Title: Lecture - Statistical Physics, PHYS 602

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Subject: Condensed Matter, Other

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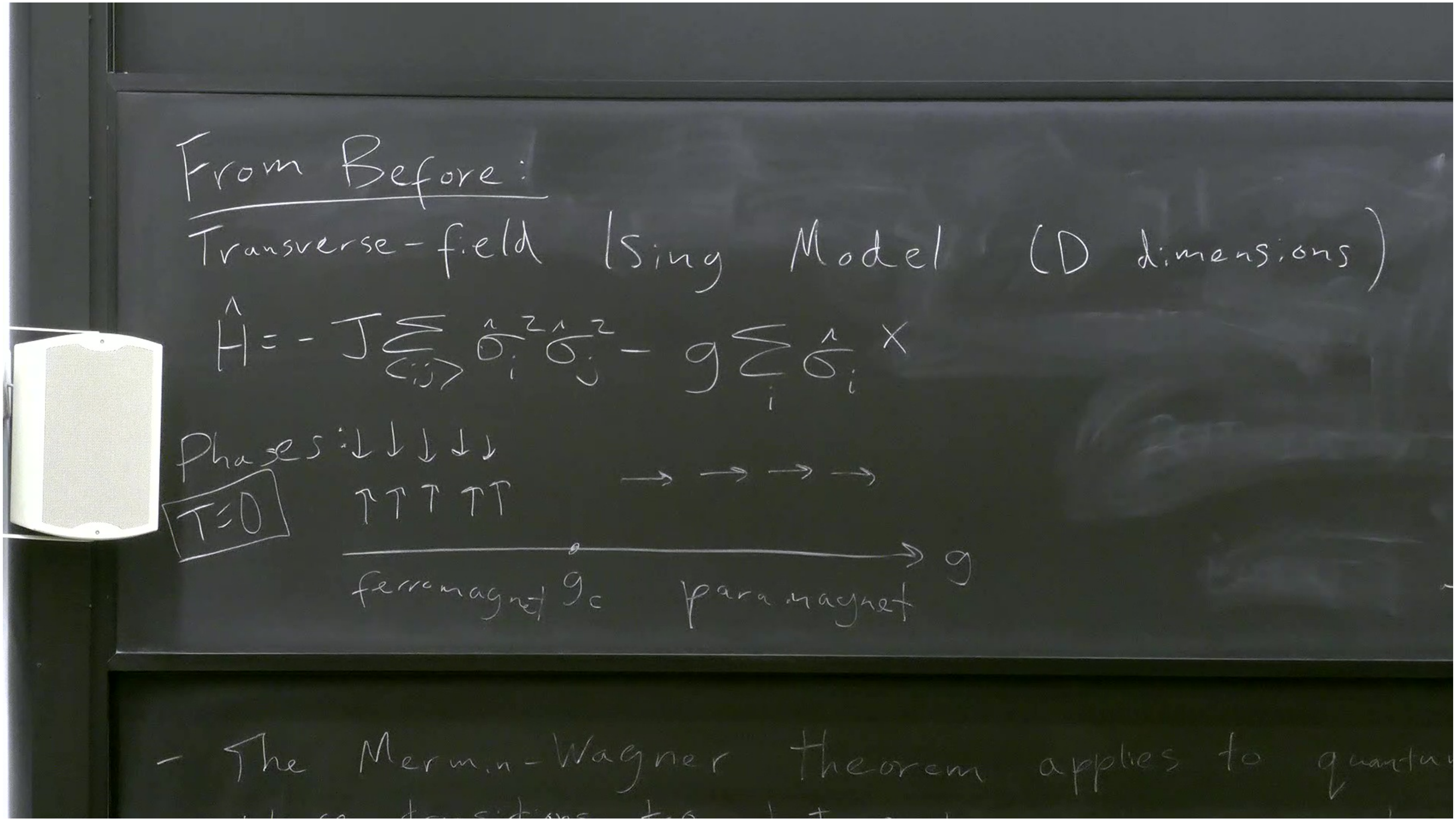


From Before:

Transverse-field Ising Model (D dimensions)

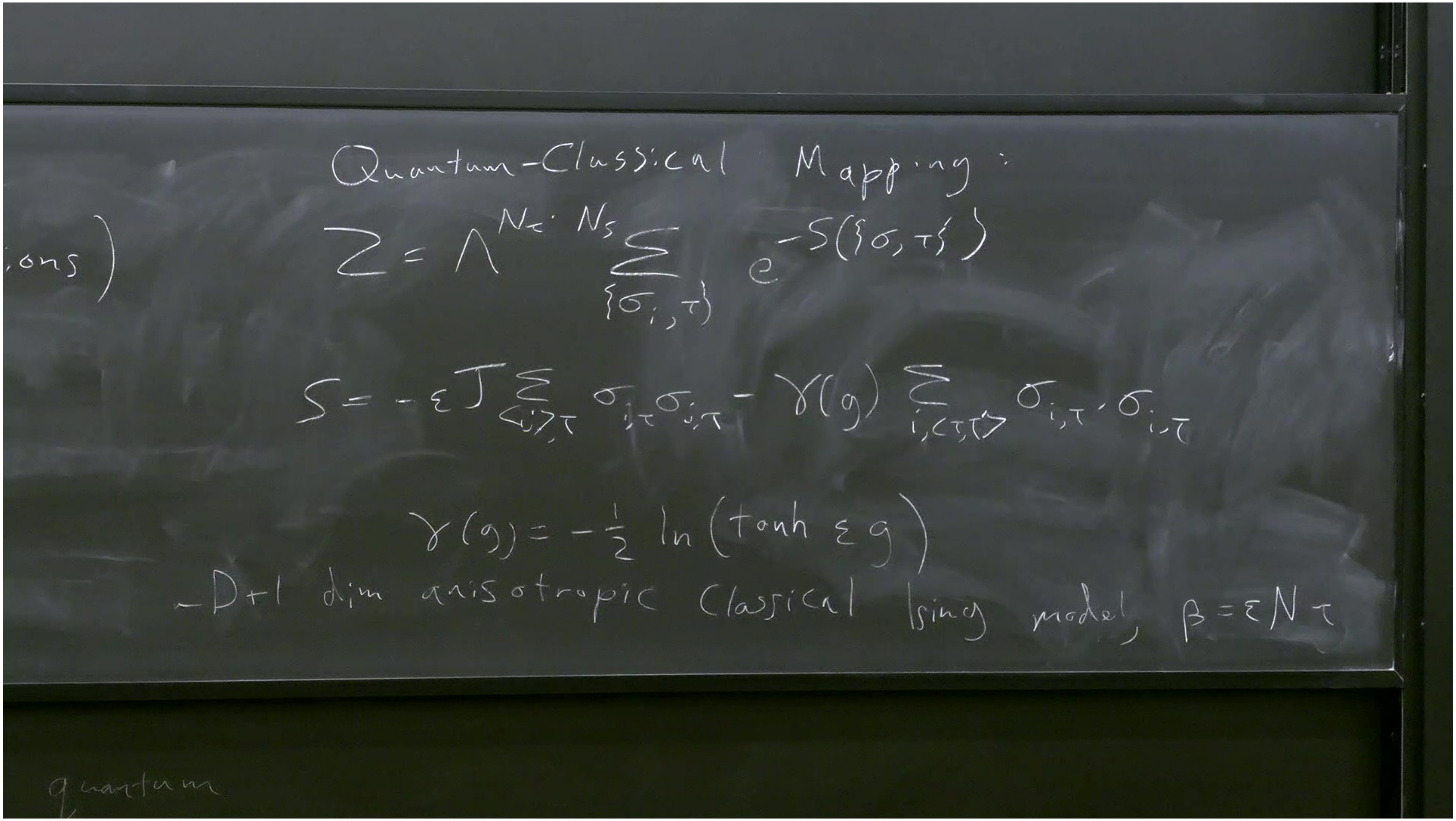
$$\hat{H} = -J\sum_{\langle i,j\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - g\sum_i \hat{\sigma}_i^x$$

Phases: ↓ ↓ ↓ ↓ ↓ / ↑ ↑ ↑ ↑ ↑ → → → →

$\boxed{T=0}$

ferromagnet g_c paramagnet $\longrightarrow g$

- The Mermin-Wagner theorem applies to quantu



ions)

Quantum-Classical Mapping:

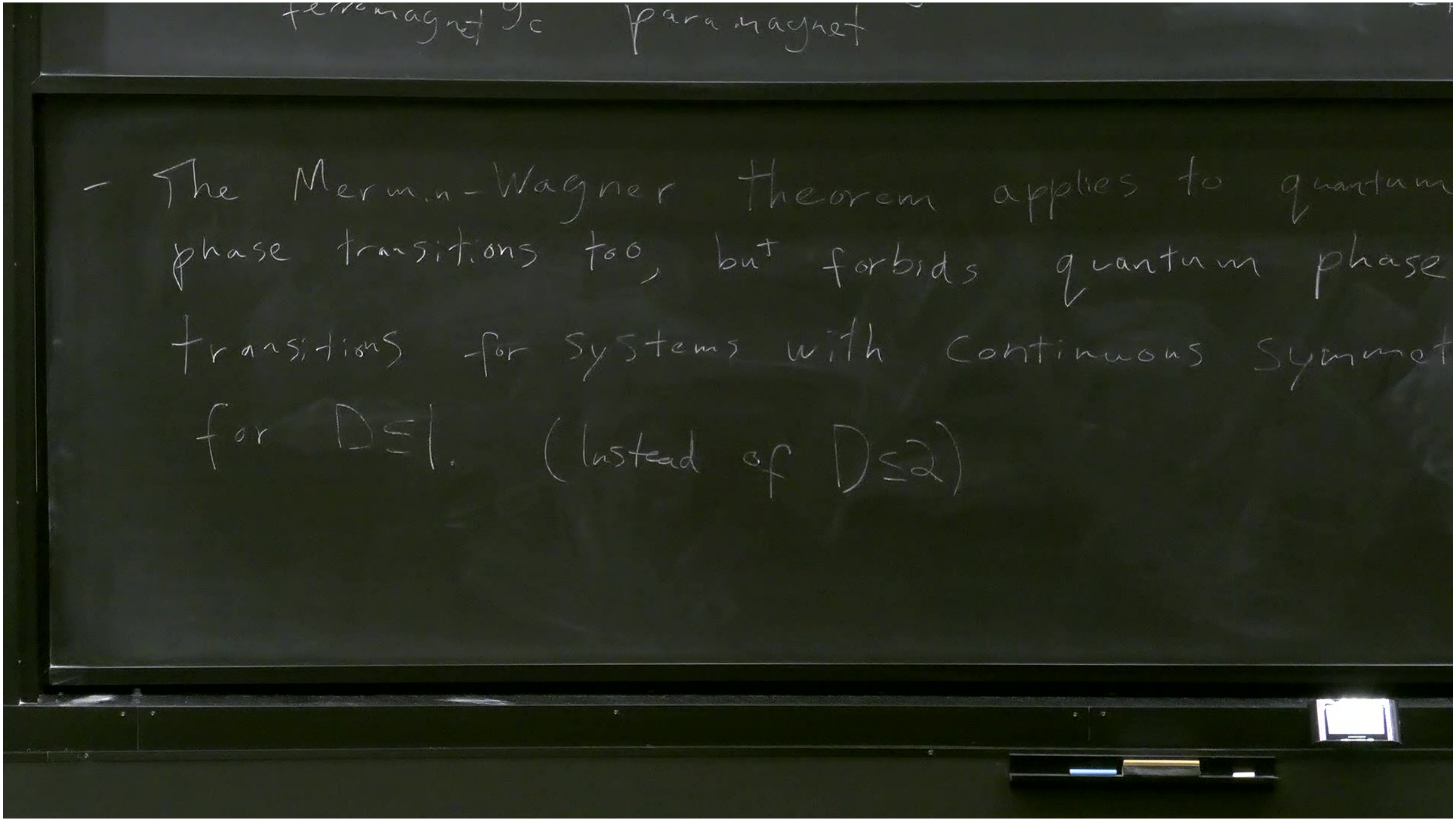
$$Z = \Lambda^{N_\tau \cdot N_s} \sum_{\{\sigma_{i,\tau}\}} e^{-S(\{\sigma,\tau\})}$$

$$S = -\varepsilon J \sum_{\langle i,j\rangle,\tau} \sigma_{i,\tau}\sigma_{j,\tau} - \gamma(g) \sum_{i,\langle\tau,\tau'\rangle} \sigma_{i,\tau} \cdot \sigma_{i,\tau'}$$

$$\gamma(g) = -\frac{1}{2} \ln(\tanh \varepsilon g)$$

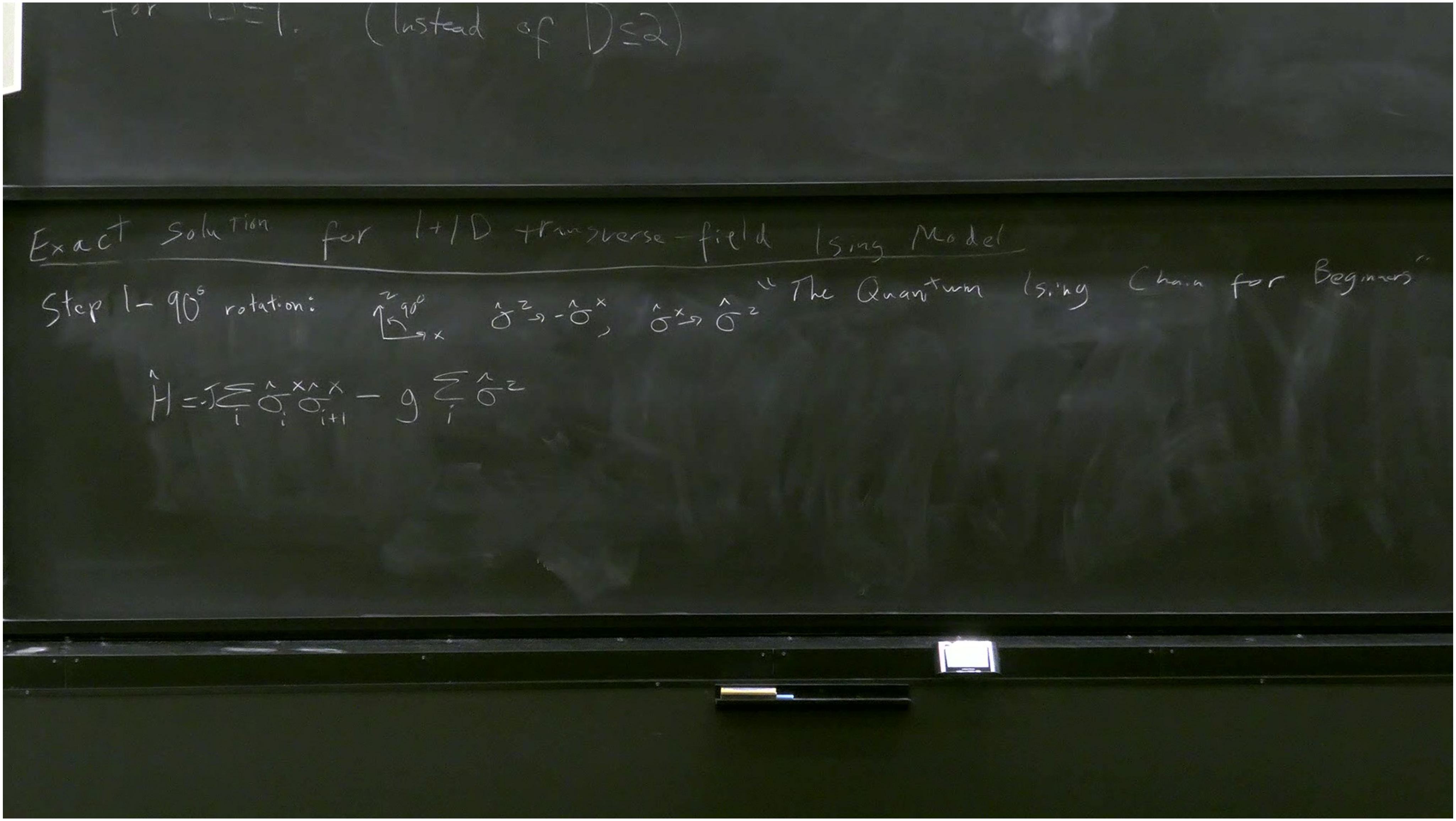
– D+1 dim anisotropic Classical Ising model, $\beta = \varepsilon N_\tau$

quantum



ferromagnet g_c paramagnet

- The Mermin-Wagner theorem applies to quantum phase transitions too, but forbids quantum phase transitions for systems with continuous symmet for $D \leq 1$. (Instead of $D \leq 2$)

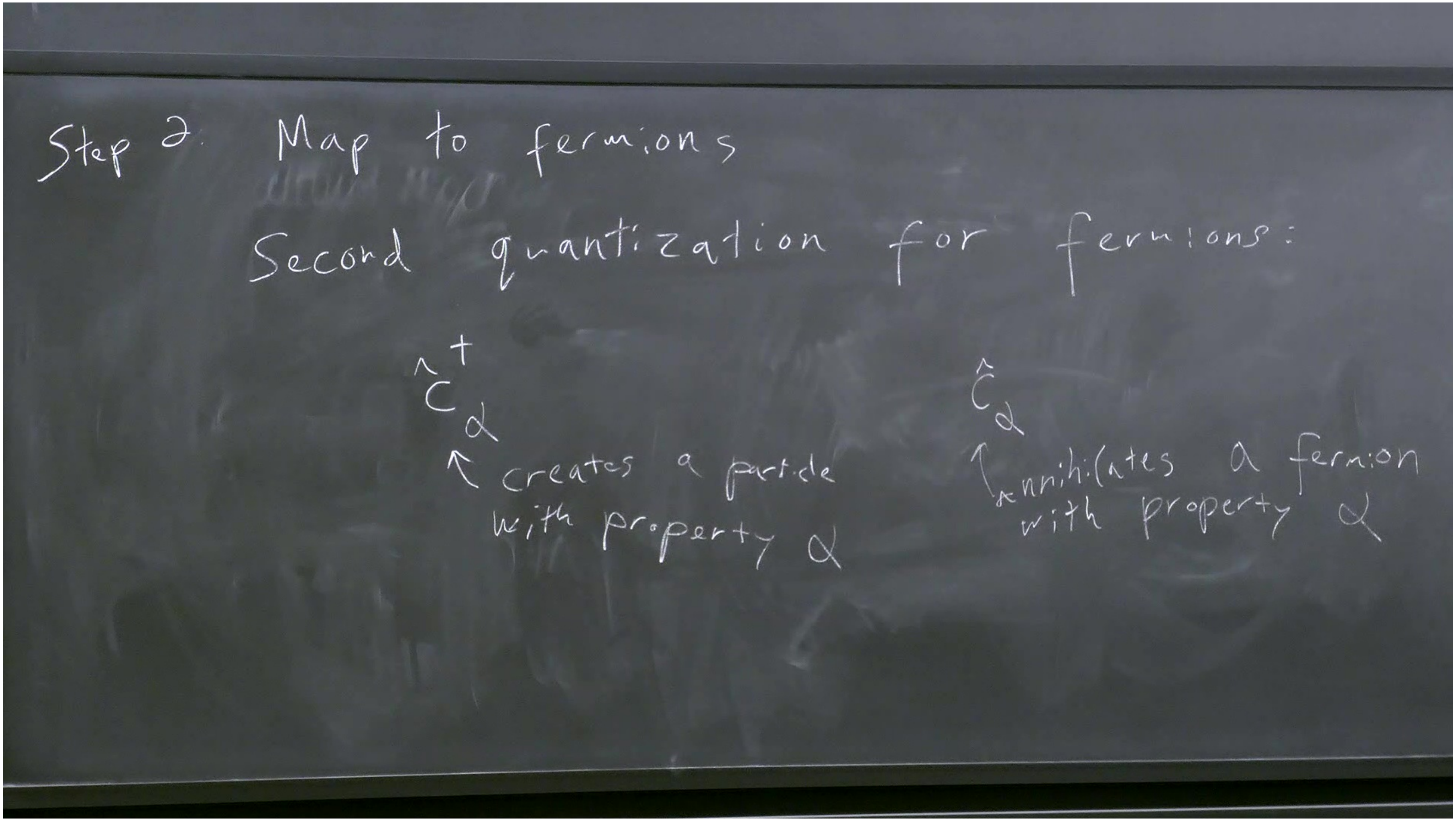


for $D=1$. (Instead of $D\leq2$)

Exact solution for 1+1D transverse-field Ising Model

Step 1 – 90° rotation: $\overset{90^\circ}{z \to x}$ $\hat{\sigma}^z \to -\hat{\sigma}^x$, $\hat{\sigma}^x \to \hat{\sigma}^z$ "The Quantum Ising Chain for Beginners"

$$\hat{H} = -J\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - g\sum_i \hat{\sigma}^z$$



Step 2: Map to fermions

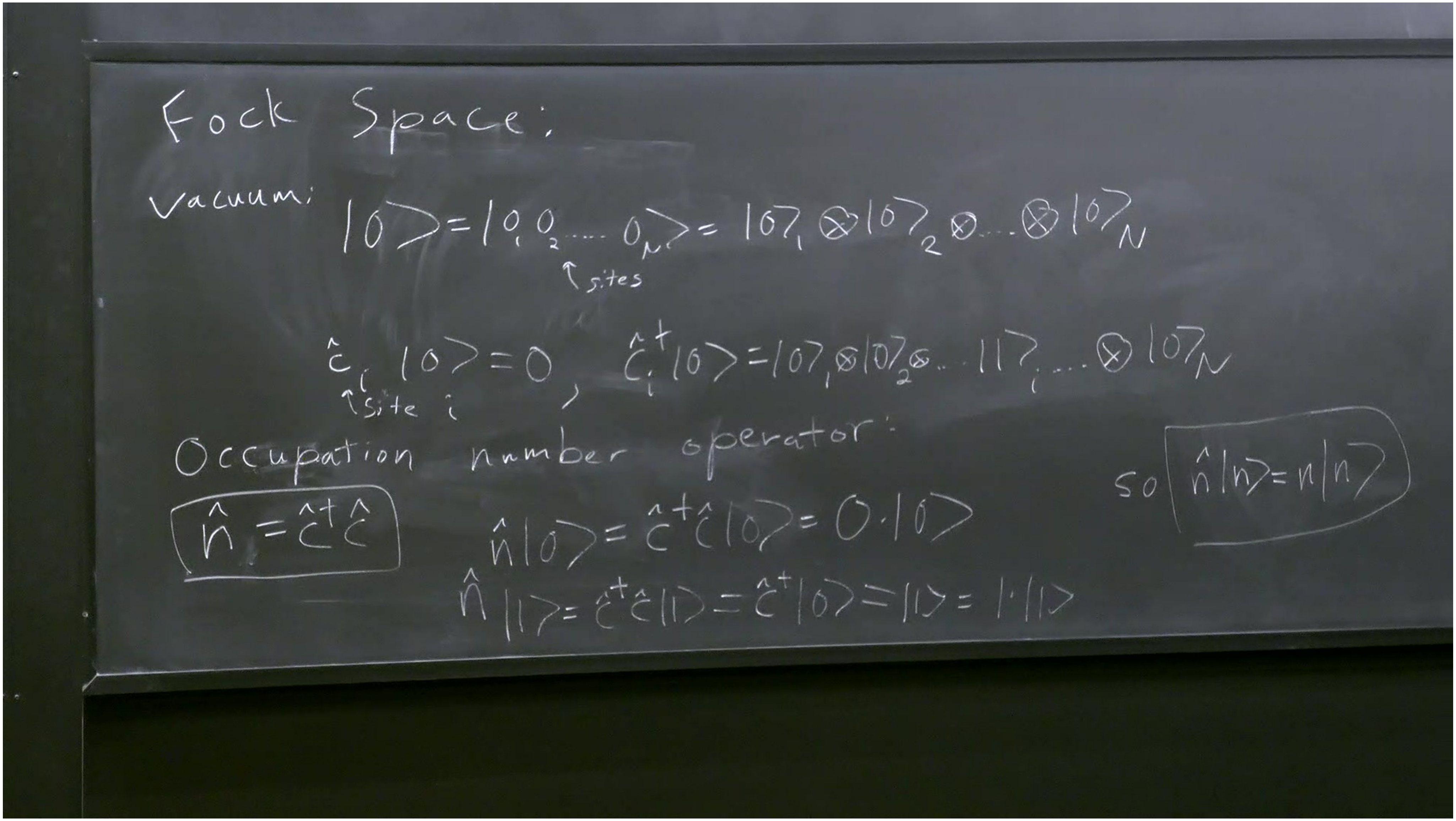
Second quantization for fermions:

$\hat{c}_\alpha^\dagger$

↖ creates a particle with property α

$\hat{c}_\alpha$

↑ annihilates a fermion with property α



Fock Space:

vacuum: $|0\rangle = |0_1 0_2 \ldots 0_N\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \ldots \otimes |0\rangle_N$

↑ sites

$\hat{c}_i |0\rangle = 0$, $\hat{c}_i^{+} |0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \ldots |1\rangle_i \ldots \otimes |0\rangle_N$

↑ site i

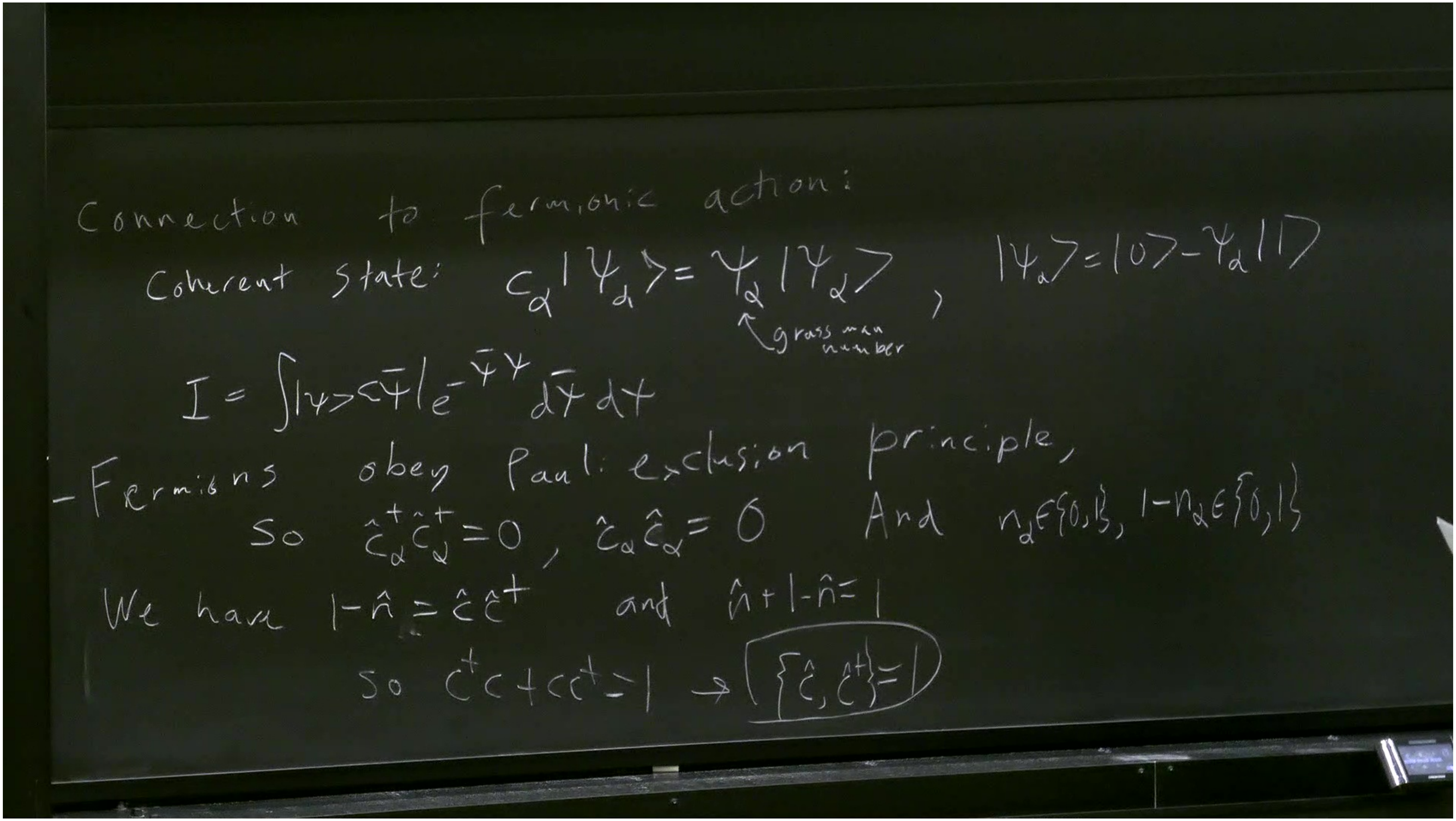
Occupation number operator:

$\boxed{\hat{n} = \hat{c}^{+} \hat{c}}$

$\hat{n}|0\rangle = \hat{c}^{+}\hat{c}|0\rangle = 0 \cdot |0\rangle$

$\hat{n}|1\rangle = \hat{c}^{+}\hat{c}|1\rangle = \hat{c}^{+}|0\rangle = |1\rangle = 1 \cdot |1\rangle$

so $\boxed{\hat{n}|n\rangle = n|n\rangle}$



Connection to fermionic action:

Coherent state: $c_\alpha |\psi_\alpha\rangle = \psi_\alpha |\psi_\alpha\rangle$, $|\psi_\alpha\rangle = |0\rangle - \psi_\alpha |1\rangle$

grassman number

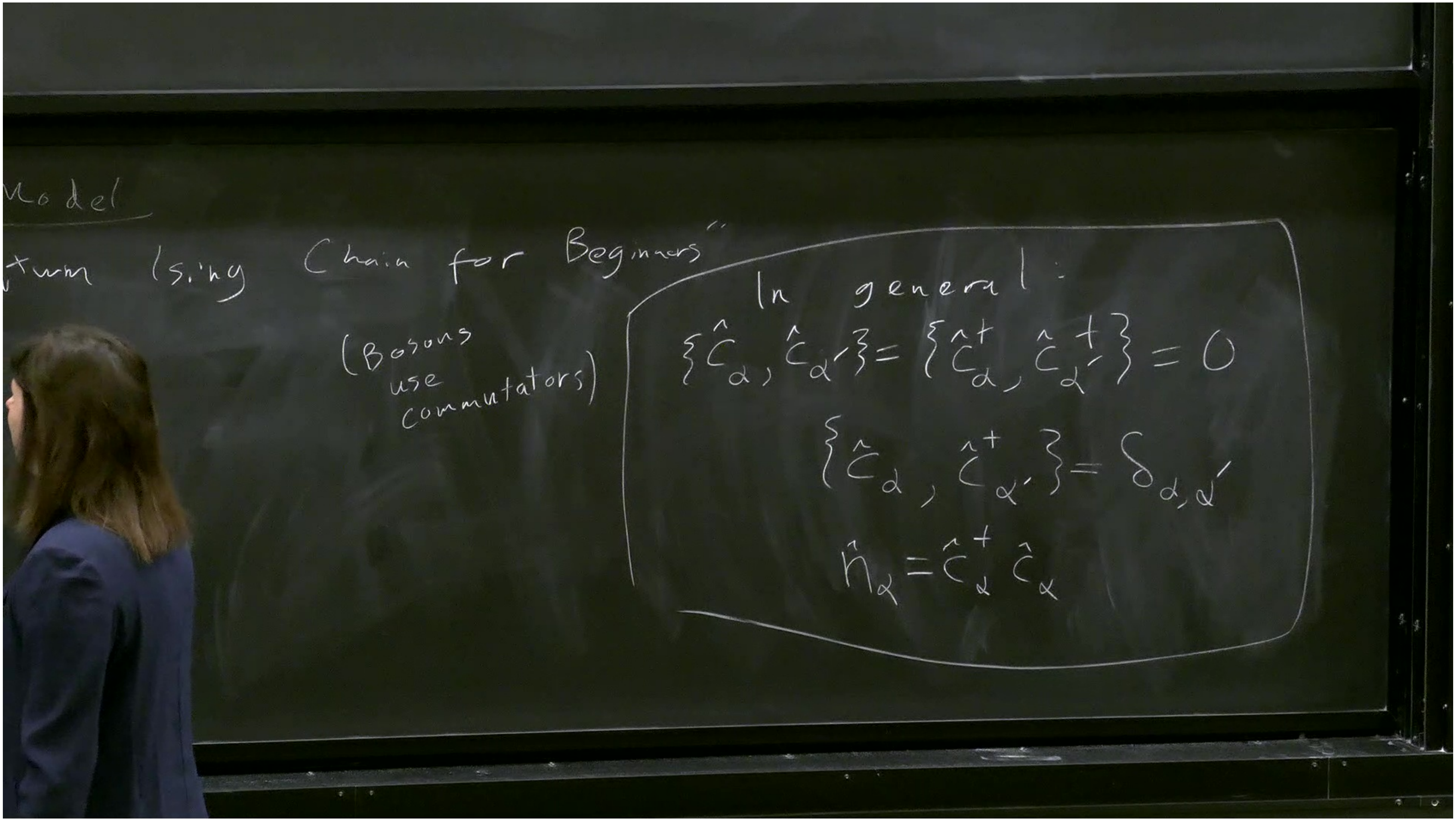
$$I = \int |\psi\rangle\langle\bar\psi| e^{-\bar\psi\psi} \, d\bar\psi \, d\psi$$

- Fermions obey Pauli exclusion principle,

So $\hat{c}^\dagger_\alpha \hat{c}^\dagger_\alpha = 0$, $\hat{c}_\alpha \hat{c}_\alpha = 0$ And $n_\alpha \in \{0,1\}$, $1-n_\alpha \in \{0,1\}$

We have $1-\hat{n} = \hat{c}\hat{c}^\dagger$ and $\hat{n} + 1 - \hat{n} = 1$

so $c^\dagger c + c c^\dagger = 1 \rightarrow \boxed{\{\hat{c}, \hat{c}^\dagger\} = 1}$



odel

tum Ising Chain for Beginners"

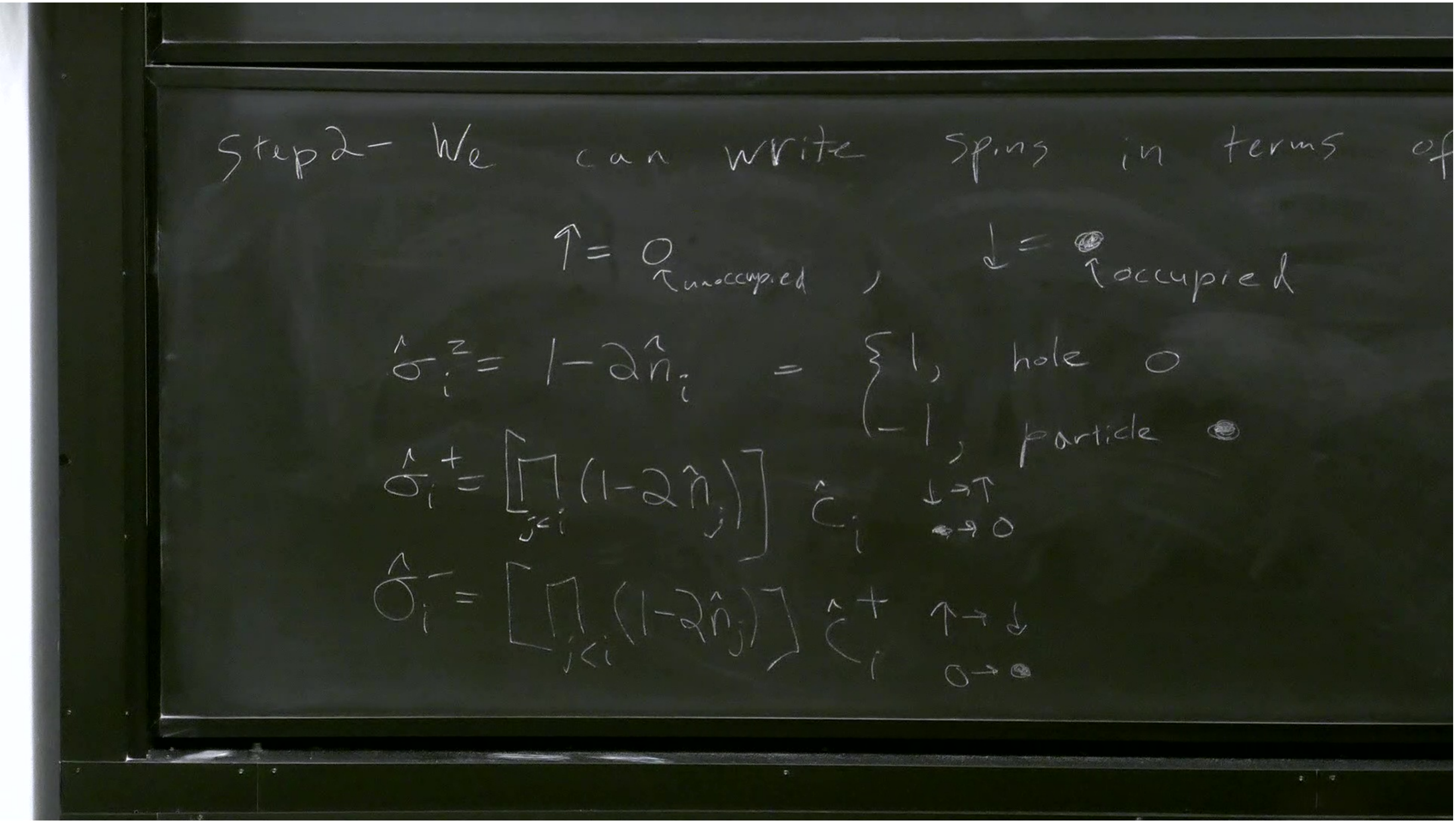
(Bosons use commutators)

In general:

$$\{\hat{c}_\alpha, \hat{c}_{\alpha'}\} = \{\hat{c}^{+}_\alpha, \hat{c}^{+}_{\alpha'}\} = 0$$

$$\{\hat{c}_\alpha, \hat{c}^{+}_{\alpha'}\} = \delta_{\alpha,\alpha'}$$

$$\hat{n}_\alpha = \hat{c}^{+}_\alpha \hat{c}_\alpha$$



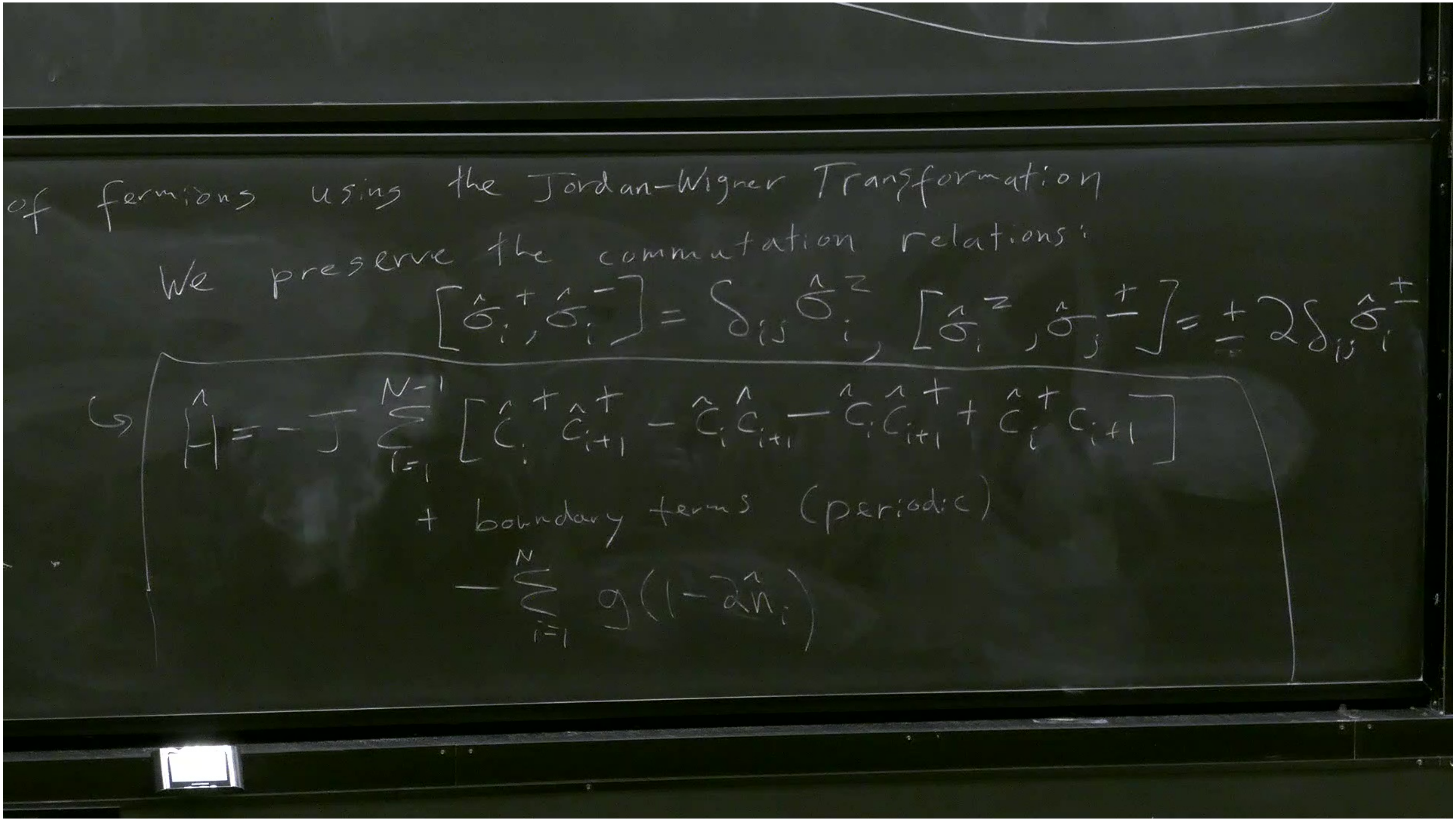
Step 2 - We can write spins in terms of

$\uparrow = \underset{\uparrow \text{unoccupied}}{○}$, $\downarrow = \underset{\uparrow \text{occupied}}{●}$

$$\hat{\sigma}_i^z = 1 - 2\hat{n}_i = \begin{cases} 1, & \text{hole } ○ \\ -1, & \text{particle } ● \end{cases}$$

$$\hat{\sigma}_i^+ = \left[\prod_{j<i} (1 - 2\hat{n}_j)\right] \hat{c}_i \qquad \begin{matrix} \downarrow \rightarrow \uparrow \\ ● \rightarrow ○ \end{matrix}$$

$$\hat{\sigma}_i^- = \left[\prod_{j<i} (1 - 2\hat{n}_j)\right] \hat{c}_i^+ \qquad \begin{matrix} \uparrow \rightarrow \downarrow \\ ○ \rightarrow ● \end{matrix}$$

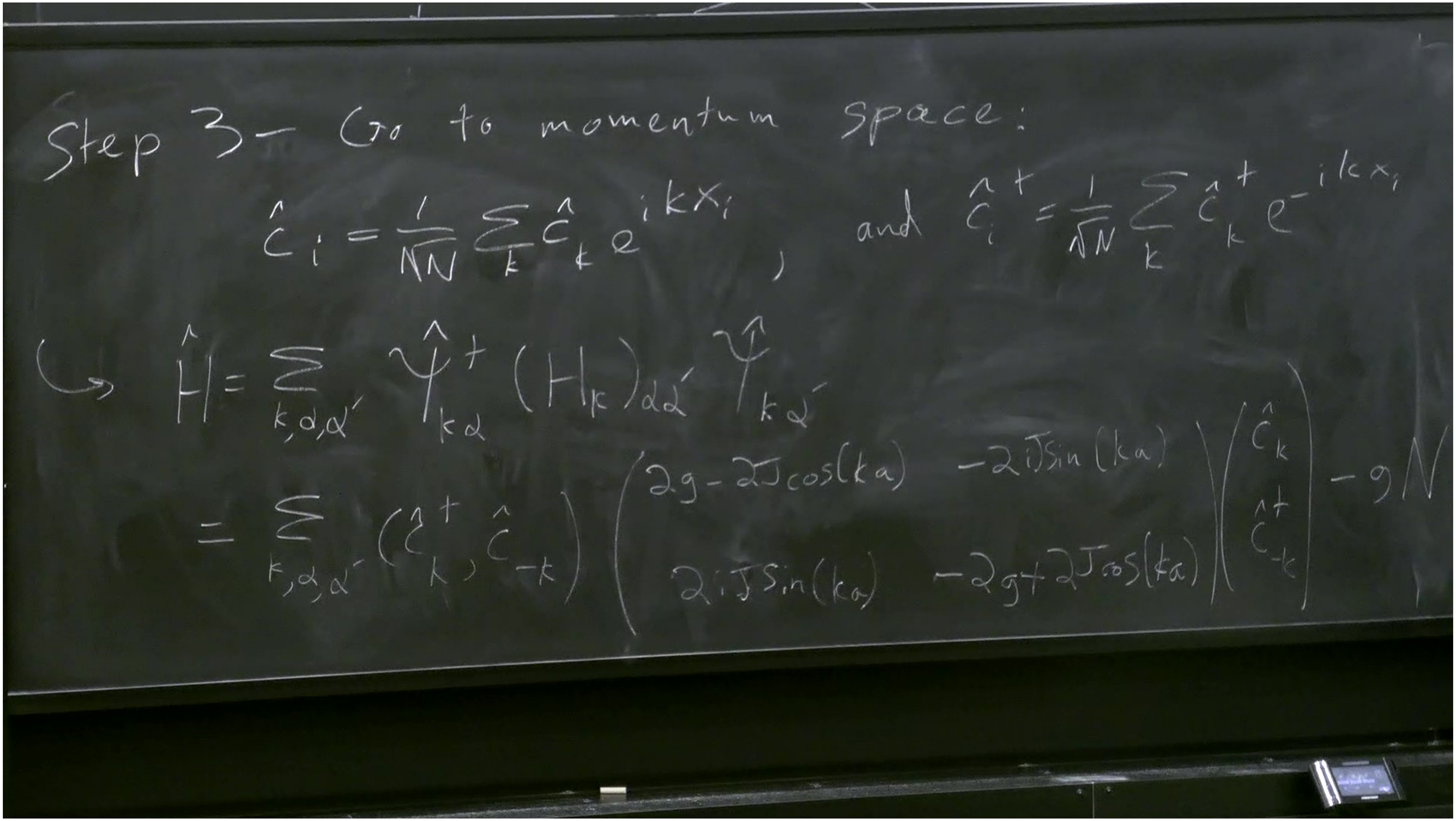


of fermions using the Jordan-Wigner Transformation

We preserve the commutation relations:

$$[\hat{\sigma}_i^+, \hat{\sigma}_i^-] = \delta_{ij}\hat{\sigma}_i^z, \quad [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2\delta_{ij}\hat{\sigma}_i^\pm$$

$$\hookrightarrow \boxed{\hat{H} = -J\sum_{i=1}^{N-1}\left[\hat{c}_i^+\hat{c}_{i+1}^+ - \hat{c}_i\hat{c}_{i+1} - \hat{c}_i\hat{c}_{i+1}^+ + \hat{c}_i^+ c_{i+1}\right] + \text{boundary terms (periodic)} - \sum_{i=1}^{N} g(1-2\hat{n}_i)}$$

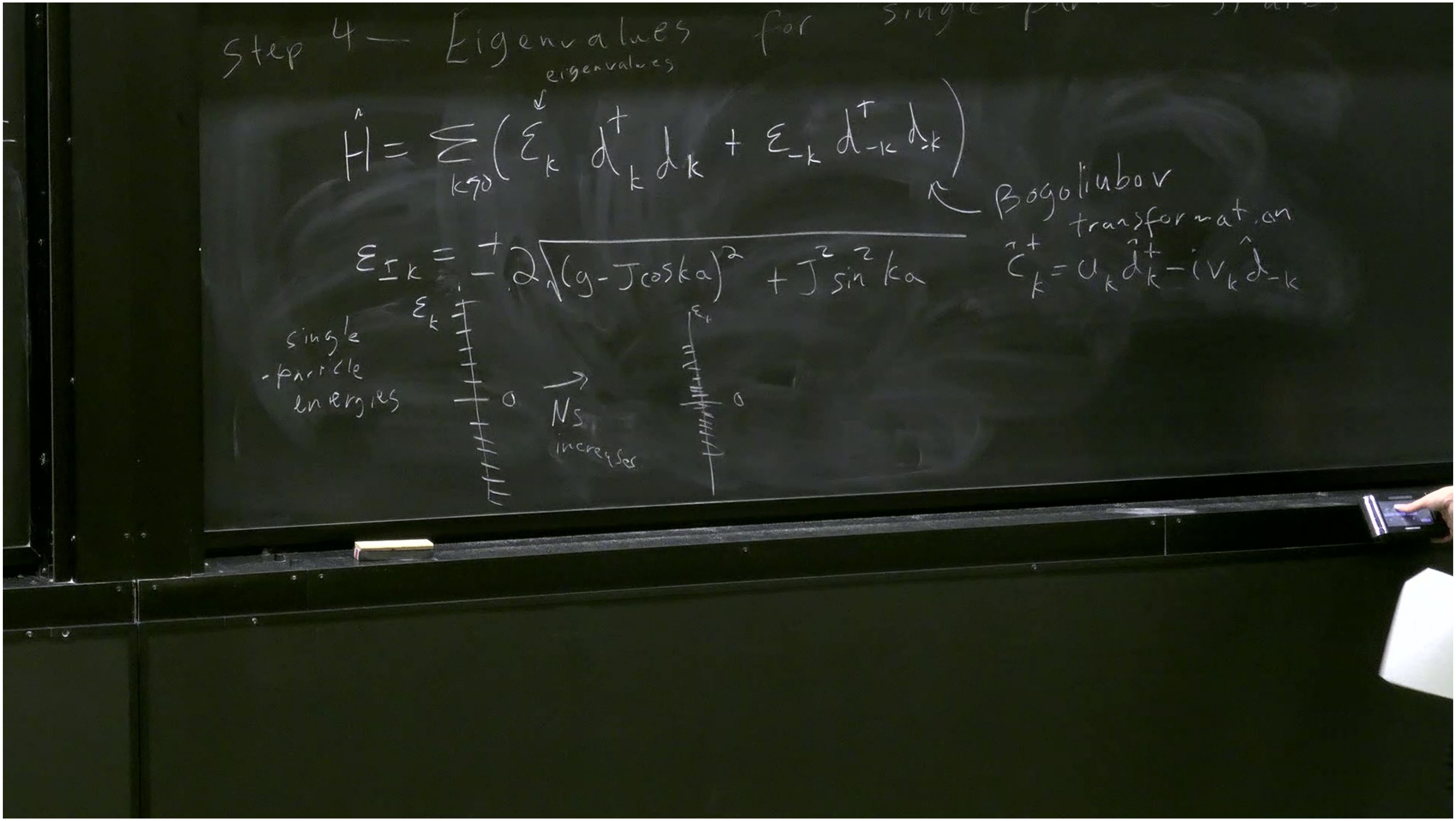


Step 3 – Go to momentum space:

$$\hat{c}_i = \frac{1}{\sqrt{N}} \sum_k \hat{c}_k e^{ikx_i} \quad , \quad \text{and} \quad \hat{c}_i^{\dagger} = \frac{1}{\sqrt{N}} \sum_k \hat{c}_k^{\dagger} e^{-ikx_i}$$

$$\hookrightarrow \quad \hat{H} = \sum_{k,\alpha,\alpha'} \hat{\Psi}^{\dagger}_{k\alpha} (H_k)_{\alpha\alpha'} \hat{\Psi}_{k\alpha'}$$

$$= \sum_{k,\alpha,\alpha'} \left(\hat{c}_k^{\dagger}, \hat{c}_{-k}\right) \begin{pmatrix} 2g - 2J\cos(ka) & -2iJ\sin(ka) \\ 2iJ\sin(ka) & -2g + 2J\cos(ka) \end{pmatrix} \begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^{\dagger} \end{pmatrix} - gN$$



Step 4 — Eigenvalues for single-

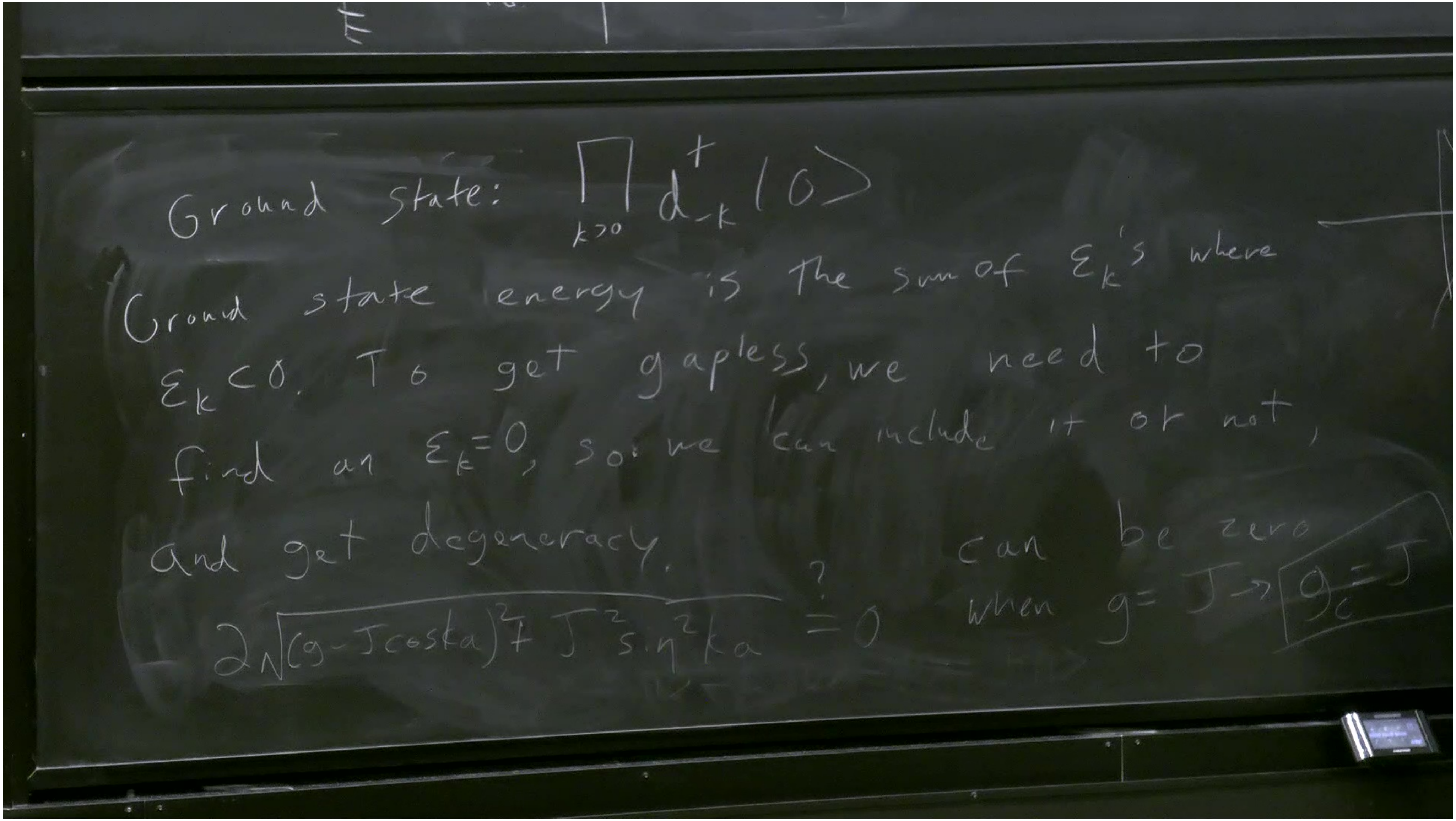
eigenvalues

$$\hat{H} = \sum_{k>0} \left(\varepsilon_k \, d^\dagger_k d_k + \varepsilon_{-k} \, d^\dagger_{-k} d_{-k} \right)$$

Bogoliubov transformation

$$\hat{c}^\dagger_k = u_k \hat{d}^\dagger_k - i v_k \hat{d}_{-k}$$

$$\varepsilon_{\pm k} = \pm 2\sqrt{(g - J\cos ka)^2 + J^2 \sin^2 ka}$$

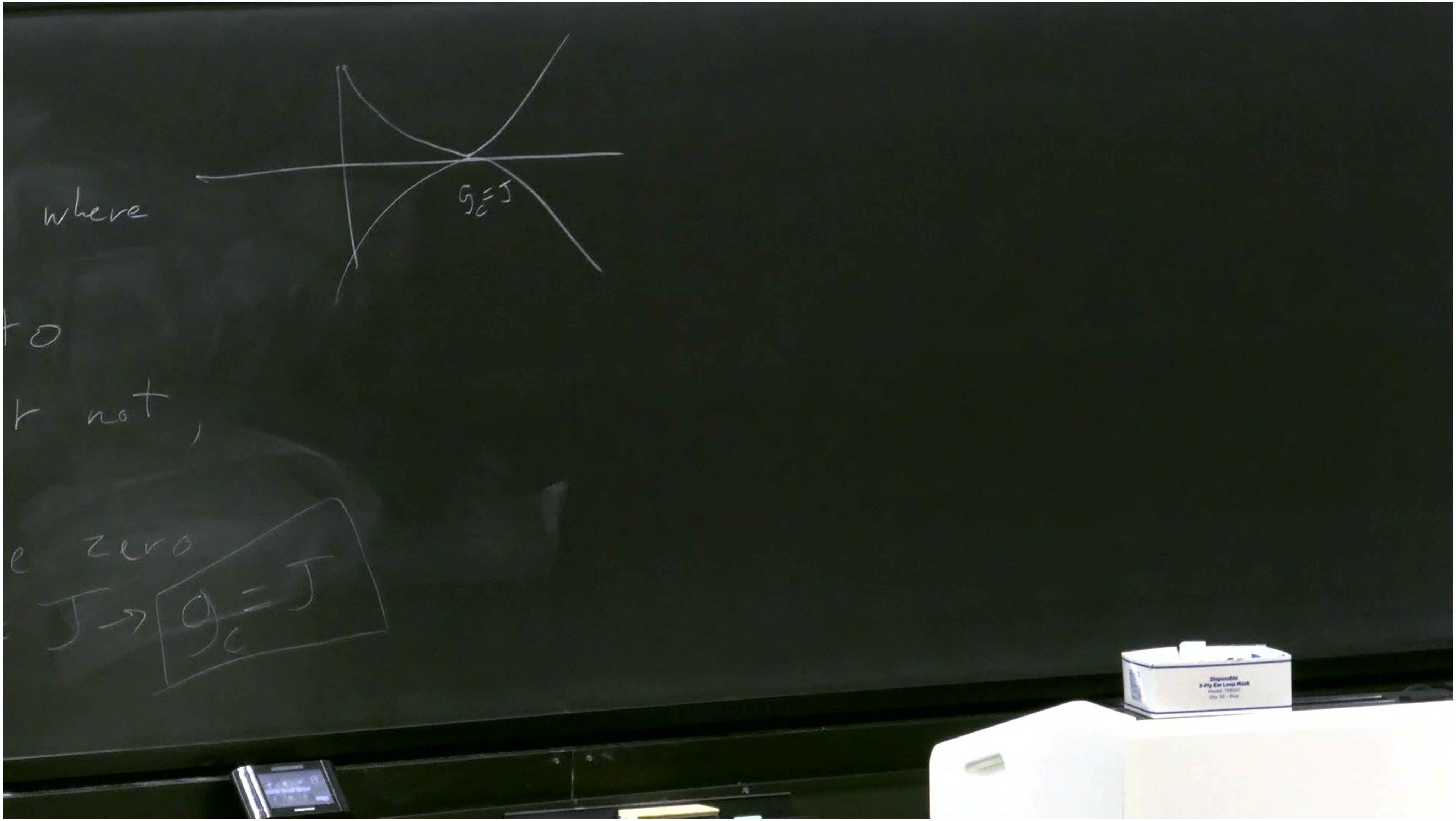


Ground state: $\prod_{k>0} d_{-k}^{\dagger} |0\rangle$

Ground state energy is the sum of ε_k's where $\varepsilon_k < 0$. To get gapless, we need to find an $\varepsilon_k = 0$, so we can include it or not, and get degeneracy.

$$2\sqrt{(g-J\cos ka)^2 + J^2\sin^2 ka} \overset{?}{=} 0$$

can be zero when $g = J \to \boxed{g_c = J}$



where

to

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e zero

$J \rightarrow \boxed{g_c = J}$

- For quantum analog of 2D classical XY,

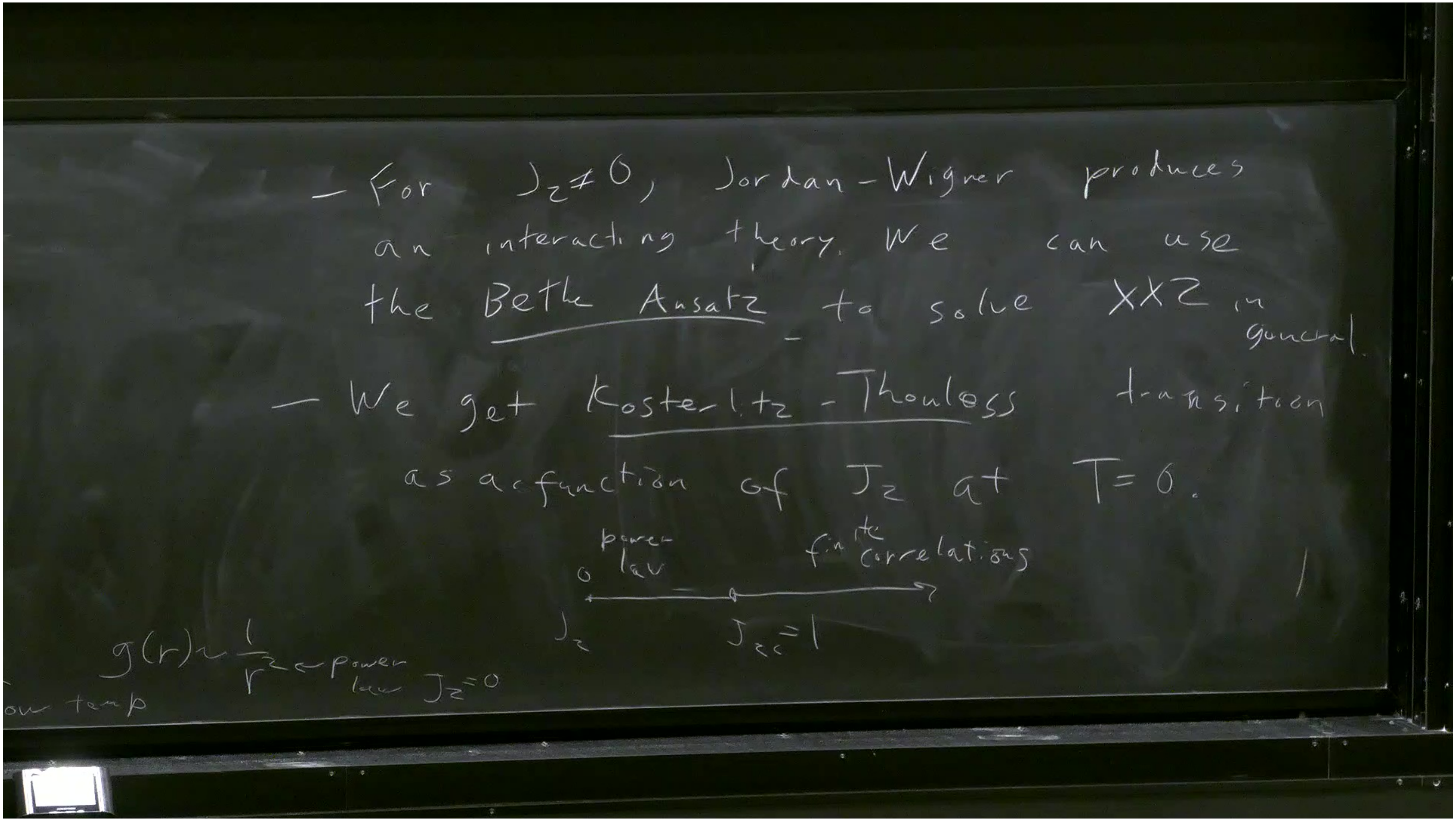
XXZ chain in 1D:

$$\hat{H} = \sum_{i=1}^{N} \left[J_x \left(\hat{S}^x_i \hat{S}^x_{i+1} + \hat{S}^y_i \hat{S}^y_{i+1} \right) \right] + J_z \hat{S}^z_i \hat{S}^z_{i+1}$$

When $J_z = 0$, Jordan-Wigner gives

$$\hat{H} = -\frac{J}{2} \sum_{i}^{N} \left[\hat{c}_i^\dagger \hat{c}_{i+1} + \hat{c}_{i+1}^\dagger \hat{c}_i \right]$$

→ free Dirac fermions in continuum limit $\varepsilon_k \sim k$ low



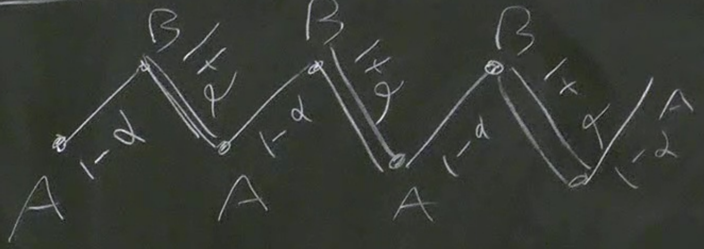
- For $J_z \neq 0$, Jordan-Wigner produces an interacting theory. We can use the **Bethe Ansatz** to solve XXZ in general.

- We get **Kosterlitz-Thouless** transition as a function of J_z at $T = 0$.

$g(r) \sim \frac{1}{r^2}$ ← power law $J_z = 0$

ow temp

– SSH Model: simple topological model



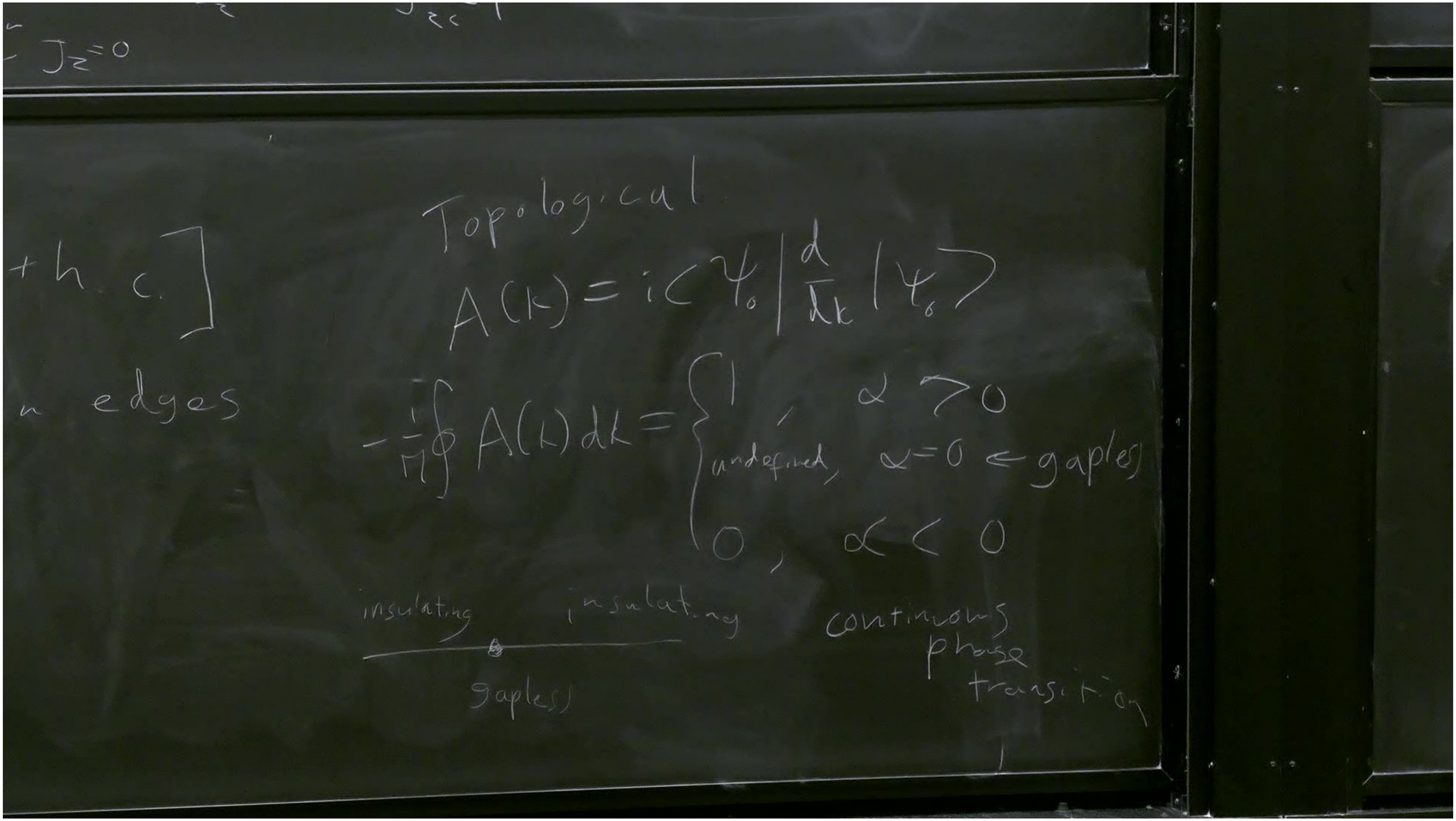
polycetylene polymer chain

open boundary conditions

$$\hat{H} = t\sum_{n=1}^{N} (1$$

$\alpha = 1 \rightarrow$ un

$\alpha = -1 \rightarrow$ all



$J_z = 0$

$+ h. c.]$

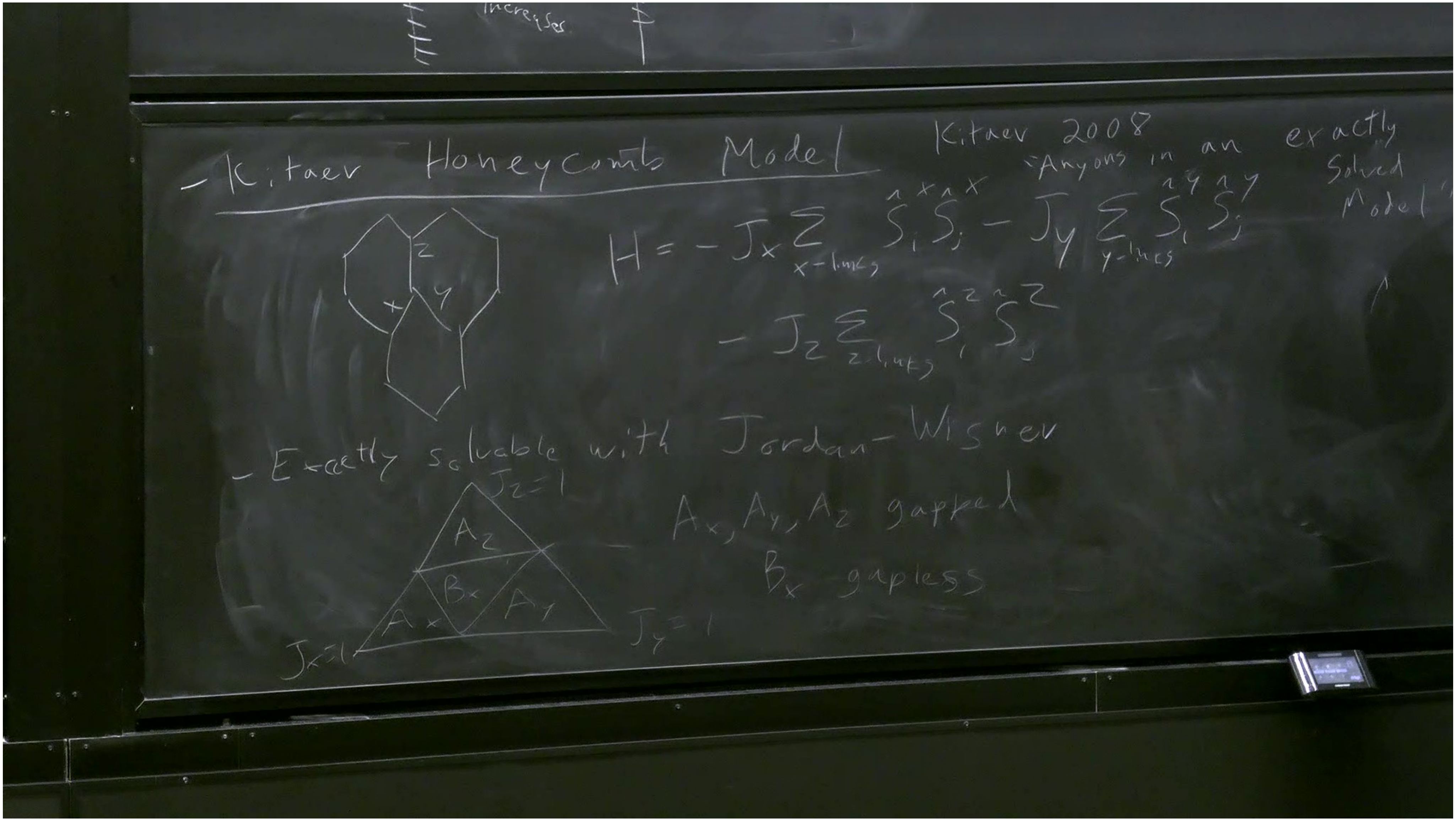
edges

Topological

$$A(k) = i\langle \psi_0 | \frac{d}{dk} | \psi_0 \rangle$$

$$-\frac{i}{\pi}\oint A(k)\,dk = \begin{cases} 1, & \alpha > 0 \\ \text{undefined}, & \alpha = 0 \leftarrow \text{gapless} \\ 0, & \alpha < 0 \end{cases}$$

continuous phase transition



- Kitaev Honeycomb Model

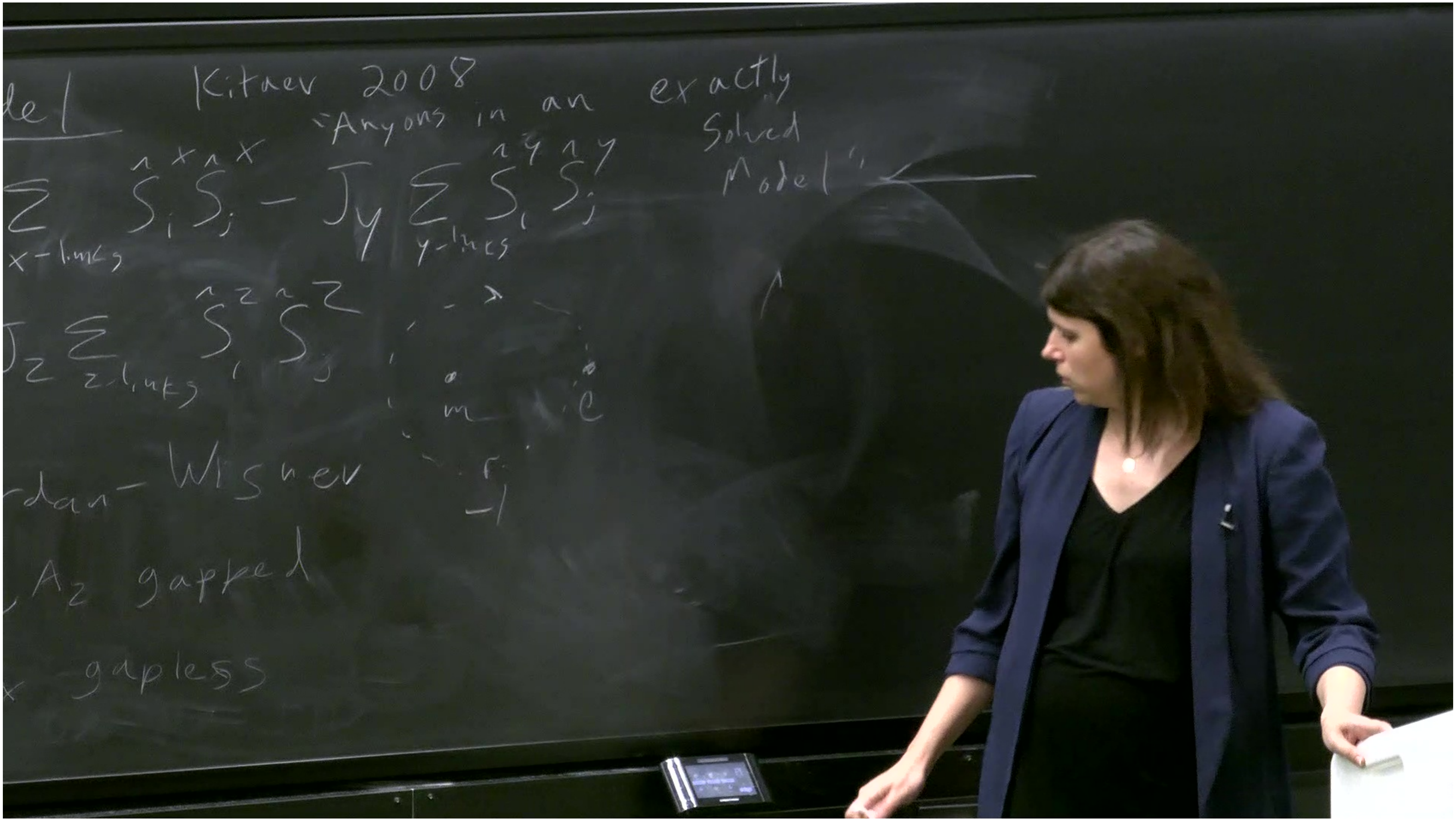
Kitaev 2008 "Anyons in an exactly Solved Model"

$$H = -J_x \sum_{x\text{-links}} \hat{S}_i^x \hat{S}_j^x - J_y \sum_{y\text{-links}} \hat{S}_i^y \hat{S}_j^y - J_z \sum_{z\text{-links}} \hat{S}_i^z \hat{S}_j^z$$

- Exactly solvable with Jordan-Wigner

A_x, A_y, A_z gapped

B_x gapless



del

Kitaev 2008 "Anyons in an exactly Solved Model"

$$\sum_{x\text{-links}} \hat{S}_i^x \hat{S}_j^x - J_y \sum_{y\text{-links}} \hat{S}_i^y \hat{S}_j^y$$

$$J_z \sum_{z\text{-links}} \hat{S}_i^z \hat{S}_j^z$$

rdan-Wigner

A_z gapped

x gapless