Title: Embeddings between Coulomb branches of quiver gauge theories

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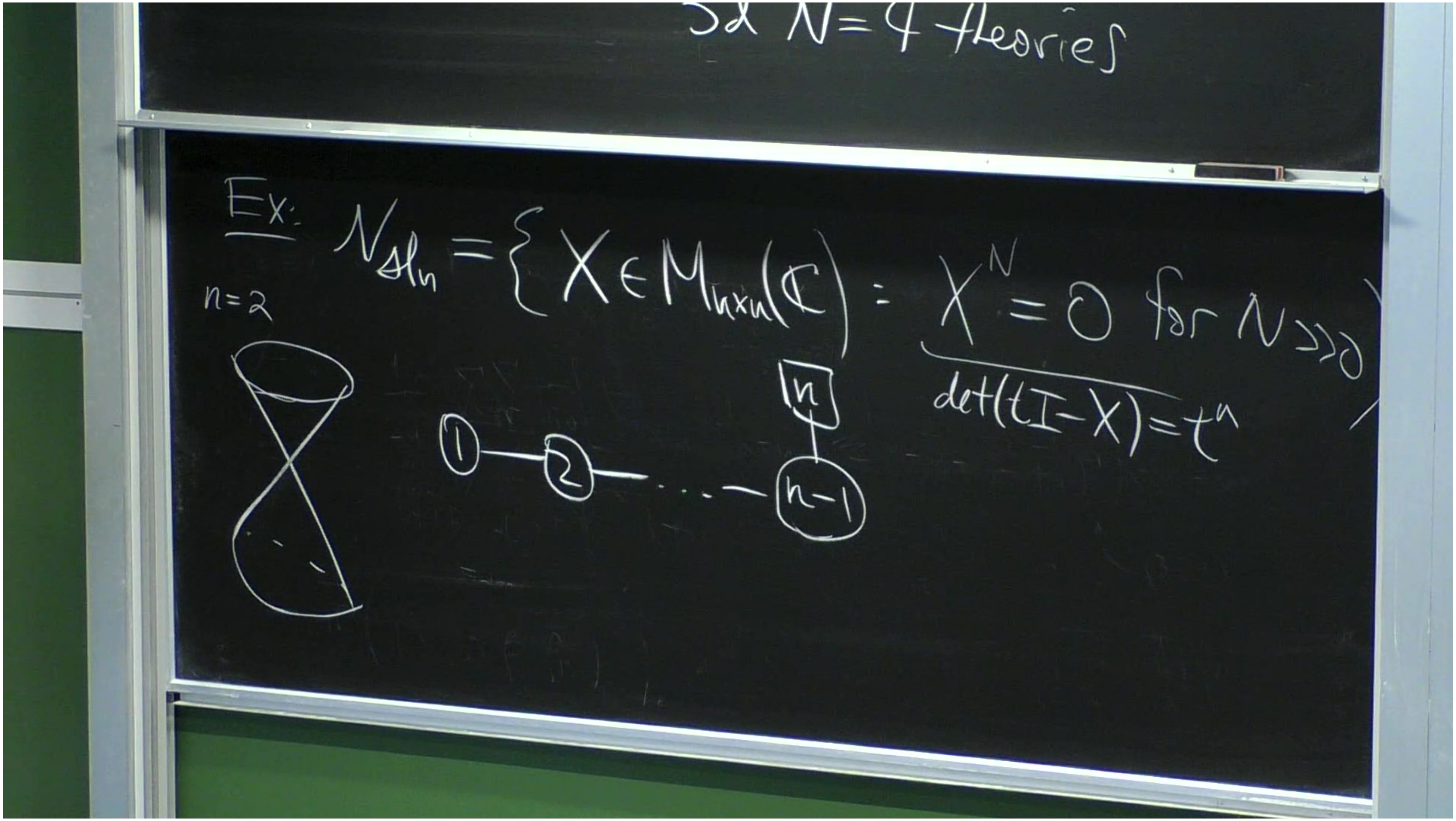
Abstract:

Many interesting spaces arise as Coulomb branches of 3d N=4 quiver gauge theories, including nilpotent orbit closures and affine Grassmannian slices. These interesting spaces often admit interesting embeddings into one another. For example, one nilpotent orbit closure might be contained inside another. That said, it is much less clear how to describe or construct such an embedding from a purely Coulomb branch perspective. I will discuss joint work with Dinakar Muthiah, where we describe a natural Coulomb branch connection with Coulomb branches: for finite ADE types, the embeddings respect monopole operators thought of as functions on the Coulomb branch. This perspective also allows us to generalize the story, and construct embeddings for arbitrary quivers which have the same property.

Embeddings between Coulomb branches of quiver gauge theories

Joint w/ Dinakar Muthiah arXiv: 2211.04788

- Many interesting singular spaces (alg. varieties) have realizations as Coulomb branches for 3d $N=4$ theories


3d $\mathcal{N}=4$ theories
Ex: $\mathcal{N}_{\mathfrak{sl}_n} = \{ X \in M_{n\times n}(\mathbb{C}) : X^N = 0 \text{ for } N \gg 0 \}$
$\det(tI - X) = t^n$
$n=2$
1
2
$n-1$
n

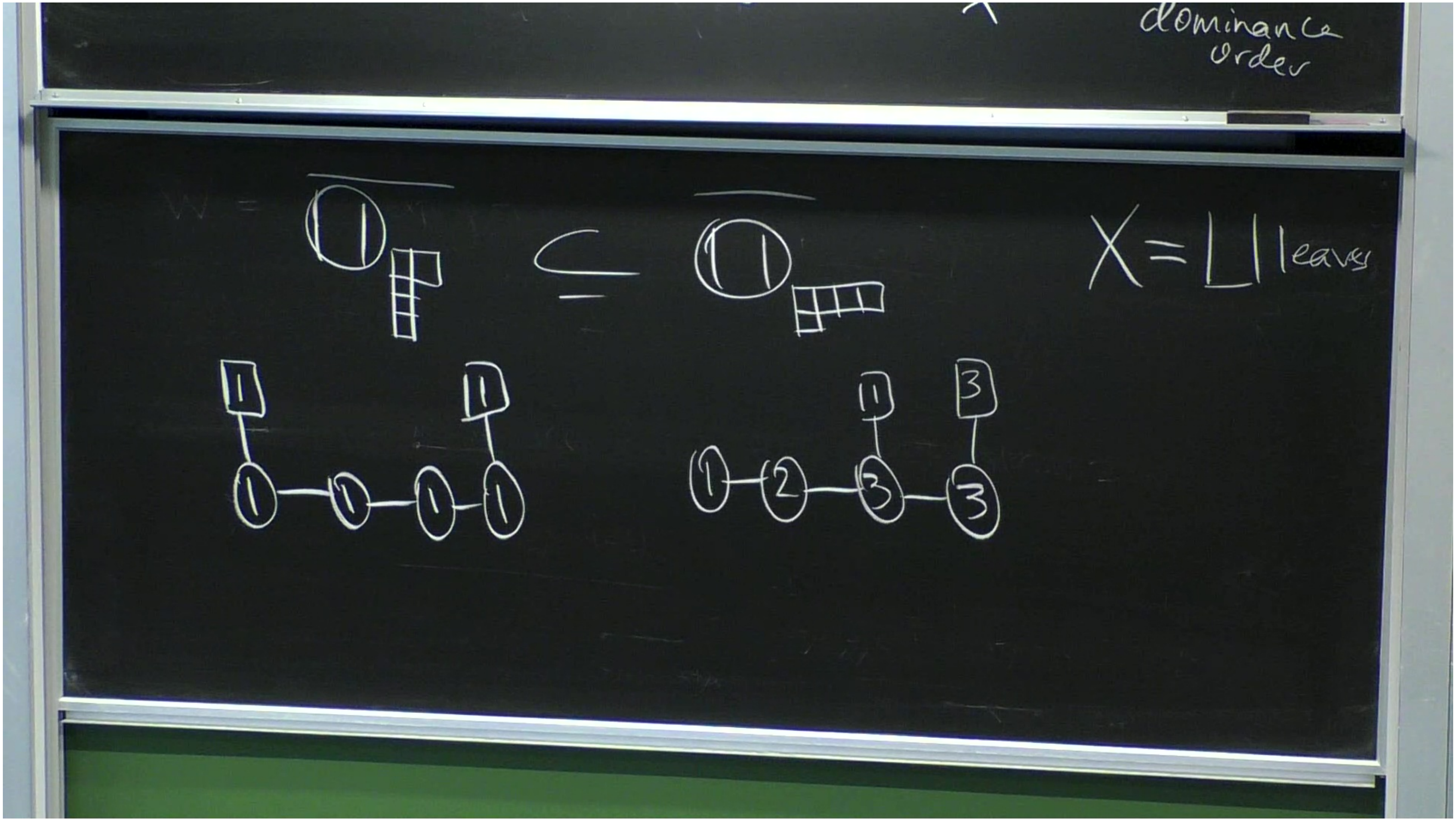
$$\mathcal{N}_{\mathfrak{sl}_n} = \bigsqcup_{\lambda \vdash n} \mathbb{O}_\lambda$$

$$\lambda = (2,2,1) \vdash 5$$

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

Each $\overline{\mathbb{O}_\lambda}$ is also a C.B.

These admit embeddings $\overline{\mathbb{O}_\nu} \subseteq \overline{\mathbb{O}_\lambda}$ iff $\nu \leq \lambda$ dominance order


dominance
order
X = ⊔ leaves

Ex: More generally, for finite ADE quiver

[Braverman-Finkelberg-Nakajima] $\mathcal{M}_C =$ generalized affine Grassmannian slice $\overline{\mathcal{W}}^{\lambda}_{\mu}$ $\overset{?}{=}$ a space of singular monopoles on $\mathbb{R}^3$

Have $\overline{\mathcal{W}}^{\nu}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$ iff $\nu \leq \lambda$.

Questions: 1) From purely Coulomb branch perspective what do these embeddings "look like"?

2) To what extent do they generalize? (To which quivers?)

* For definitess we use BFN construction of C.B.

G = complex reductive group = gauge group

M = "matter" representation of G

<u>But</u>: assume $M = N \oplus N^*$

N: rep'n of G

(1) — (2) — . . . — (n−1)

$(G,N) \rightsquigarrow \mathcal{M}_C = \operatorname{Spec} H_*^{G_\mathcal{O}}(\mathcal{R}_{G,N})$

* $\mathcal{M}_C$ is an algebraic variety/$\mathbb{C}$ with Poisson structure
* $\mathcal{M}_C$ has symplectic singularities (Bellamy)

Focus: (Unitary) Quiver gauge theories

$$\begin{tikzcd} \boxed{W_1} \arrow[d] & \\ V_1 \arrow[r] \arrow[loop below] & V_2 \end{tikzcd}$$

$$G = GL(V_1) \times GL(V_2) \quad \text{or } U(V_1) \times U(V_2)$$

$$N = \mathrm{Hom}(\mathbb{C}^{W_1}, \mathbb{C}^{V_1}) \oplus \mathrm{Hom}(\mathbb{C}^{V_1}, \mathbb{C}^{V_1}) \oplus \mathrm{Hom}(\mathbb{C}^{V_1}, \mathbb{C}^{V_2})$$

$\oplus \mathrm{Hom}(\mathbb{C}^{\cdot}, \mathbb{C}^{\cdot})$

Natural functions on $\mathcal{M}_C$:

- gauge invariant polynomials $f \in \mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W = H^\bullet_G(pt)$
- monopole operators, labelled by charge γ = dominant coweight for G

$$M_\gamma(f) \quad + \text{ dressing } f \in \mathbb{C}[\mathfrak{t}]^{W_\gamma}$$

In BFN construction:

$K = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$

$$R_{G,N} \subseteq G_K \times N_{\mathcal{O}} / G_{\mathcal{O}}$$

$$\downarrow$$

$$G_K / G_{\mathcal{O}} = Gr_G = \text{affine Grassmannian}$$

stratification by $G_{\mathcal{O}}$-orbits

$$Gr_G = \bigsqcup_{\gamma} Gr^{\gamma}$$

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Then: $M_\gamma(\mathcal{F}) \in H^{G_{\mathcal{O}}}_{\bullet}\left(\pi^{-1}\left(\overline{Gr^{\gamma}}\right)\right)$

There are choices involved!

Best case: γ is minuscule, meaning $\overline{Gr^{\gamma}} = Gr^{\gamma}$

[BFN]: can construct canonical $M_\gamma(\mathcal{F})$

* for quivers: $G = \prod_i GL(v_i)$

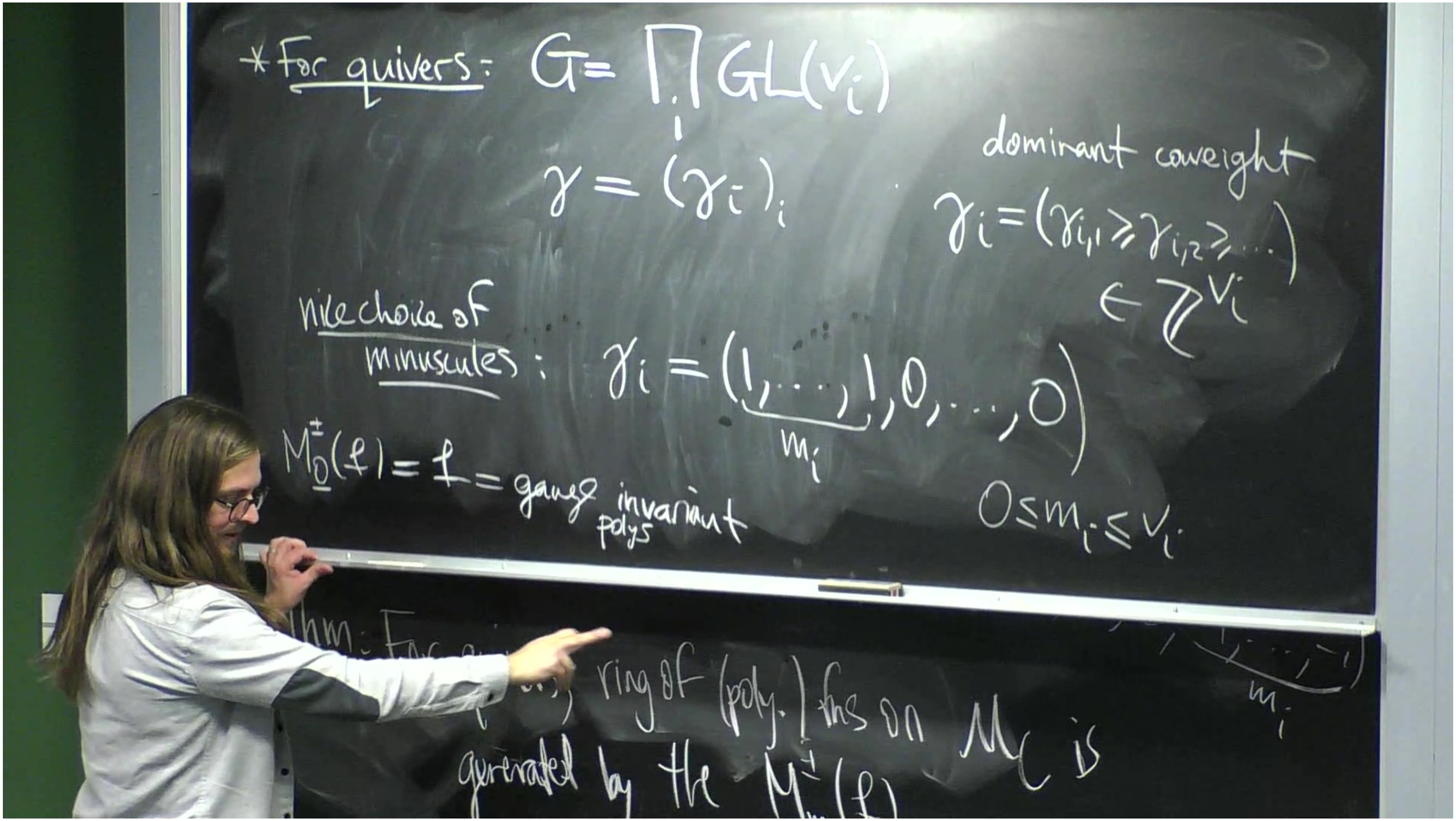
$$\gamma = (\gamma_i)_i$$

dominant coweight

$$\gamma_i = (\gamma_{i,1} \geq \gamma_{i,2} \geq \dots) \in \mathbb{Z}^{v_i}$$

nice choice of minuscules: $\gamma_i = (\underbrace{1, \dots, 1}_{m_i}, 0, \dots, 0)$

$$0 \leq m_i \leq v_i$$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nice choice of minuscules: $\gamma_i = (\underbrace{1,\dots,1}_{m_i},0,\dots,0)$
$0 \leq m_i \leq v_i$
$M^{\pm}_{\underline{0}}(f) = f$ = gauge invariant polys
Thm: For
ring of (poly.) fns on $\mathcal{M}_C$ is
generated by the $M^{\pm}_{m}(f)$

For a finite ADE quiver:

- We can associate G
- Dimension vectors $\underline{v}=(v_i)$, $\underline{w}=(w_i)$

$\updownarrow$

coweights for G

$$\lambda = \sum w_i \varpi_i^\vee$$

$$\mu = \lambda - \sum v_i \alpha_i^\vee$$

if 

$G = PGL_4$

$$\boxed{W_1} \to (V_1)$$

labelled by charge

γ = dominant coweight for G

$M_0(f)$ + dressing

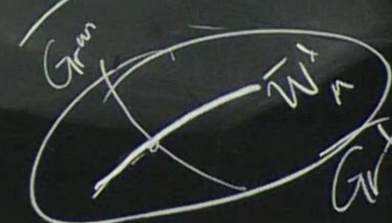
$\mu = \lambda - \sum v_i \alpha_i^\vee$)

$\rightsquigarrow$ generalized affine Grassmannian slice for G

$$\overline{W}^\lambda_\mu = U_1^+[[z^{-1}]]\, T_1[[z^{-1}]]\, z^\mu\, U_1^-[[z^{-1}]] \cap \overline{G[z]\, z^\lambda\, G[z]}$$

If μ is dominant: inside Gr_G have $\overline{Gr^\mu} \subseteq \overline{Gr^\lambda}$

$\overline{W}^\lambda_\mu$ is a transverse slice

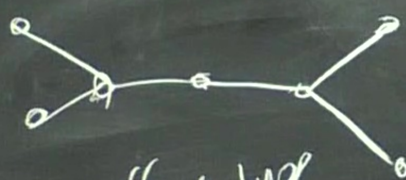


Grassmannian

$\boxed{\underline{\text{Thm}}: \text{(BFN)} \quad \mathcal{M}_C \cong \overline{\mathcal{W}}^{\lambda}_{\mu}}$

One reason this is interesting: quiver doesn't need to be ADE

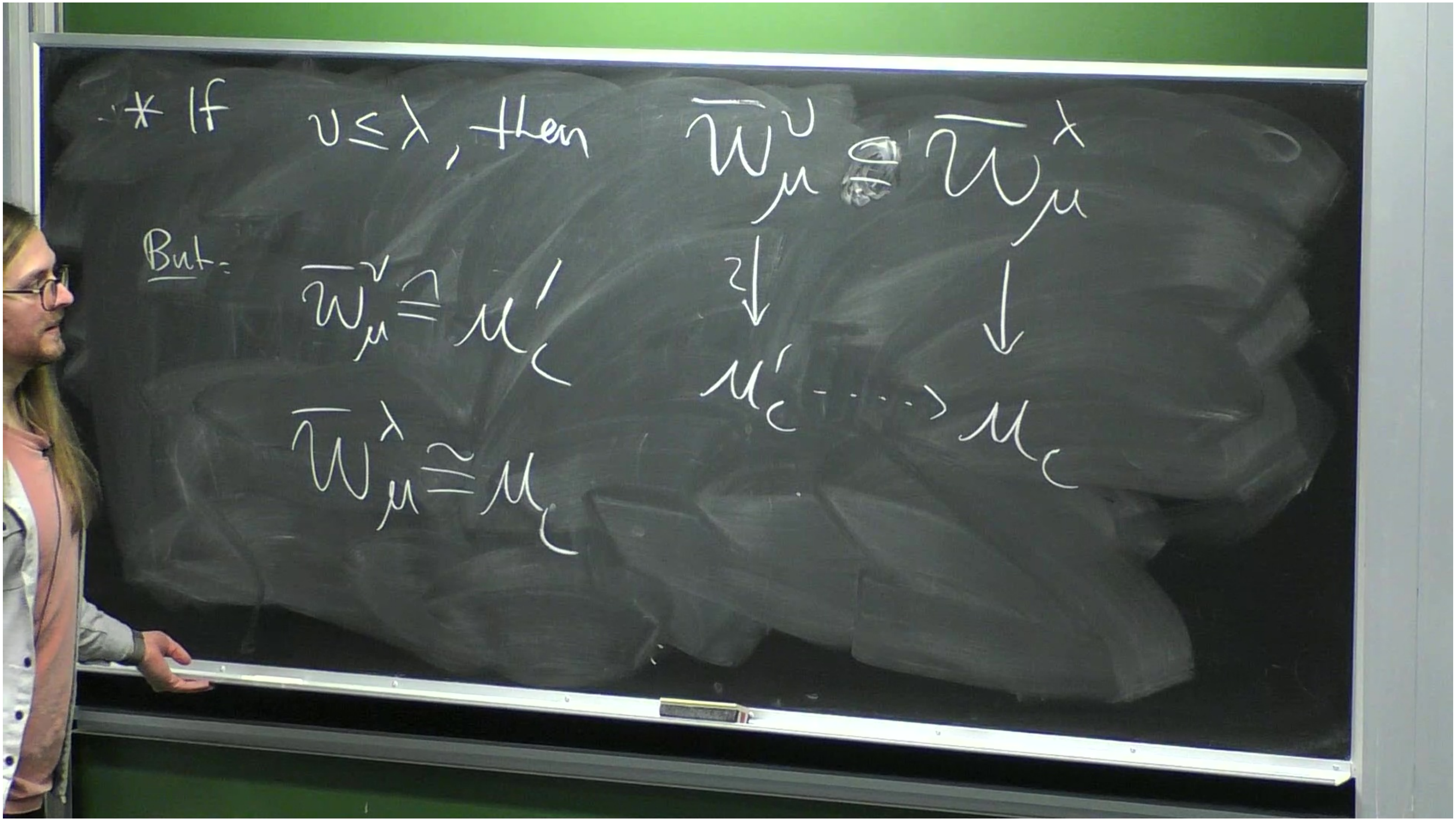
G = Kac-Moody group



affine type D

$\underline{\text{BFN}}$: For general symmetric KM

$\underline{\underline{\text{define}}}$ $\overline{\mathcal{W}}^{\lambda}_{\mu} := \mathcal{M}_C$



* If $\nu \leq \lambda$, then $\overline{\mathcal{W}}^{\nu}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$
But: $\overline{\mathcal{W}}^{\nu}_{\mu} \cong \mathcal{M}'_{C}$
$\overline{\mathcal{W}}^{\lambda}_{\mu} \cong \mathcal{M}_{C}$
$\mathcal{M}'_{C} \dashrightarrow \mathcal{M}_{C}$

Theorem: (Muthiah–W.) Under corresponding map

$$\mathbb{C}[\mathcal{U}_c] \longrightarrow \mathbb{C}[\mathcal{U}'_c]$$

We have

$$M^{\pm}_{\underline{m}}(f) \longmapsto \begin{cases} M^{\pm}_{\underline{m}}(\tilde{f}), & \text{if } \underline{m} \leq \underline{v}' \\ 0, & \end{cases}$$

Thm: (MW) For any quiver, have

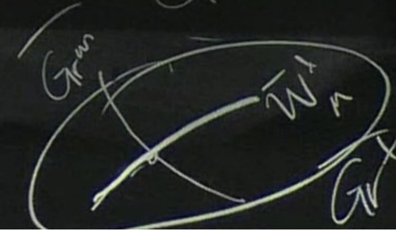
$$\overline{\mathcal{W}}^{\nu}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$$

if $\overline{\mathcal{W}}^{\lambda}_{\nu}$ is good (in sense of Gaiotto-Witten)

It is defined by $M^{\pm}_{\underline{m}}(f) \longmapsto \ldots$

$\mathbb{G}$ have $Gr \subseteq Gr$

$\overline{\mathcal{W}}^{\lambda}_{\mu}$ is a transverse slice



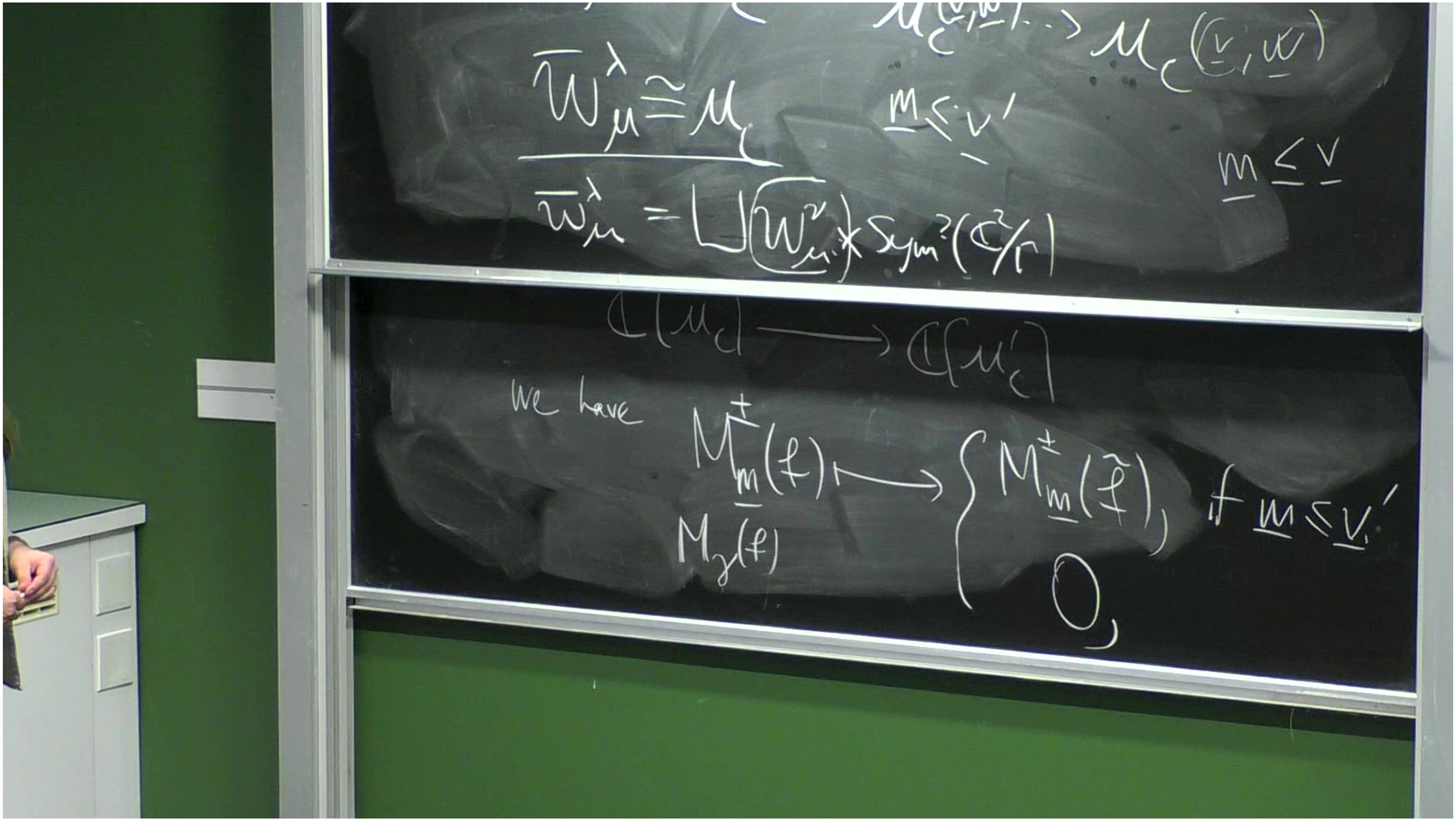
Warning/Open problem: Don't know what $M^{\pm}_{\underline{m}}(f)$ even in some simple examples

Ex: $\mathcal{N}_{\mathfrak{sl}_7} \ni X = \begin{array}{cc} & i \\ i & \left(\boxed{Y} \right) \end{array}$

$\det(t \cdot I - Y)$

$\mathcal{M}_L$ $\quad$ $\boxed{7}$ — 6 — $\cdots$ — 2 — 1

- gauge invariant f


$\mathcal{M}_C(\underline{v},\underline{w}) \to \mathcal{M}_C(\underline{v},\underline{w})$
$\overline{\mathcal{W}}^\lambda_\mu \cong \mathcal{M}_C$
$\underline{m} \le \underline{v}'$
$\underline{m} \le \underline{v}$
$\overline{\mathcal{W}}^\lambda_\mu = \bigsqcup \mathcal{W}^\nu_\mu \times \mathrm{Sym}^?(\mathbb{C}^2/\Gamma)$
$\mathbb{C}[\mathcal{M}_C] \longrightarrow \mathbb{C}[\mathcal{M}_C']$
We have $M^\pm_{\underline{m}}(f) \longmapsto \begin{cases} M^\pm_{\underline{m}}(\tilde{f}), & \text{if } \underline{m} \le \underline{v}' \\ 0, & \end{cases}$
$M_\gamma(f)$