Title: Lecture - QFT II, PHYS 603

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Collection/Series: Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

Subject: Condensed Matter, Particle Physics, Quantum Fields and Strings

Date: November 14, 2024 - 9:00 AM

URL: https://pirsa.org/24110010

① Scalar Free Field $\phi(x)$ Real Field set $\hbar = 1$

"Physical time" $t \longleftrightarrow$ "Euclidean time" $\tau \leftarrow\rightarrow$ quantum theo[ry] at finite t[emperature] with period

$t \leftarrow\rightarrow -i\tau$

period τ =

correlation function
2 points funct

$$\langle \phi(x_1)\phi(x_2)\rangle_E := \frac{\int D_E[\phi]\, \exp(-S_E[\phi])\, \overbrace{\phi(x_1)\phi(x_2)}^{\text{observable}}}{\int D_E[\phi]\, \exp(-S_E[\phi])} \longrightarrow \mathrm{Tr}\left[e^{-\beta\hat{H}}\phi(x_1)\phi\right.$$

$\mathrm{Tr}(\rho_\beta \hat{\phi}(x_1)\hat{\phi}(x_2))$ $\hat{\phi}(x_1)\,\hat{\phi}(x_2)$ Field local operators ρ_β density matrix Gibbs

Gaussian integrals $-1/2$ $\int d\phi\ \exp$

(unit system) and $c = 1$ $\quad S_E[\phi] = \int_{\mathbb{R}^d} d^dx \; \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{m^2}{2}\phi^2$

ny

mperature Temp $\quad (+++\;+)$ $\quad h^{(E)}_{\mu\nu} = \delta_{\mu\nu}$ Euclidean metric

c τ.

$\frac{\hbar}{k_B \,\mathrm{Temp}}$

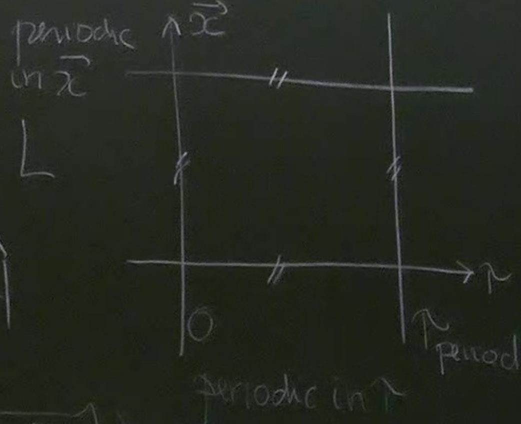
$(x_2)]$

start from "space-time" periodic in τ (1+1 dim)

also a periodic space size L

(torus of dimension $d-1$, $\boxed{\text{size } L}$

$\tau_{\text{period}} = \beta = \frac{1}{k_B T}$



tate $\rho_\beta = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\,\exp(-\beta\hat{H})}$

② Limit period $\tau_{period} \to \infty \longleftrightarrow$ Temp $\searrow 0$

$$\rho_\beta = \frac{|n\rangle\langle n| e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}$$

eigenstate basis

$\hat{H}|n\rangle = E_n|n\rangle$

$E_0 < E_1 \ldots$

$\beta \to \infty$ only E_0 remains

$|0\rangle$ = vacuum state of the system

$$\boxed{\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle \quad \text{vacuum expectation value}}$$

Compute

physical h

How to

$S_E[$

starts from Euclidean theory

compute 2pt function

$\downarrow$

physical time using Poincaré Invariance

$X_1 = (0, \vec{x}_1)$ $X_2 = (0, \vec{x}_2)$ equal time corelators

How to compute this ?

$$S_E[\phi] = \int d^dx \, \frac{1}{2} \phi(x) (-\Delta_x + m^2) \phi(x)$$

differential operator

quadratic form

$$= \frac{1}{2} \phi \cdot (-\Delta + m^2) \cdot \phi$$

short hand notation

Δ Laplace Beltrami operator

$$\Delta_x = \sum_{\mu=0}^{d-1} \frac{\partial^2}{\partial x_\mu^2}$$

$\phi = \{\phi(x)\}$ vector ∞ dim

$-\Delta + m^2$ operator matrix ∞ dim

Gaussian integrals

$$Z = \int D_E[\phi] \exp\left(-\tfrac{1}{2}\phi\cdot(-\Delta+m^2)\cdot\phi\right) = \left(\det(-\Delta+m^2)\right)^{-1/2}$$

$\uparrow$ ∞-dim determinant (in fact ∞)

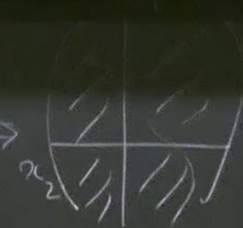
$$\int_{-\infty}^{+\infty} d\phi\ e$$

$$\int D[\phi] \exp\left(-\tfrac{1}{2}\phi(-\Delta+m^2)\phi\right)\phi(x_1)\phi(x_2)$$

$$= \left[\det(-\Delta+m^2)\right]^{-1/2} (-\Delta+m^2)^{-1}_{x_1 x_2} = \mathrm{Minor}_{x_1 x_2}(-\Delta+m^2)$$

...y Matrix Gibbs State $\frac{1}{\beta}\,\mathrm{Tr}\exp(-\beta H)$ periodic in β

$$\int_{-\infty}^{+\infty} d\phi \, \exp\left(-\frac{1}{2}A\phi^2\right) = \sqrt{\frac{2\pi}{A}}$$

$(-\Delta + m^2) \Rightarrow$  2 pt propagator

(in fact ∞) $\Rightarrow$

$$\boxed{\langle \phi(x_1)\phi(x_2)\rangle = \left(-\Delta+m^2\right)^{-1}_{x_1 x_2} = \left(\frac{1}{-\Delta+m^2}\right)_{x_1 x_2} = G(x_1,x_2)}$$

$$\boxed{\left(-\Delta_{x_1} + m^2\right) G(x_1,x_2) = \delta^{(d)}(x_1-x_2)} \; *$$

Space time periodic $T_{\text{period}} \to \infty$, $L \to \infty$

$G(x_1,x_2) \searrow 0$ when $|X_1 - X_2| \to \infty$

Euclidean space

④ $G(x_1 - x_2) = G(x_1 - x_2)$

Translation invariance

$$G(x) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} \hat{G}(k)$$ Inverse Fourier Transform

$x \in \mathbb{R}^d$

$x = (x^\mu)_{\mu=0,\ldots,d-1}$ vector $k^2 = \sum_\mu k_\mu k_\mu$

$k = (k_\mu)_{\mu=0,\ldots,d-1}$ co-vector Euclidean

$$(k^2 + m^2) \cdot \hat{G}(k) = 1$$

Check

$$\hat{G}(k) = \frac{1}{k^2 + m^2}$$ Lorentzian Function

$$G(x) = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{k^2 + m^2}$$ in co space

Rotation invariance + translation

Euclidean invariance

$= \frac{1}{2}\,\phi\cdot(-\Delta+m^2)\cdot\phi$ differential operator
short hand notation

$\phi = \{\phi(x)\}$ vector ∞ dim

$-\Delta+m^2$ operator matrix ∞ dim

$(K^2+m^2)\cdot\hat{G}(k) = 1$ Check $m^2>0$ finite integral $K \to 0$ computed

$\hat{G}(k) = \frac{1}{K^2+m^2}$ Lorentzian Function

$G(x) \simeq e^{-m|x|}$ $|x| \to \infty$

$|x| = \sqrt{x^\mu x^\mu}$ exponential decay

$$G(x) = \frac{1}{2\pi}\left(\frac{2\pi|x|}{m}\right)^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(|x|m)$$

K_ν = modified Bessel of 2^{nd} kind

$K_\nu(z) = \mathrm{epx}(-z)$

$$G(x) = \int_{\mathbb{R}^d} \frac{d^dk}{(2\pi)^d}\,\frac{e^{ik\cdot x}}{K^2+m^2}$$ in ∞ spacetime

Rotation invariance + translation

Euclidean invariance

(5) Euclidean → Minkowski $\quad X = (x^0, \vec{x}) \quad K = (k_0, \vec{k})$

$$\langle \phi(X)\phi(0) \rangle_E = G_E(X) = \int \frac{d^d K}{(2\pi)^K} \frac{e^{iK X}}{K^2 + m^2}$$

$K^2 = k_0^2 + \vec{k}^2$

$K \cdot X = k_0 x^0 + \vec{k} \cdot \vec{x}$

Wick Rotation $\quad x^0 = i t \quad e^{iKX} = e^{i\vec{k}\cdot\vec{x}} \, e^{i k_0 x^0} \quad d^d K = d$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{(-i)\, e^{i(\omega t + \vec{k}\cdot\vec{x})}}{-\omega^2 + \vec{k}^2 + m^2}$$

(5) Euclidean → Minkowski $\quad X = (x^0, \vec{x}) \quad K = (k_0, \vec{k}) \quad \vec{x}, \vec{k}$ (d-1) di

$$\langle \phi(X)\phi(0)\rangle_E = G_E(X) = \int \frac{d^d K}{(2\pi)^K} \frac{e^{iK X}}{K^2 + m^2}$$

$$K^2 = k_0^2 + \vec{k}^2$$

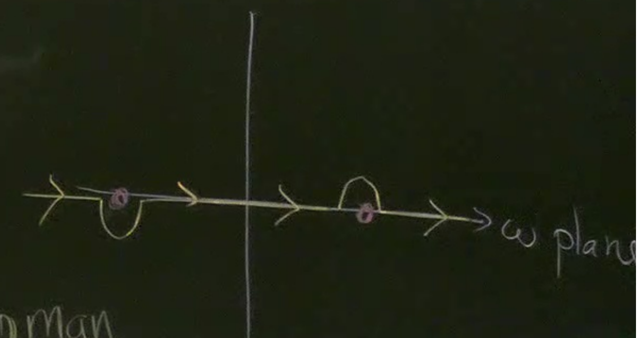
$$K \cdot X = k_0 x^0 + \vec{k} \cdot \vec{x}$$

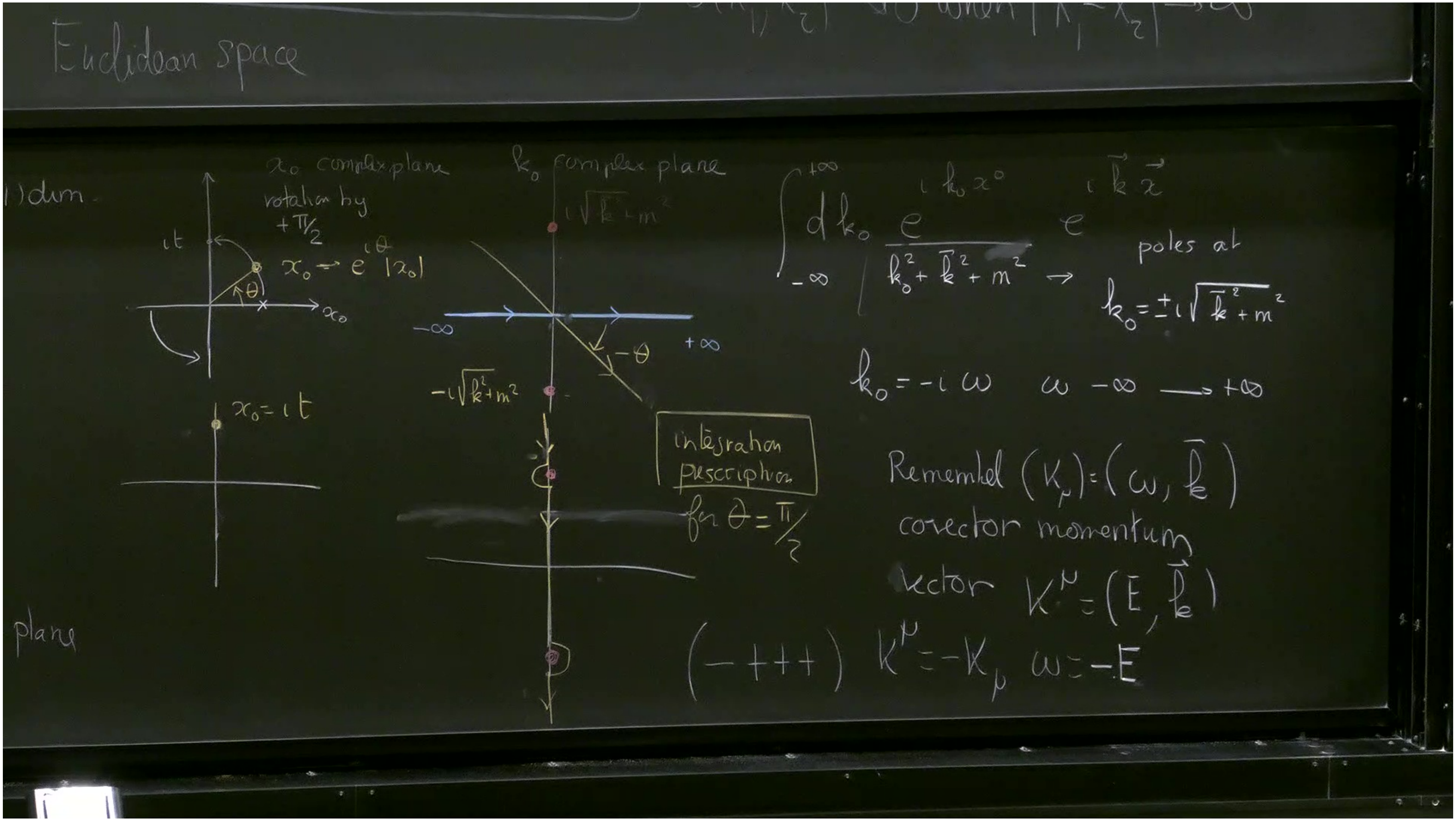
Wick Rotation $\quad x^0 = i t$

$$e^{iKX} = e^{i \vec{k} \cdot \vec{x}} \, e^{i k_0 x^0} \qquad d^d K = d k_0 \, d^{d-1}\vec{k}$$

$$\boxed{G(t, \vec{x}) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{(-i)\, e^{i(\omega t + \vec{k} \cdot \vec{x})}}{-\omega^2 + \vec{k}^2 + m^2 - i\epsilon_+}}$$

↖ Feynman





Euclidean space
1) dim.
plane
x_0 complex plane
rotation by $+\pi/2$
it
$x_0 \to e^{i\theta}|x_0|$
θ
x_0
$x_0 = it$
k_0 complex plane
$i\sqrt{\vec{k}^2+m^2}$
$-\infty$
$+\infty$
$-\theta$
$-i\sqrt{\vec{k}^2+m^2}$
integration prescription
for $\theta = \pi/2$
$\int_{-\infty}^{+\infty} dk_0 \frac{e^{ik_0x^0}}{k_0^2+\vec{k}^2+m^2} e^{i\vec{k}\vec{x}} \to$ poles at $k_0 = \pm i\sqrt{\vec{k}^2+m^2}$
$k_0 = -i\omega \quad \omega \; -\infty \longrightarrow +\infty$
Remember $(K_\mu) = (\omega, \vec{k})$
covector momentum
vector $K^\mu = (E, \vec{k})$
$(-+++) \quad K^\mu = -K_\mu \quad \omega = -E$

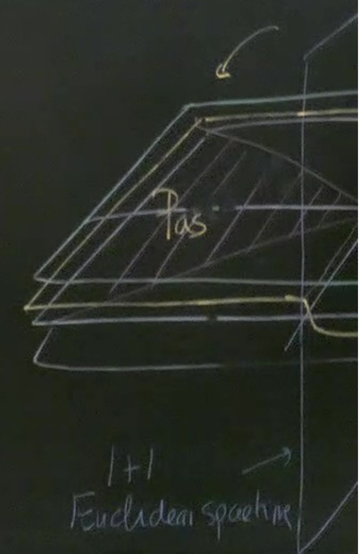
$$\boxed{G(t,\vec{x}) = G_F(t,\vec{x}) = \langle 0 | T\, \hat{\phi}(x)\hat{\phi}(0) | 0 \rangle}$$

Feynman Causal Propagator

time-oriented product

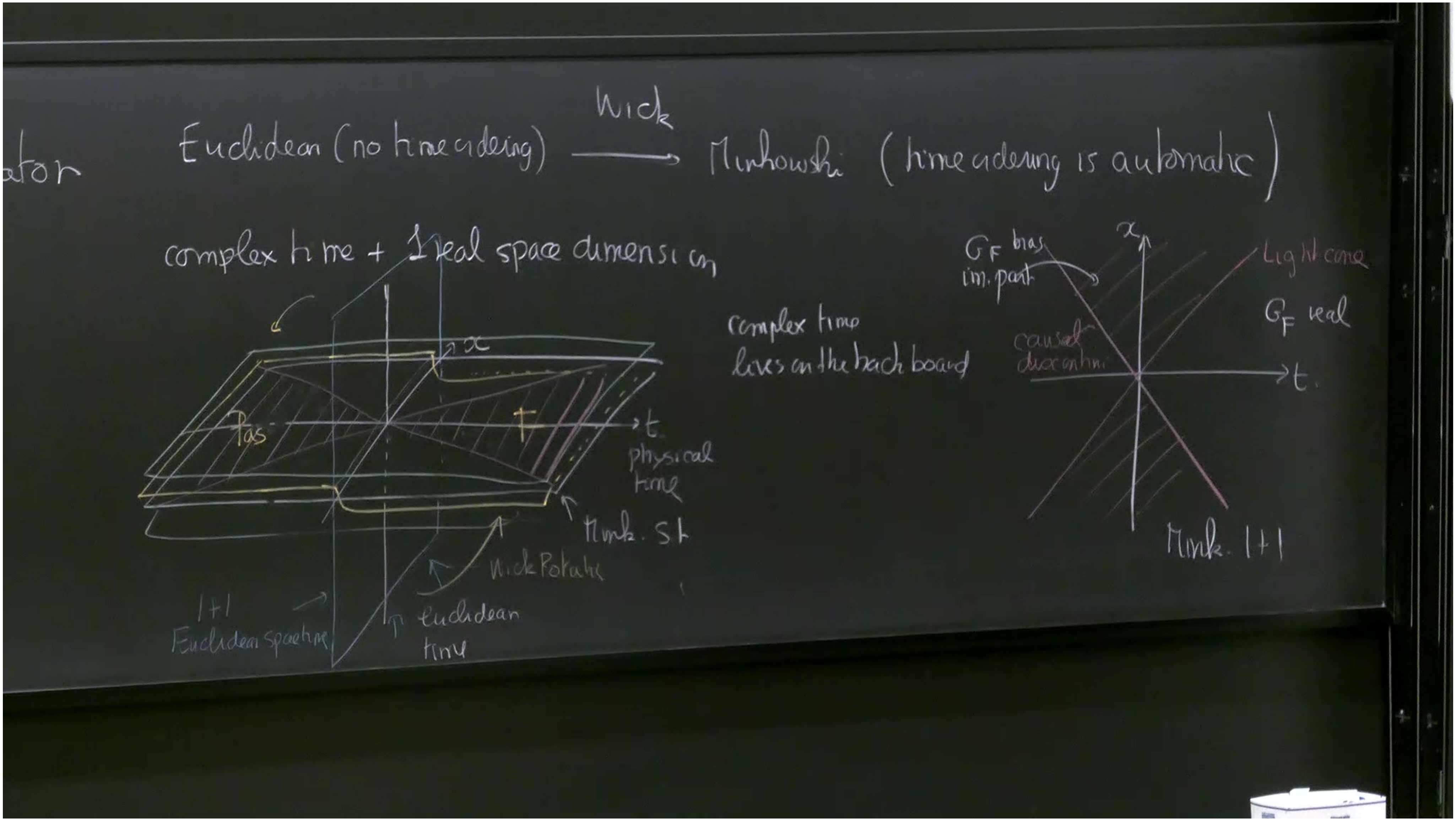
Euclidean (n

complex time



Funct Int Quantization $\Longleftrightarrow$ Canonical Quantization

$G_{adv} + G_{ret}$


Wick
Euclidean (no time ordering) → Minkowski (time ordering is automatic)
complex time + 1 real space dimension
x
t
physical time
Mink. St
Wick Rotation
1+1
Euclidean spacetime
Euclidean time
complex time
lives on the black board
G_F bias
im. part
Light cone
G_F real
t.
Mink. 1+1

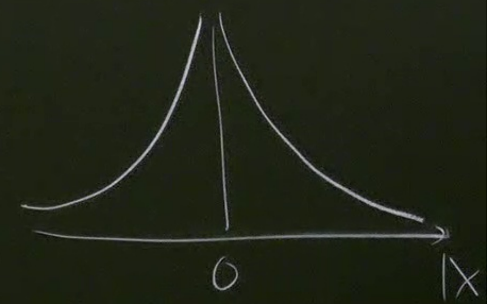
$\omega + k + m - i\epsilon_+$ ↖ Feynman

UV and IR behaviour of $G_E(x)$ or $G_F(x)$

when $|x| \searrow 0$ $\quad G(x) \sim \cdot\, |x|^{2-d}$

$$\int \frac{d^dk}{(2\pi)^d} \frac{e^{ikx}}{k^2} \sim$$

large k

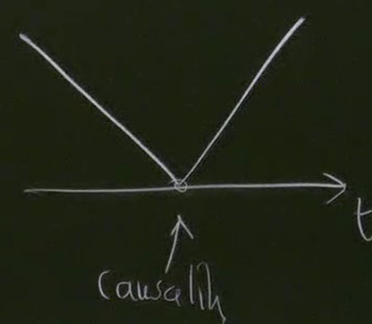


$d > 2$ singularity at $x = 0$

$d = 2$ $\quad -\frac{1}{2\pi} \log|x|$

$d < 2$ fine $\quad G(x) = |x|$

UV singularity



← Feynman

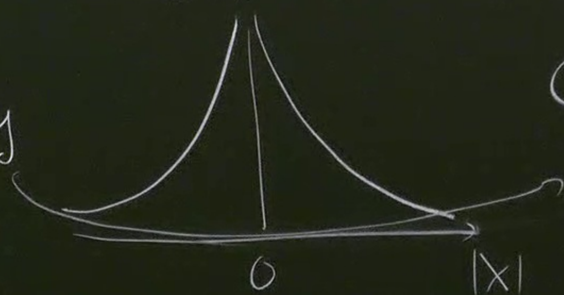
$(-+++)$ K

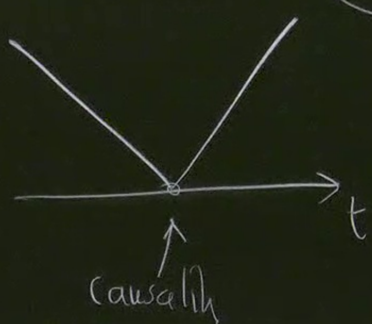
or $G_F(x)$ $\int \frac{d^d k}{(2\pi)^d} \frac{e^{ikx}}{k^2} \sim |\text{momentum}|^{d-2} \sim (\text{length})^{2-d}$

$2-d$

large K

UV singularity

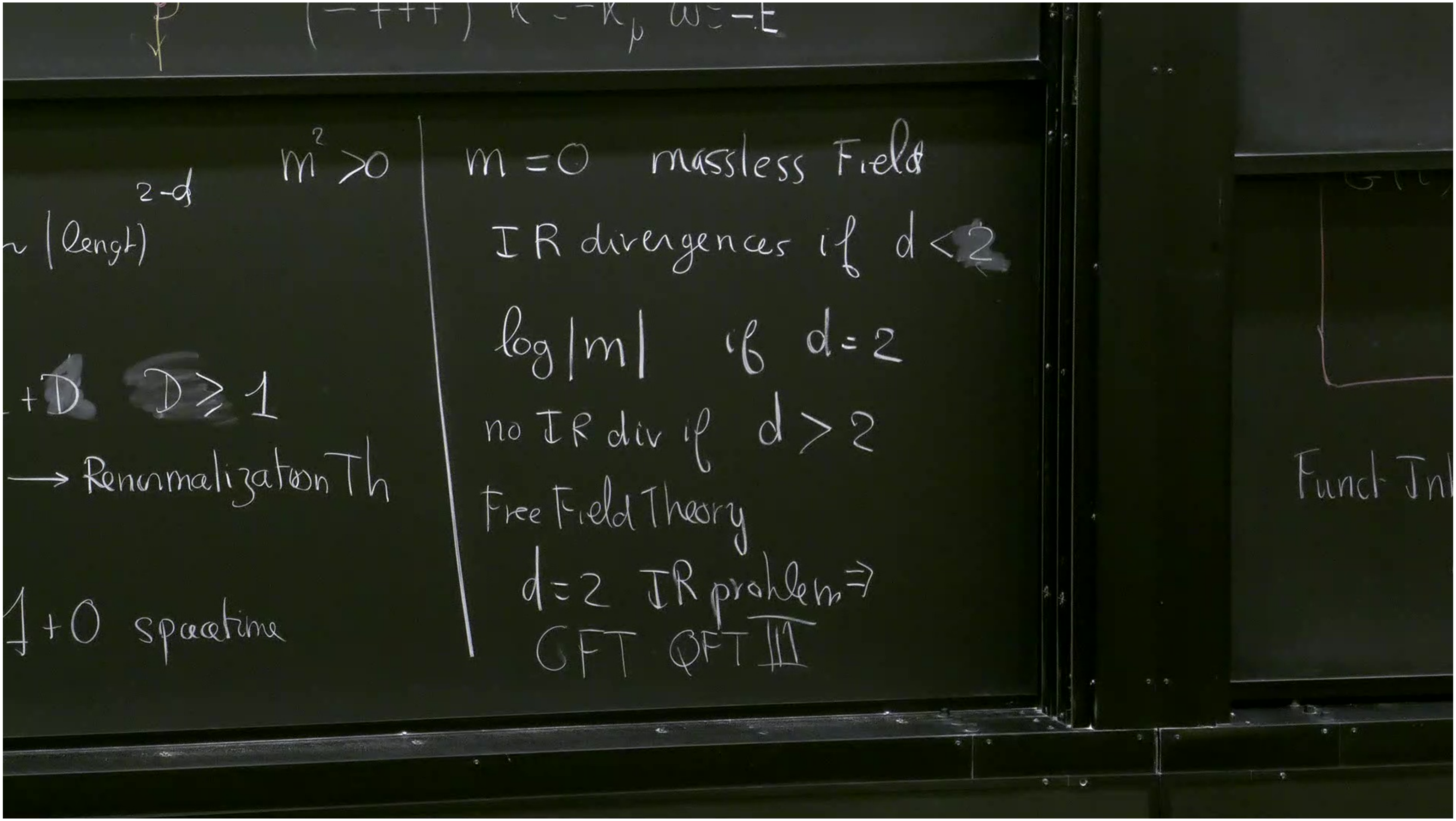




QFT $1+D$ $D \geq 1$

feature → Renormalization Th

QM $1+0$ spacetime


m = 0 massless Field
IR divergences if d < 2
log|m| if d = 2
no IR div if d > 2
Free Field Theory
d = 2 IR problem ⇒
GFT QFT
m² > 0
→ Renormalization Th
1+0 spacetime
Funct Int