Title: Cluster Reductions, Mutations, and q-Painlev'e Equations

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Abstract:

In my talk I will explain how to extend the Goncharov-Kenyon class of cluster integrable systems by their Hamiltonian reductions. In particular, this extension allows to fill in the gap in cluster construction of the q-difference Painlev'e equations. Isomorphisms of reduced Goncharov-Kenyon integrable systems are given by mutations in another, dual in non-obvious sense, cluster structure. These dual mutations cause certain polynomial mutations of dimer partition functions and polygon mutations of the corresponding decorated Newton polygons.

Cluster Reductions, Mutations, and q-Painlevé Equations

Mykola Semenyakin

based on upcoming paper with
Mikhail Bershtein, Pavlo Gavrylenko and Andey Marshakov

PI, October 2024

q-Painlevé equations

q-Painlevé equations are classified by their symmetries

$$\begin{array}{ccccccccccccccccc} & & & & & & & & & & & & & \nearrow & (E_1^{(1)})_{|\alpha^2|=8} & \searrow & \\ E_8^{(1)} & \longrightarrow & E_7^{(1)} & \longrightarrow & E_6^{(1)} & \longrightarrow & E_5^{(1)} & \longrightarrow & E_4^{(1)} & \longrightarrow & E_3^{(1)} & \longrightarrow & E_2^{(1)} & \longrightarrow & E_1^{(1)} & & E_0^{(1)} \end{array}$$

where

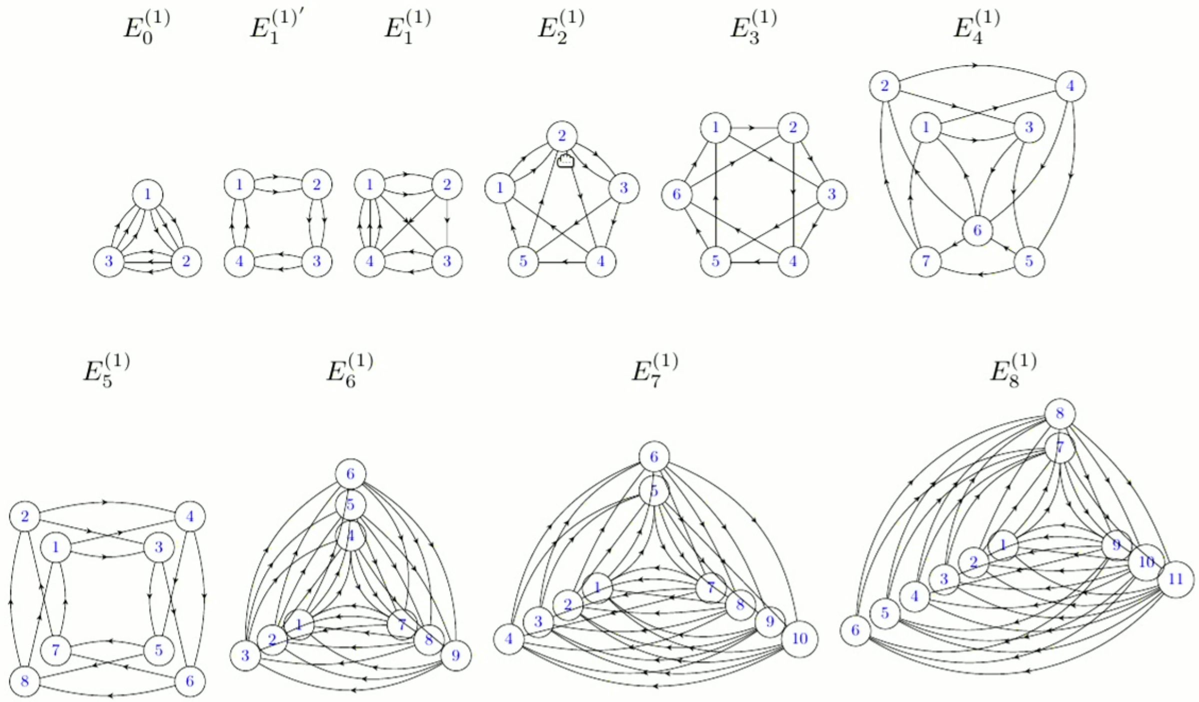
$$E_5^{(1)} = D_5^{(1)},\ E_4^{(1)} = A_4^{(1)},\ E_3^{(1)} = (A_2+A_1)^{(1)},\ E_2^{(1)} = (A_1+A_1)^{(1)},\ E_1^{(1)} = A_1^{(1)}$$

Reflections s_i act on root variables a_j by

$$s_i(a_j) = a_j a_i^{-C_{ij}},\ i = 0, \dots, n,$$

and birationally on (log-)Darboux coordinates $(\lambda, \mu) \in (\mathbb{C}^*)^2$

Painlevé quivers



Theorem (BGM). All q-Painlevé equations can be realized by actions of cluster MCGs of $\mathcal{X}$-cluster varieties corresponding to quivers above

Goncharov-Kenyon integrable systems

GK: Newton polygons $\to \mathcal{X}$ cluster variety + Hamiltonians $\mathcal{H}_{a,b}$ on $\mathcal{X}$

Spectral curve $\overline{\mathcal{C}}$ is compactification of

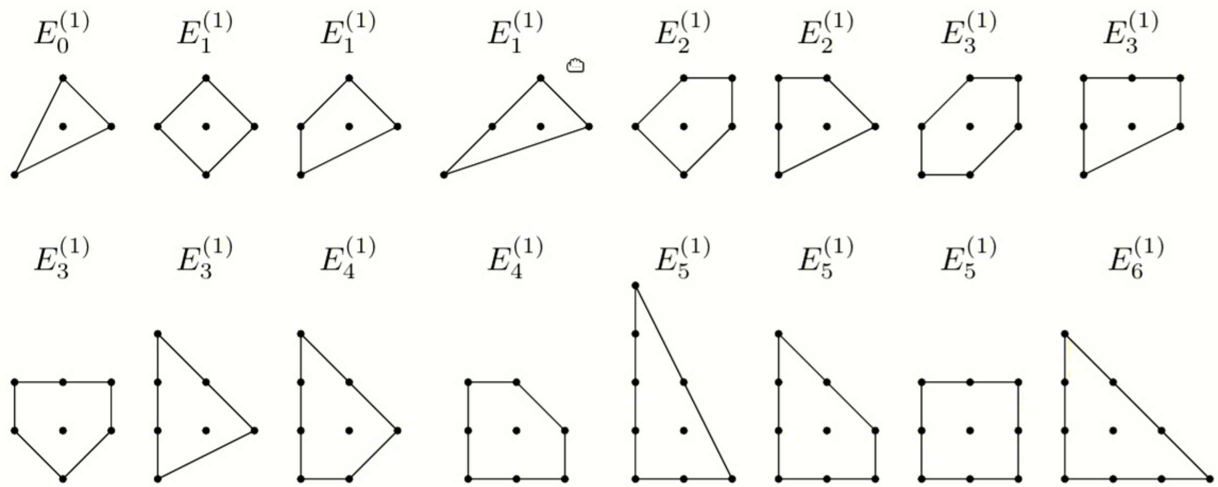
$$\mathcal{C} = \{P(\lambda, \mu) = 0\} \subset \mathbb{C} \times \mathbb{C}$$

$$P(\lambda, \mu) = \sum_{(a,b)\in N} \mathcal{H}_{a,b} \lambda^a \mu^b$$

Genus of the curve: $g(\overline{\mathcal{C}}) =$ number of internal points of $N = I$

This curves are Seiberg-Witten curves for $5d$ $\mathcal{N} = 1$ theories.
Cluster varieties $\mathcal{X}$ are conjecturally Coulomb branches on $\mathbb{R}^3 \times (S^1)^2$.

Reflexive polygons



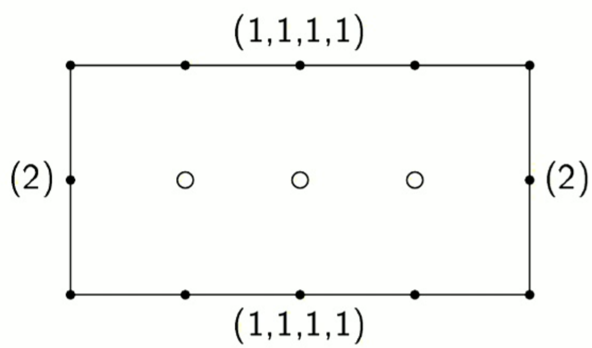
Question: where are the Newton polygons for $E_7^{(1)}$ and $E_8^{(1)}$?

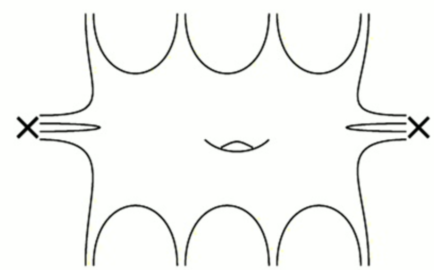
Decorated Newton polygons

A *decorated Newton polygon* is a pair (N, H)

- Convex integral polygon N
- Set $\mathrm{H} = (H_E \mid E \in \text{sides of } N)$ of partitions $H_E = \{h_{E,i}\}$ of $|E|_{\mathbb{Z}}$.

Decorations prescribe singularities on $\overline{C}$ of type $x^{h_{E,i}} = y^{h_{E,i}}$





Genus of the curve: $g(\overline{C}) = I - \sum_{E,i} h_{E,i}(h_{E,i} - 1)/2$

Reduced Goncharov-Kenyon integrable systems

Conjecture. Under the certain conditions for decorated Newton polygon there exists integrable system such that

- It is reduction of Goncharov-Kenyon integrable system corresponding to Newton polygon N
- The dimension of the phase space is $\dim \mathcal{X}_{N,\mathrm{H}} - 1$, where

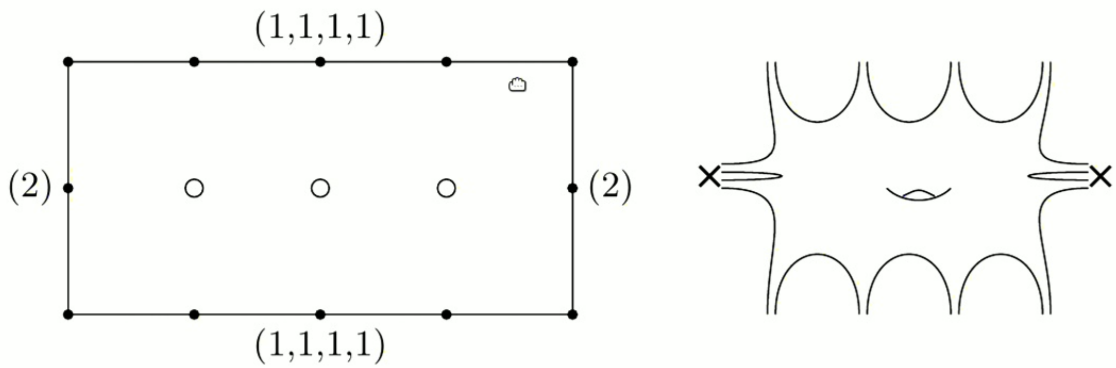
$$\dim \mathcal{X}_{N,\mathrm{H}} = 2\mathrm{Area}(N) - \sum_{E,i} (h_{E,i}^2 - 1)$$

- The rank of the Poisson bracket is

$$\mathrm{rk}\{\cdot,\cdot\}_{\mathcal{X}_{N,\mathbf{H}}} = 2I - \sum_{E,i} h_{E,i}(h_{E,i} - 1)$$

- The phase space is a subvariety given by equation $q = 1$ in the $\mathcal{X}$-cluster variety $\mathcal{X}_{N,\mathrm{H}}$, where q is certain Casimir function

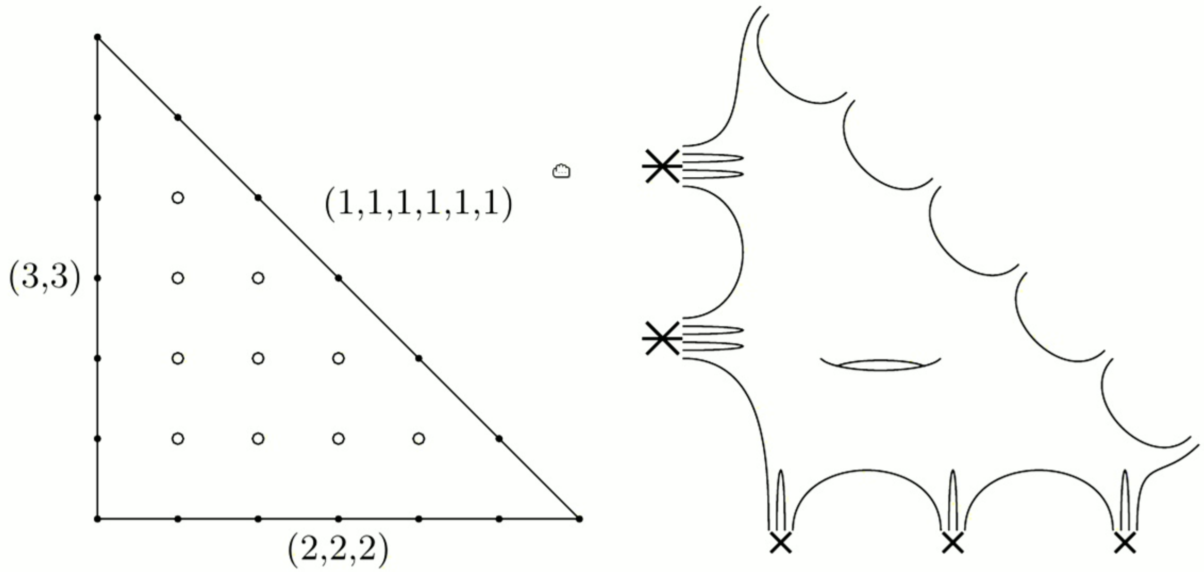
Example: $E_7^{(1)}$



$$\dim \mathcal{X}_{N,\mathbf{H}} = 2\cdot 8 - 2\cdot(2^2-1) = 10 = 8+2$$

$$\mathrm{rk}\{\cdot,\cdot\}_{\mathcal{X}_{N,\mathbf{H}}} = 2\cdot 3 - 2\cdot 2\cdot 1 = 2$$

Example: $E_8^{(1)}$



$$\dim \mathcal{X}_{N,\mathbf{H}} = 2 \cdot 18 - 3 \cdot (2^2 - 1) - 2 \cdot (3^2 - 1) = 11 = 9 + 2$$

$$\mathrm{rk}\{\cdot,\cdot\}_{\mathcal{X}_{N,\mathbf{H}}} = 2 \cdot 10 - 2 \cdot 3 \cdot 2 - 3 \cdot 2 \cdot 1 = 2$$

Analogy: character varieties

Choose the following data

- Rank n local system on $\mathbb{P}^1$ with k punctures p_i
- Conjugacy class C_i of matrices in $\mathrm{GL}(n,\mathbb{C})$ for each p_i

Parametrized by *character varieties*

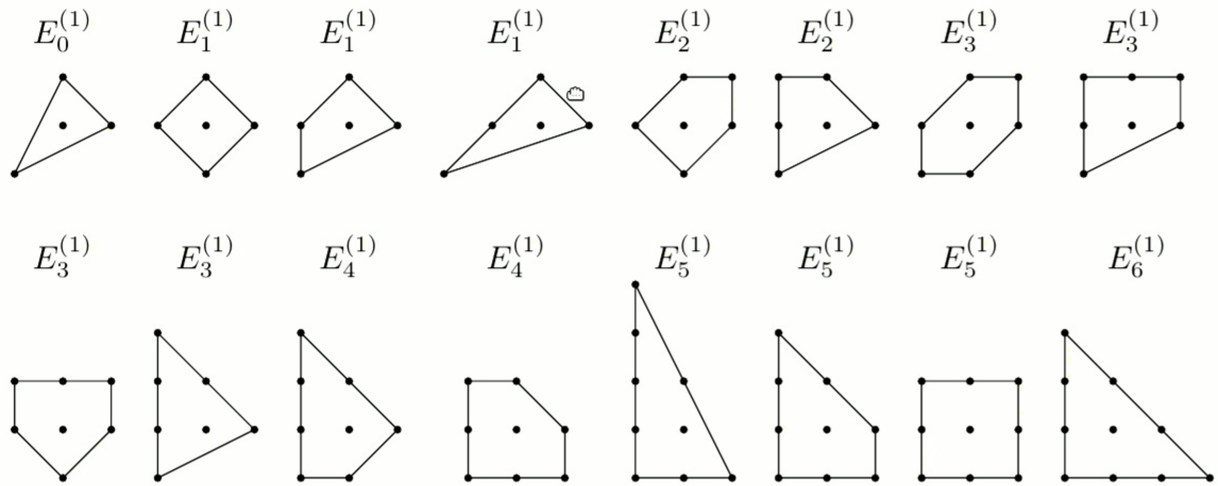
$$\mathcal{M}_C = \{(M_1, ..., M_k) | M_i \in C_i, \prod_i M_i = 1\} / \mathrm{GL}(n,\mathbb{C})$$

Deligne-Simpson problem: when $\mathcal{M}_C$ is non-empty?

Spaces $\mathcal{M}_C$ are Coulomb branches for $4d$ $\mathcal{N}=2$ theories on $\mathbb{R}^3 \times S^1$

Spaces $\mathcal{M}_C$ also have conjectural cluster structure (ongoing by Bershtein, Di Francesco, Ip, Kedem, Prokushkin, Schrader, Shapiro,...)

Reflexive polygons



Question: why there are multiple polygons for one quiver?

Remark. Such issue is known for Seiberg-Witten curves

Polynomial mutations

Decorated Newton polygons $\Leftrightarrow$ Curves with reduction conditions

Singularity of $x^h = y^h$ type on $\mathcal{C}$ $\Leftrightarrow$ exists $\mathrm{SL}_2(\mathbb{Z})$ frame s.t.

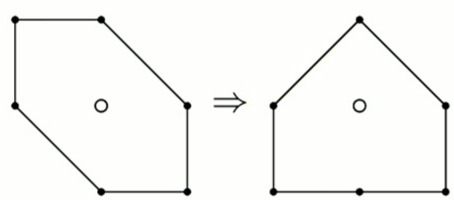
- $P(\lambda,\mu) = \sum_{k=-h'}^{h} \mu^k P_k(\lambda)$
- There exists C such that

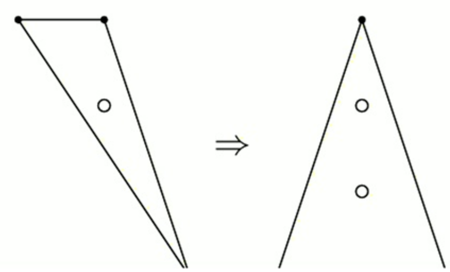
$$(1 + C\lambda^{-1})^k \text{ divides } P_k(\lambda), \quad \text{for all } k > 0$$

The *mutation of the polynomial* P is polynomial $\tilde{P}$ defined by

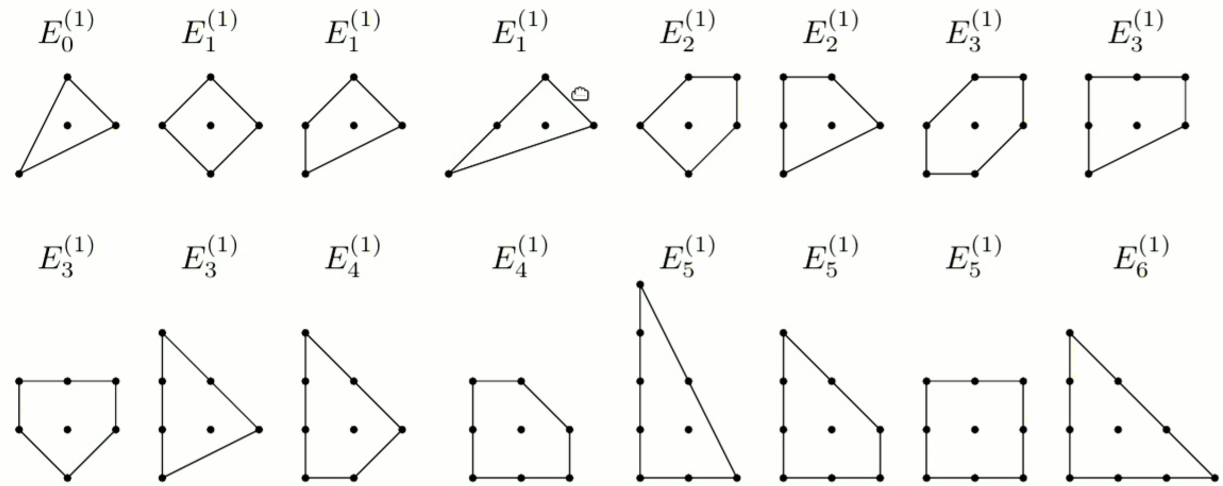
$$\tilde{P}(\lambda,\nu) = P(\lambda,\mu), \quad \text{where } \mu = \frac{\nu}{1 + C\lambda^{-1}}$$

Mutation of the polygon is a a corresponding transformation of N





Reflexive polygons



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"Dual" quivers and their mutations

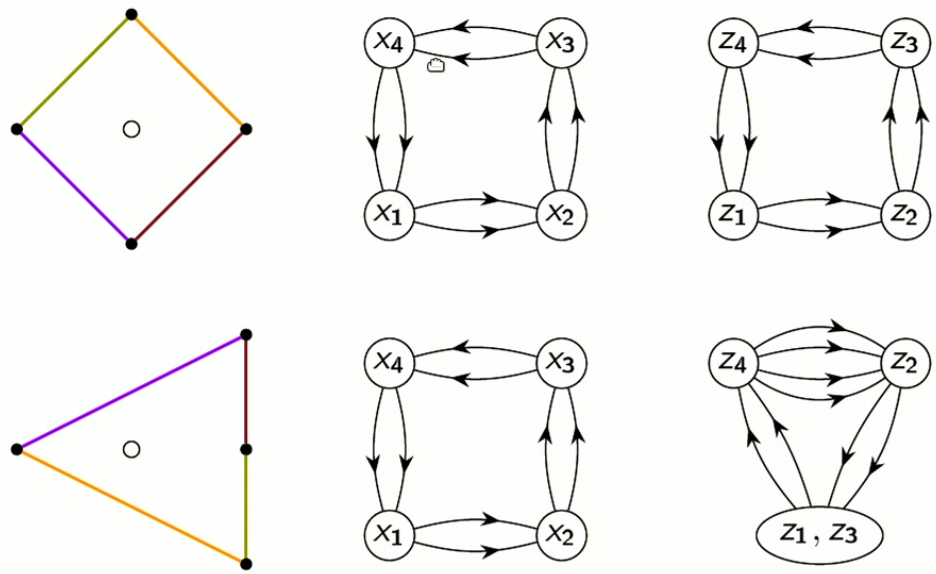
For a decorated Newton polygon (N, H) *dual quiver* is defined as

- Vertices correspond to the sides of N. For any side E there are $\ell(H_E)$ vertices
- Number of edges between vertices corresponding to E and vertices corresponding to E' is $\dfrac{\det(E, E')}{|E|_{\mathbb{Z}}|E'|_{\mathbb{Z}}}$.

Mutations of polygons give rise to mutations of dual quivers of their decorated Newton polygons

Conjecture. "Dual" mutations correspond to isomorphisms of cluster varieties $\mathcal{X}_{N,\mathrm{H}}$ and $\mathcal{X}_{\widetilde{N},\widetilde{\mathrm{H}}}$. Furthermore, they induce isomorphisms of reduced Goncharov-Kenyon integrable systems.

Example of dual mutation



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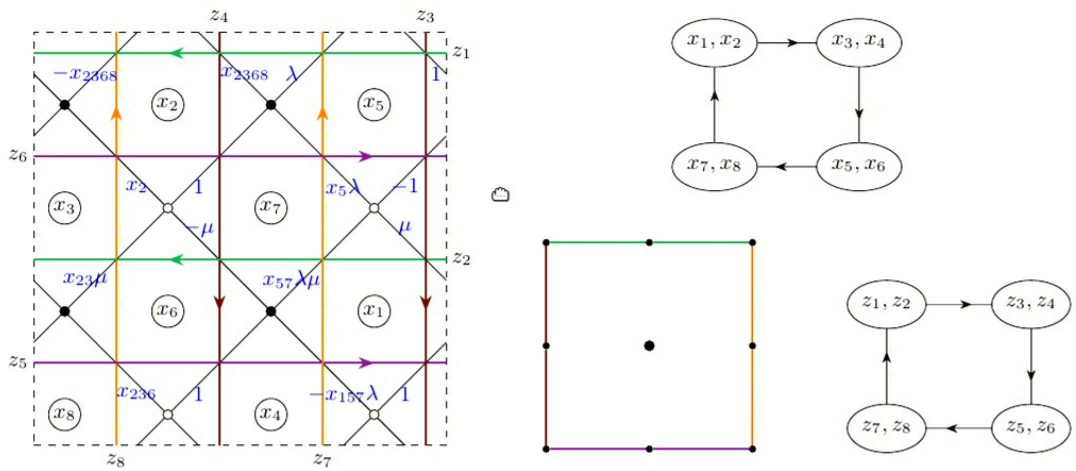
Painlevé self-duality

Theorem. Initial and dual quivers for Painlevé decorated Newton polygons are mutation equivalent

Theorem. For each Painlevé case there is an action of elliptic Weyl group $W \ltimes (P \oplus P)$ on (phase space) $\times (\lambda, \mu)$ that preserves spectral curve polynomial P

Dragons live here



$$\mathcal{Z}(\mathrm{a}|\lambda_d, \mu_d; \lambda, \mu) = f(\mathrm{a}|\lambda, \mu) - f(s_{1523452}\mathrm{a}|\lambda_d, \mu_d),$$

where

$$f(\mathrm{a}|\lambda,\mu) = a_5^{1/4} a_1^{2/4} a_3^{2/4} a_4^{3/4} a_2^{4/4} \Big(a_0^2 a_1 a_2^2 a_3 \lambda \mu^{-1} + a_0 a_3 a_5 \lambda \mu +$$
$$+ a_5 a_3 \lambda^{-1} \mu + a_3 \lambda^{-1} \mu^{-1} + a_3(1+a_5)\lambda^{-1} + a_0(1 + a_0 a_1 a_2^2 a_3^2 a_5)\lambda +$$
$$+ a_0 a_2 a_3 (1+a_1)\mu^{-1} + a_3 a_5 (1+a_0)\mu \Big)$$

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Outline

- There exists an extension of the class of Goncharov-Kenyon integrable systems by their Hamiltonian reductions
- Isomorphisms of such reductions are mutations in a "dual" cluster structure
- All q-Painlevé equations are deautonomizations of reduced GK integrable systems
- Initial and dual cluster structures coincide in Painlevé cases. The groups of automorphisms of corresponding integrable systems contain elliptic Weyl groups