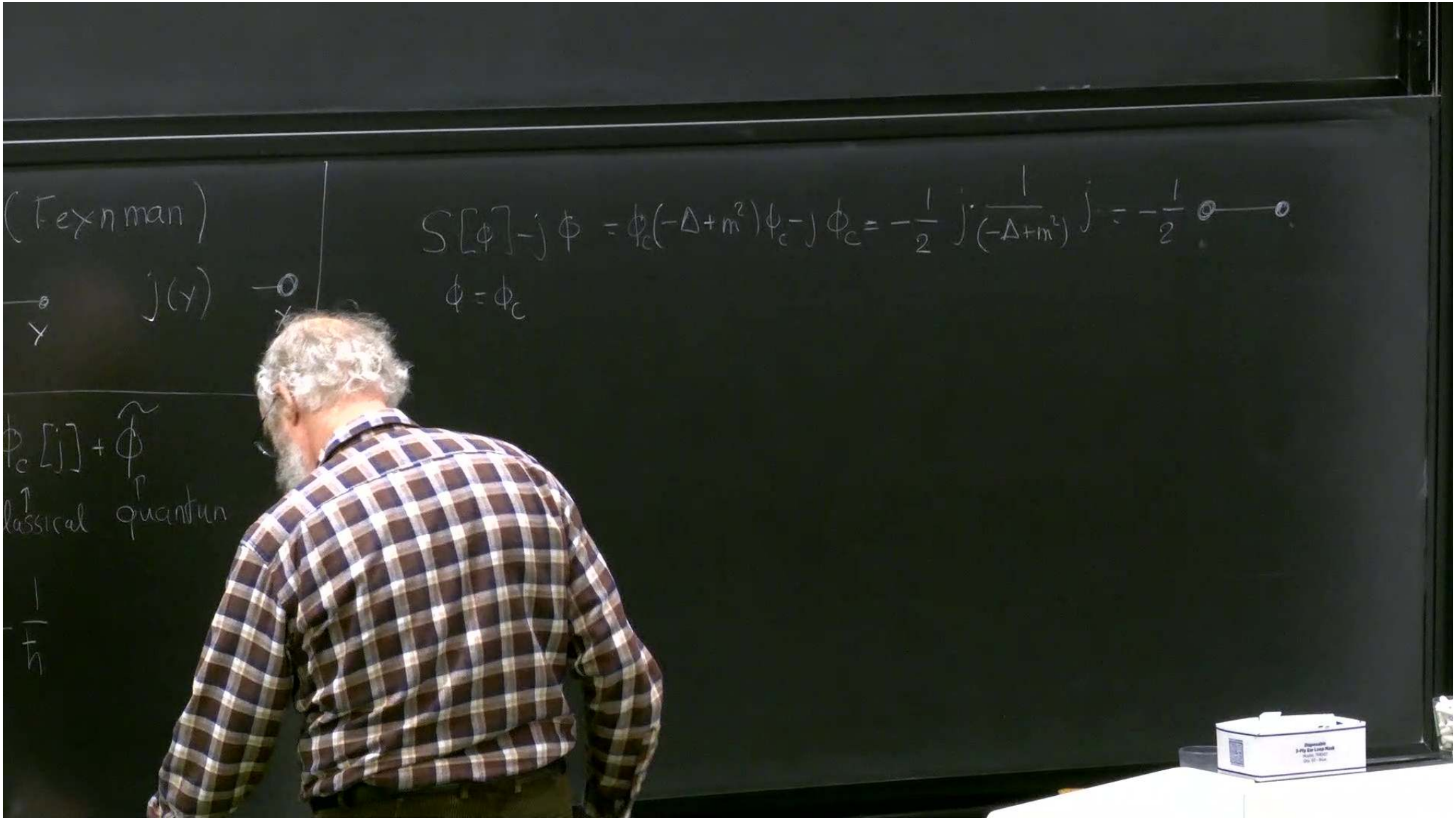
replace $\phi = \phi_c[j] + \tilde{\phi}$

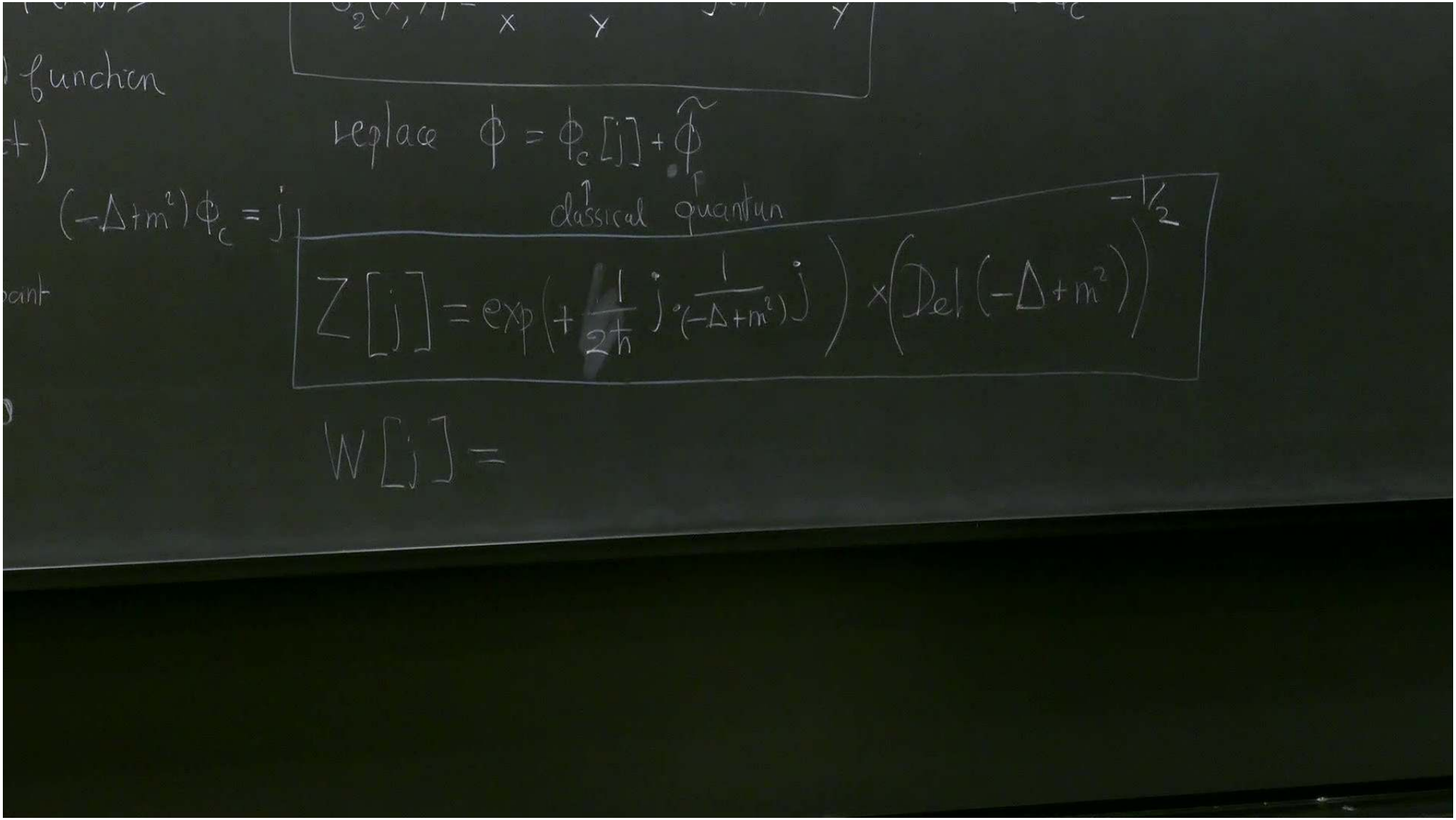
classical quantum

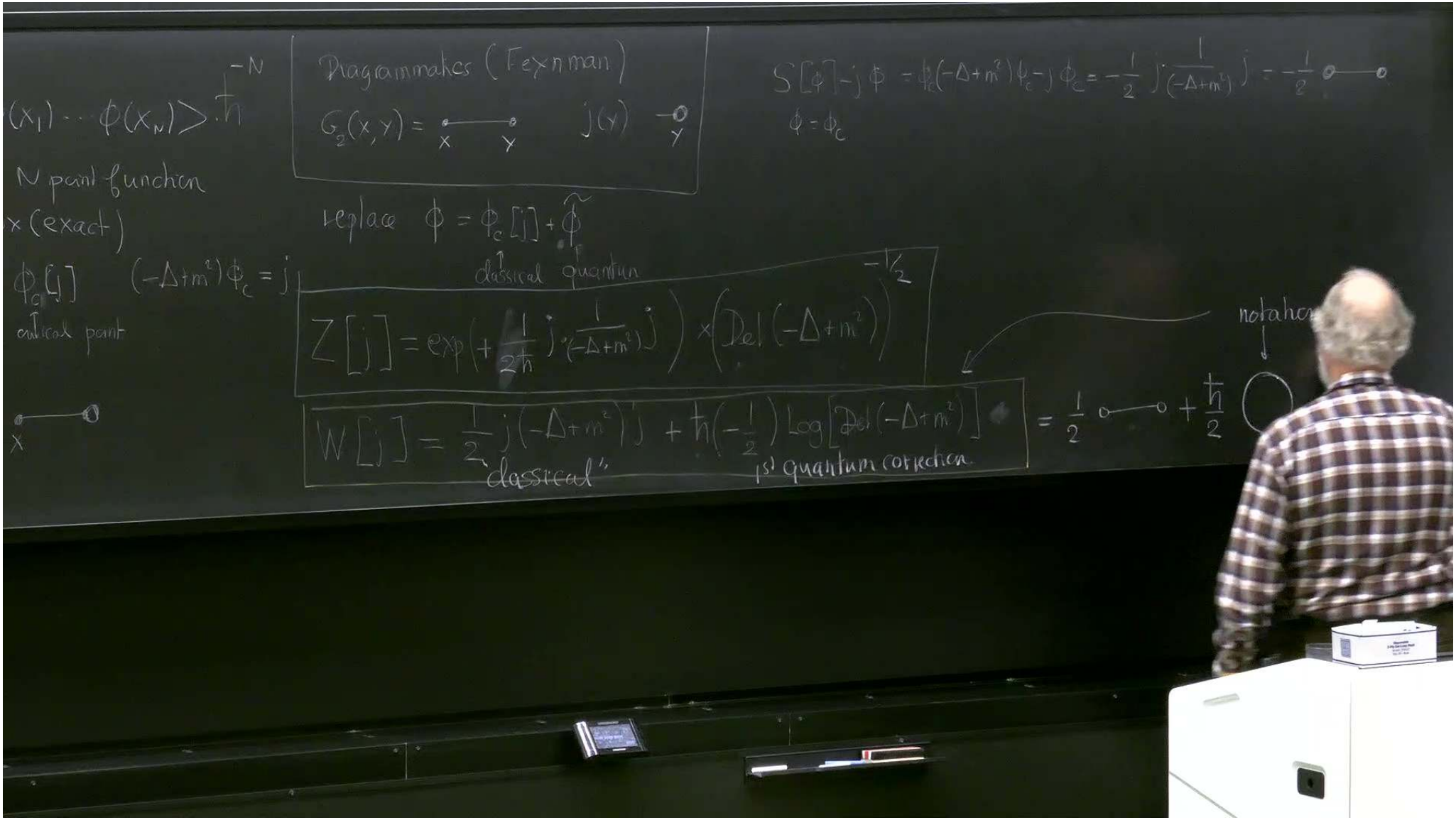
$$Z[j] = \exp\left(-\frac{1}{\hbar}\right.$$

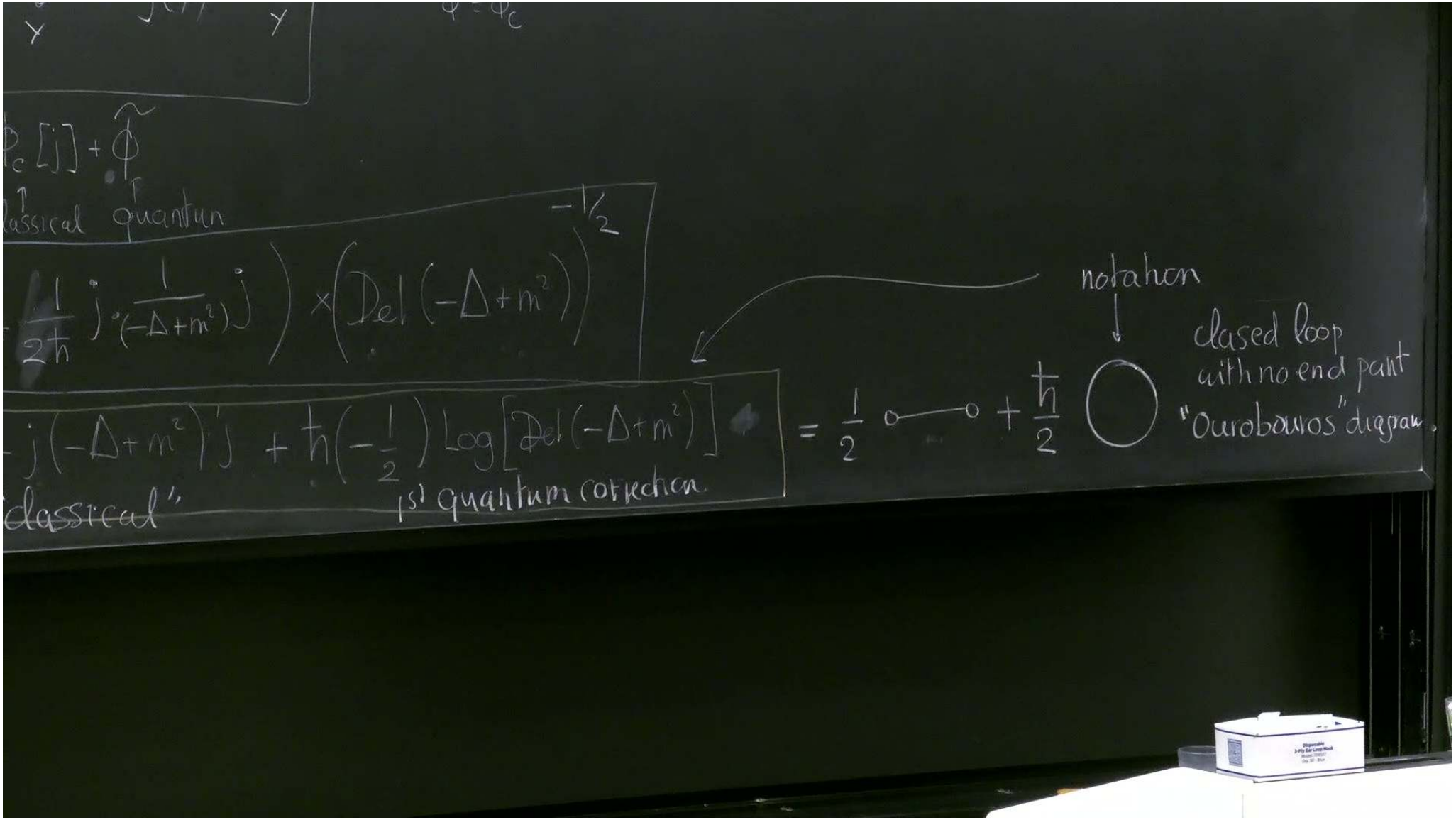
$S[\phi]$

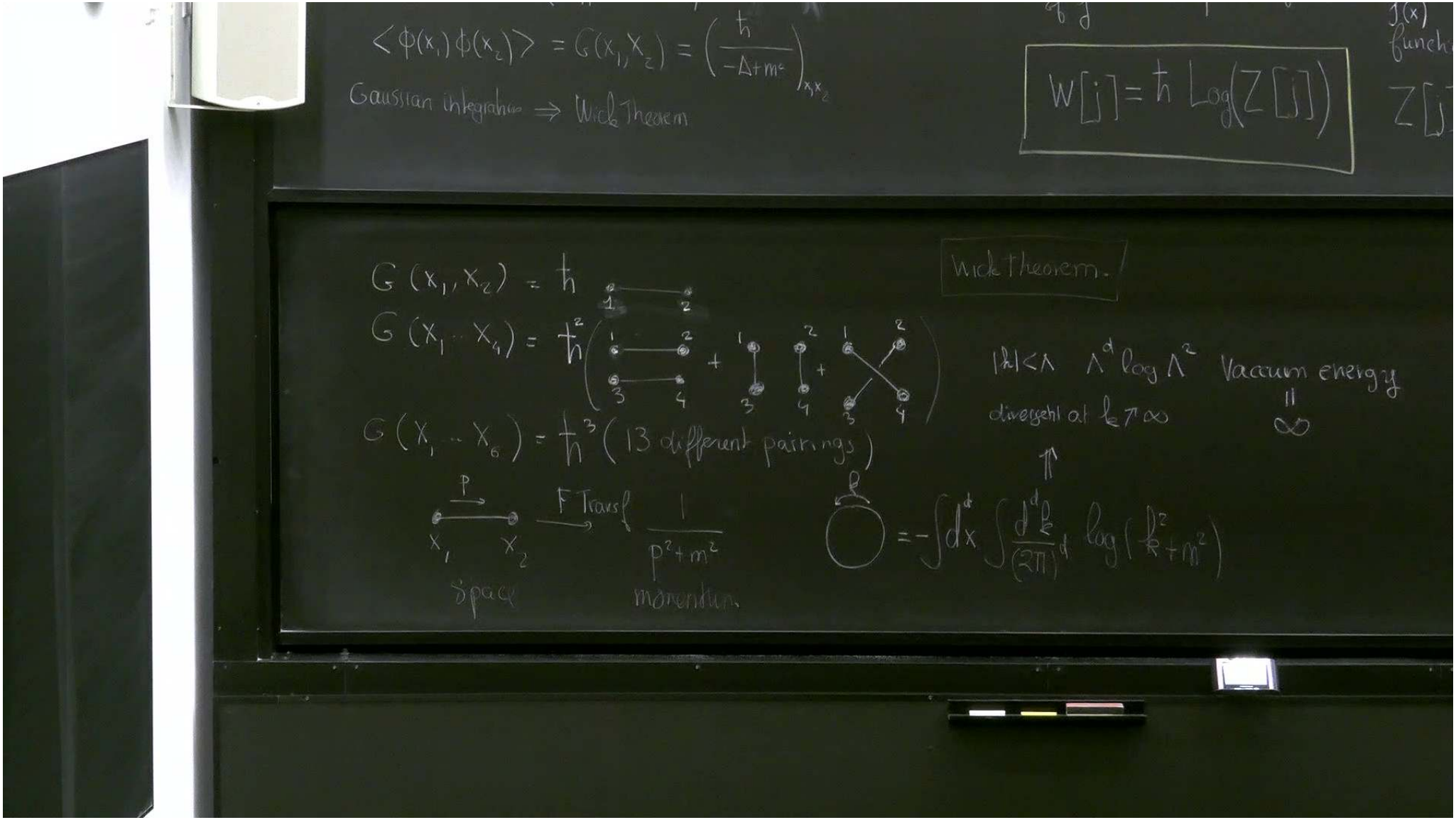
$\phi = \phi_c$

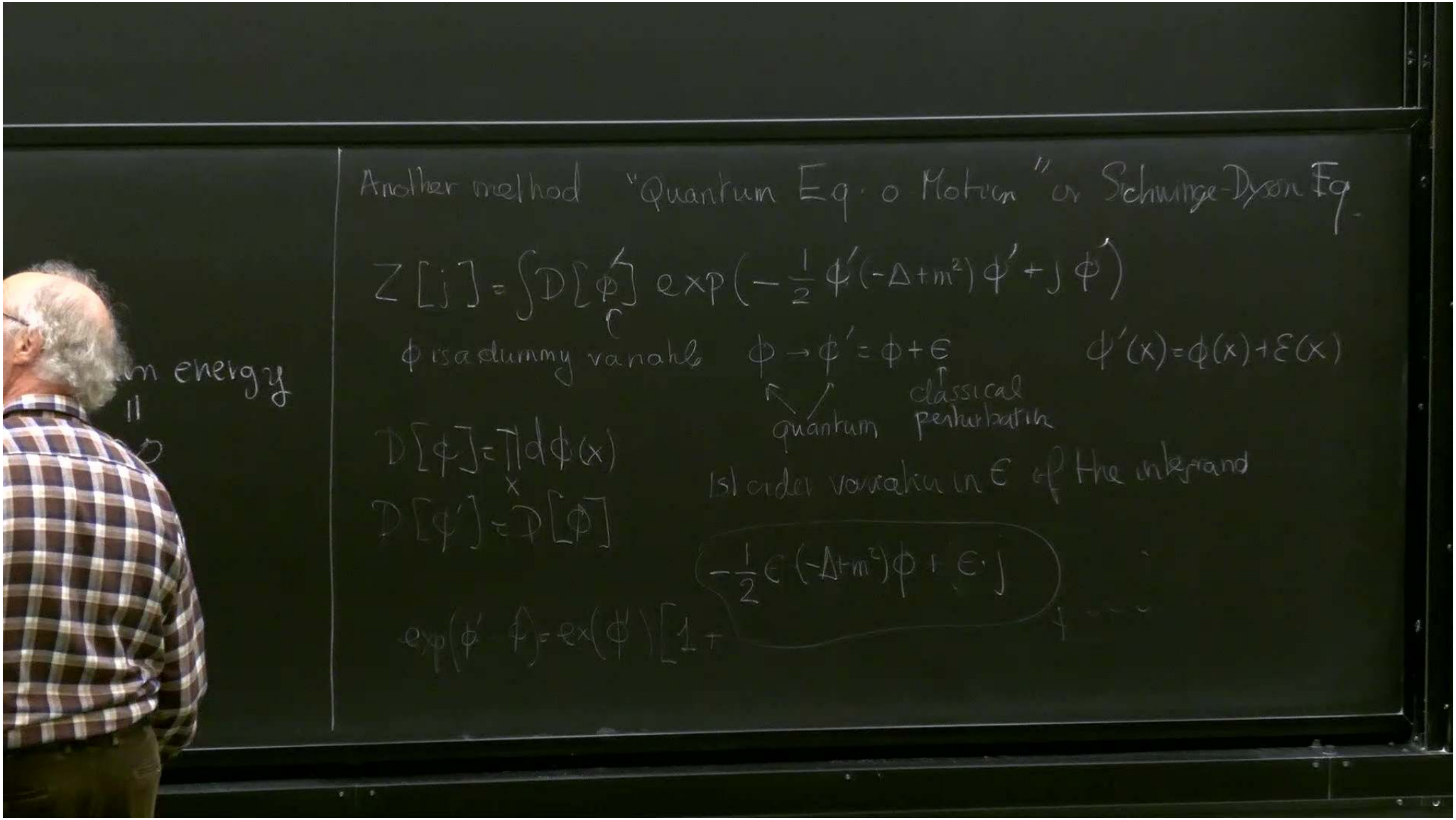

(Feynman)
$j(y)$
y
x
$S[\phi]-j\phi = \phi_c(-\Delta+m^2)\phi_c - j\phi_c = -\frac{1}{2} j \frac{1}{(-\Delta+m^2)} j = -\frac{1}{2}$
$\phi = \phi_c$
$\phi_c[j] + \tilde{\phi}$
classical quantum
$\frac{1}{\hbar}$


function
$(-\Delta + m^2)\phi_c = j$
replace $\phi = \phi_c[j] + \tilde{\phi}$
classical
quantum
$Z[j] = \exp\left(+\frac{1}{2\hbar} j \cdot \frac{1}{(-\Delta+m^2)} j\right) \times \left(\mathrm{Det}(-\Delta+m^2)\right)^{-1/2}$
$W[j] =$

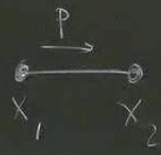

Diagrammatics (Feynman)
N point function
(exact)
critical point
classical
quantum
classical
1st quantum correction
notaha


notation
closed loop
with no end point
"Ourobouros" diagram.
classical"
1st quantum correction.


$\langle \phi(x_1)\phi(x_2)\rangle = G(x_1,x_2) = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1x_2}$
Gaussian integrals $\Rightarrow$ Wick Theorem
$W[j] = \hbar \, \mathrm{Log}(Z[j])$
$J(x)$
$Z[j]$
Wick theorem.
$G(x_1,x_2) = \hbar$
$G(x_1 \dots x_4) = \hbar^2$
$G(x_1 \dots x_6) = \hbar^3$ (13 different pairings)
$|k|<\Lambda$ $\Lambda^d \log \Lambda^2$ Vacuum energy
diverges at $k \to \infty$
$= \infty$
P
F Transf
$\frac{1}{p^2+m^2}$
x_1 x_2
space
momentum
$= -\int d^dx \int \frac{d^dk}{(2\pi)^d} \log(k^2+m^2)$


Another method "Quantum Eq. o Motion" or Schwinge-Dyson Eq.
$Z[j] = \int D[\phi]\, \exp\left(-\frac{1}{2}\phi'(-\Delta+m^2)\phi' + J\phi'\right)$
ϕ is a dummy variable
$\phi \to \phi' = \phi + \epsilon$
$\phi'(x) = \phi(x) + \epsilon(x)$
quantum
classical perturbation
$D[\phi] = \prod_x d\phi(x)$
$D[\phi'] = D[\phi]$
Is I a fct of ϵ of the integrand
$\exp(\phi' \ldots) = \exp(\phi)\left[1 + \left(-\frac{1}{2}\epsilon(-\Delta+m^2)\phi + \epsilon \cdot J\right)\right]$

$G(x_1 \dots x_6) = \hbar^3$ (13 different pairings)



space

$\frac{1}{p^2+m^2}$

momentum

$= -\int dx \int \frac{d^d k}{(2\pi)^d} \log\left(k^2 + m^2\right)$

For any $\epsilon(x)$ $\quad \epsilon(x)\left\langle (-\Delta + m^2)\phi + j \right\rangle_{j \neq 0} = 0$

$(-\Delta_x + m^2) = x$

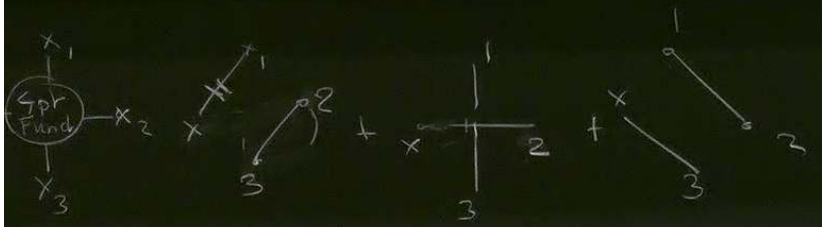
$(-\Delta_x + m^2) \int \mathcal{D}[\phi] e^{-S+j\phi} \phi(x) = \int \mathcal{D}[\phi] e^{-S+j\phi} j(x)$ $\quad \Rightarrow$ SD equation $\Rightarrow$ Wick Theorem

Application $N = 2$

$(-\Delta_x + m^2)\langle \phi(x)\phi(x_1)\rangle = \delta(x - x_1) \Rightarrow$ propagator

$(-\Delta_x + m^2)\langle \phi(x)\phi(x_1)\phi(x_2)\phi(x_3)\rangle = \delta(x-x_1)\langle \phi(x_2)\phi(x_3)\rangle + \delta(x-x_2)\langle \phi(x_1)\phi(x_3)\rangle + \delta(x-x_3)\langle \phi(x_1)\phi(x_2)\rangle$

$\exp(\phi' \cdot \phi) = \exp(\phi)\left[1 + \right.$ ... $+$ ...



$(-\Delta + m^2)\phi = 0$ KG equation

$\langle (-\Delta + m^2)\phi \rangle$ = source term. SD eq.

↓

other operators in the theory

... $\Longrightarrow$ Wick Theorem

$\phi(x_3) + \delta(x - x_3)\langle \phi(x_1, x_2) \rangle$

From Free Scalar Field to Interactions

Euclidean

$$S_E[\phi] = \int d^d x \, \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4(x)$$

free term — symmetry reason — coupling constant — Interaction term

ϕ^4 theory

$$+ \frac{h}{3!}\phi^3(x)$$

perturbative expansion in powers of g c.c

apply Wick theorem

1 particle mass m

contact strength g 2→2 process

repulsive interaction

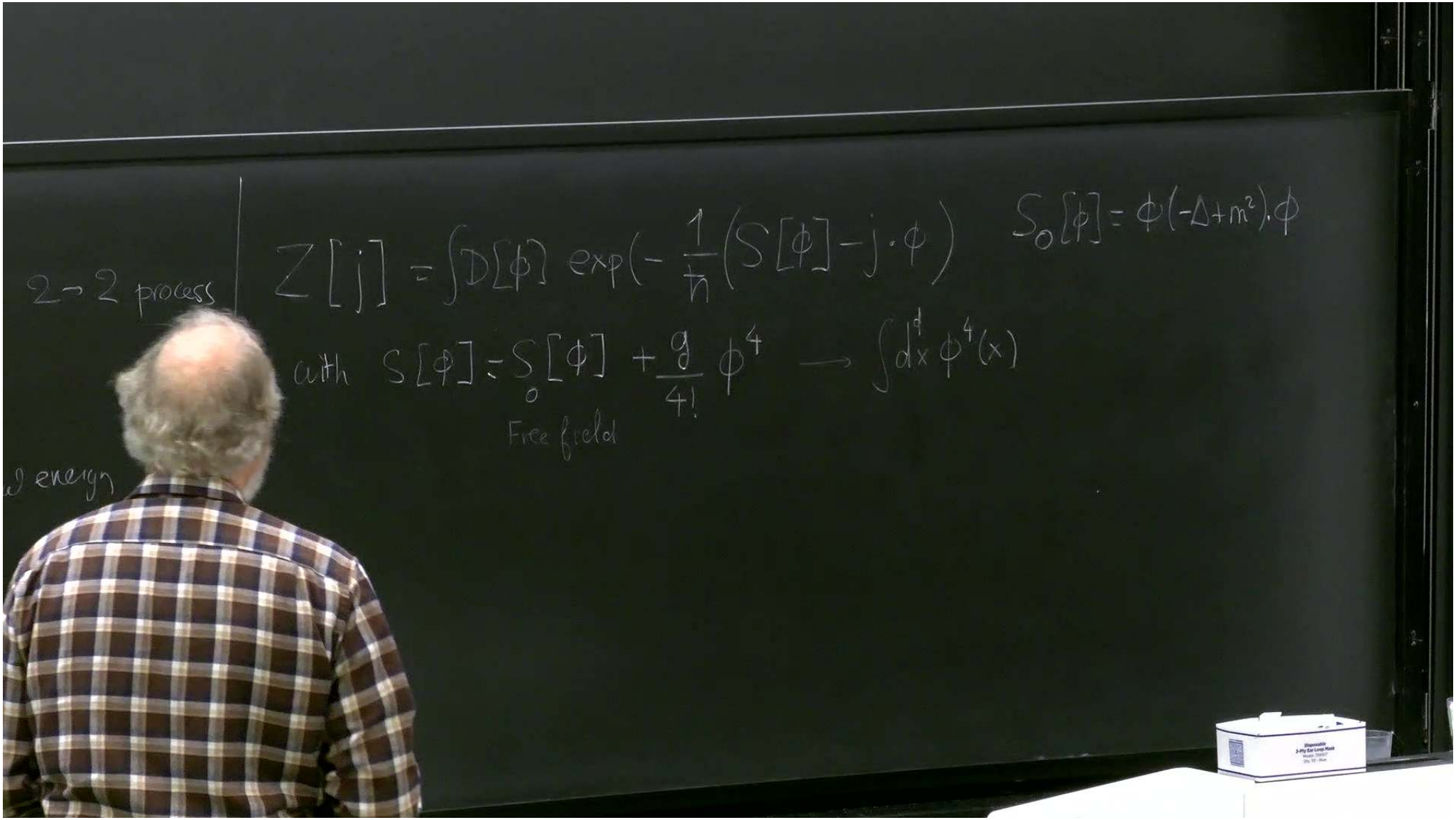
attractive → Bose Einstein Cond

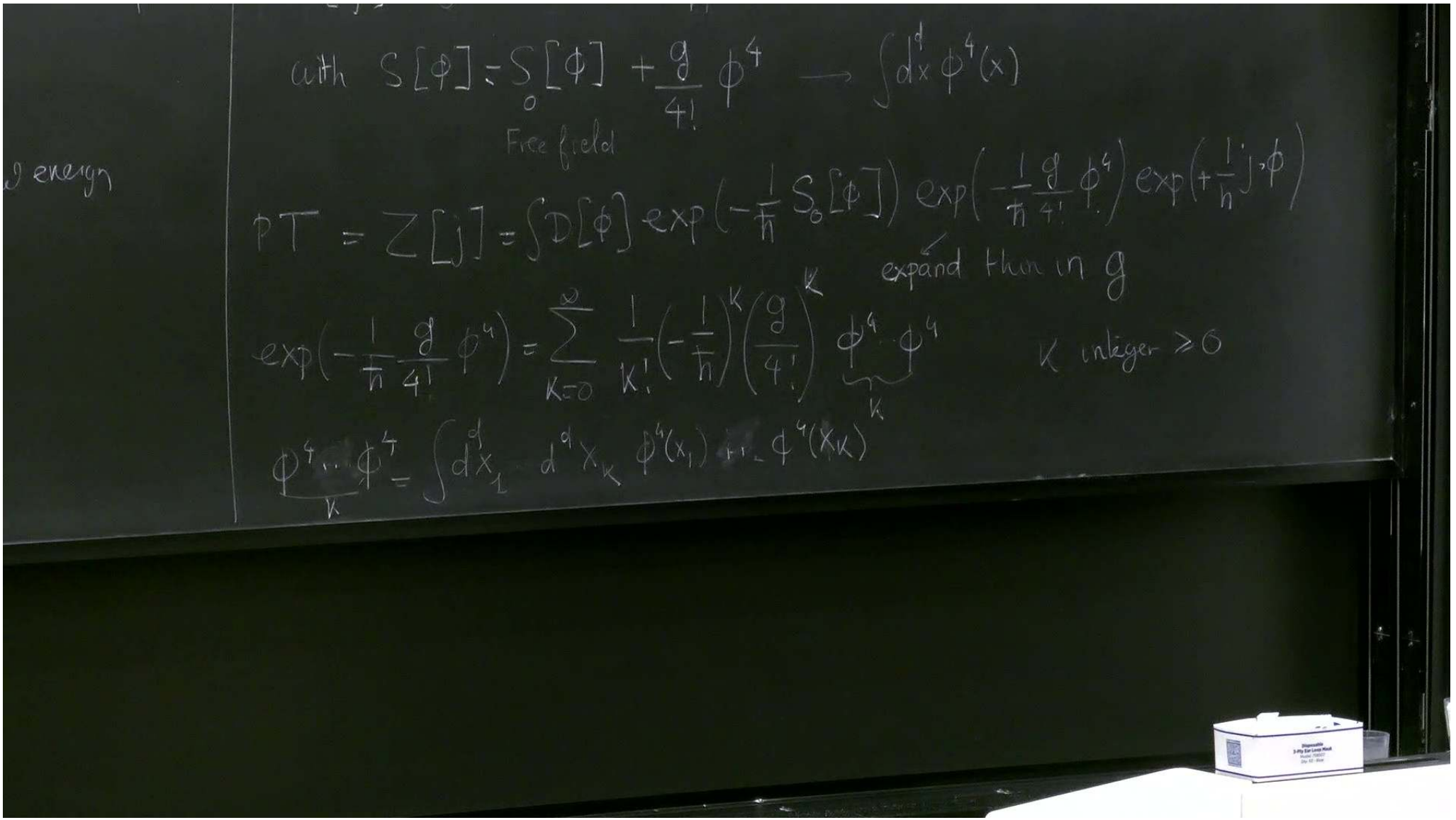
no $|0\rangle$ with finite classical energy

2→1 process

statistical mechanics

dynamical processes


2→2 process
$Z[j] = \int D[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$
$S_0[\phi] = \phi(-\Delta + m^2)\cdot\phi$
with $S[\phi] = S_0[\phi] + \frac{g}{4!}\phi^4 \longrightarrow \int d^4x\, \phi^4(x)$
Free field


with $S[\phi] = S_0[\phi] + \frac{g}{4!}\phi^4 \longrightarrow \int d^d x\, \phi^4(x)$
Free field
$PT = Z[J] = \int D[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \exp\left(-\frac{1}{\hbar}\frac{g}{4!}\phi^4\right) \exp\left(+\frac{1}{\hbar} J\cdot\phi\right)$
expand then in g
$\exp\left(-\frac{1}{\hbar}\frac{g}{4!}\phi^4\right) = \sum_{K=0}^{\infty} \frac{1}{K!}\left(-\frac{1}{\hbar}\right)^K \left(\frac{g}{4!}\right)^K \underbrace{\phi^4 \cdots \phi^4}_{K}$
K integer ≥ 0
$\underbrace{\phi^4 \cdots \phi^4}_{K} = \int d^d x_1 \cdots d^d x_K\, \phi^4(x_1) \cdots \phi^4(x_K)$

perturbative expansion in powers of g cc
apply Wick theorem

statistical mechanics
dynamical processes

$$\exp\left(-\frac{1}{\hbar}\frac{g}{4!}\phi^4\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{\hbar}\right)^k \left(\frac{g}{4!}\right)^k$$

$$\underbrace{\phi^4 \cdots \phi^4}_{k} = \int d^d x_1 \; d^d x_k \; \phi^4(x_1) \cdots \phi^4$$

commute $\int D[\phi]$ with $\sum_k$

$$Z[J] = \sum_{k=0}^{\infty} \int D[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \exp\left(+\frac{1}{\hbar} J\cdot\phi\right) \cdot \frac{1}{k!}\left(-\frac{1}{\hbar}\right)^k \left(\frac{g}{4!}\right)^k \underbrace{\phi^4 \cdots \phi^4}_{k}$$

by taking $\frac{\delta}{\delta J} \prod_{i=1}^{N} \frac{\delta}{\delta J(z_i)}$

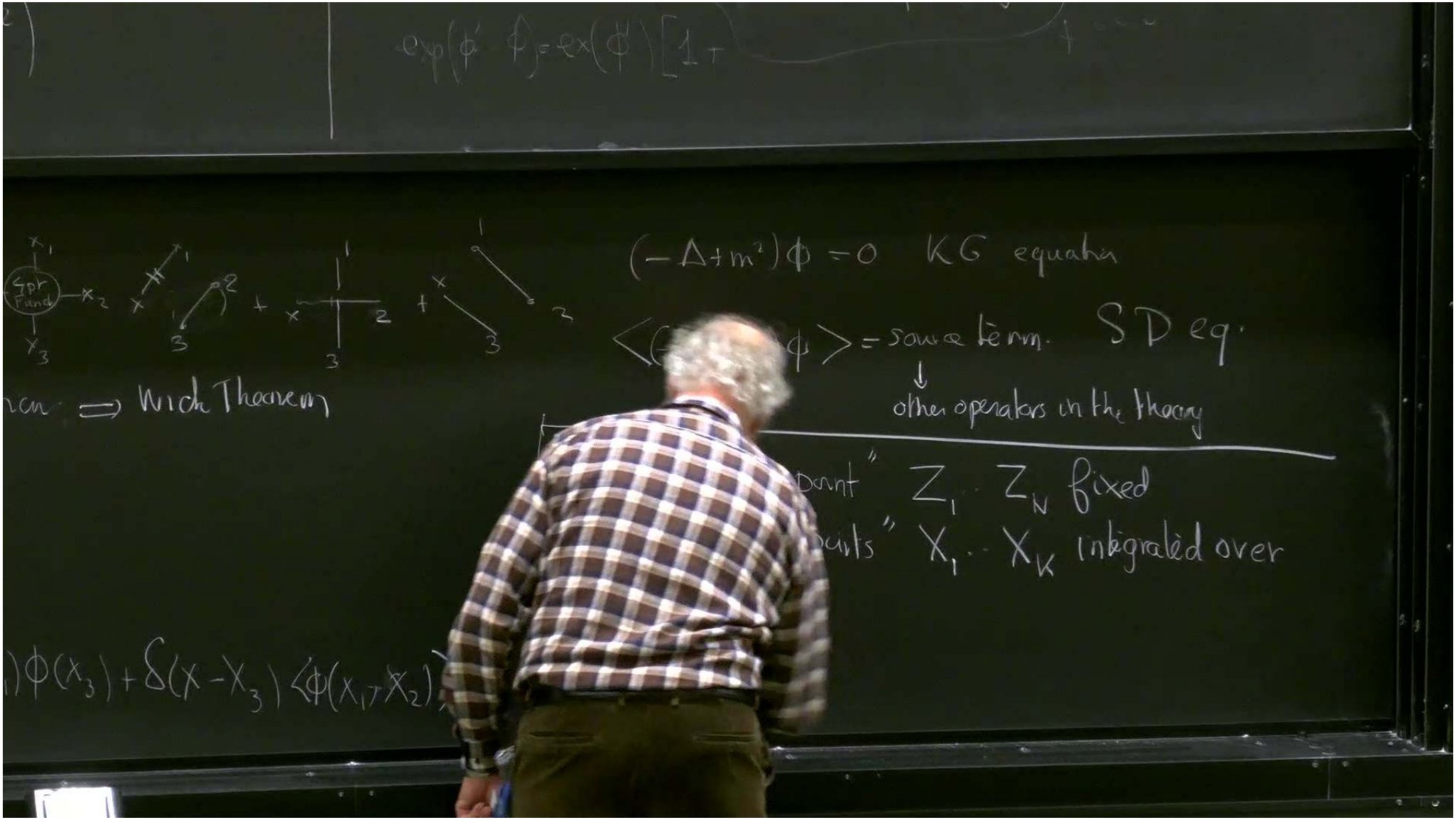
$$Z[J]\Big|_{J=0} / Z[0] = \boxed{\hbar^{-N} \langle \phi(z_1) \cdots \phi(z_N) \rangle_{\text{interacting theory}}}$$

as for Free theory expand $\sum_k$ both

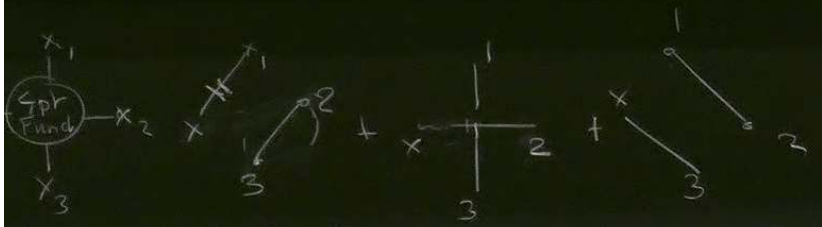
$$= \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{\hbar 4!}\right)^k \int d^d x_1 \; d^d x_k \langle \phi(z_1) \cdots \phi(z_N) \phi^4(x_1) \cdots \phi^4(x_N) \rangle_{\text{Free theory}}}{\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-g}{\hbar 4!}\right)^k \int d^d x_1 \cdots d^d x_k \langle \phi^4(x_1) \cdots \phi^4(x_N) \rangle}$$

∞ series in g^k

$\langle \phi \cdots \phi \, \phi^4 \cdots \phi^4 \rangle_{\text{Free theory}}$


$(-\Delta + m^2)\phi = 0$ KG equation
= source term. SD eq.
other operators in the theory
Wick Theorem
$Z_1 \cdots Z_N$ fixed
$X_1 \cdots X_K$ integrated over

$\exp(\phi' \cdot \phi) = \exp(\phi)\left[1 + \right.$



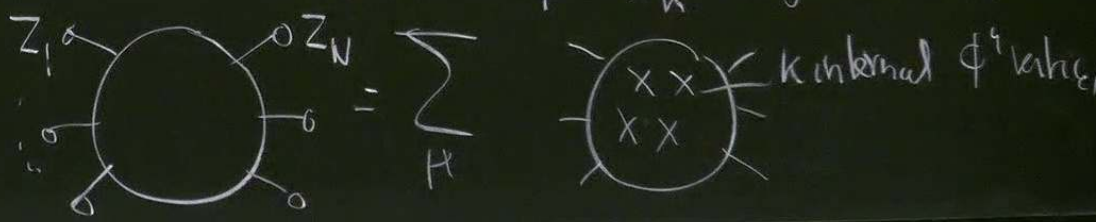
$(-\Delta + m^2)\phi = 0$ KG equation

$\langle(-\Delta + m^2)\phi\rangle$ = source term. SD eq.

↓ other operators in the theory

$\Rightarrow$ Wick Theorem

$\phi(x_3) + \delta(x - x_3)\langle\phi(x_1, x_2)\rangle$

N "external point" $z_1 \dots z_N$ fixed

K "internal points" $x_1 \dots x_K$ integrated over