Title: Gauge theories and boundaries: from superselection to soft modes and memory

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Collection/Series: Quantum Gravity

Subject: Quantum Gravity

Date: October 31, 2024 - 2:30 PM

URL: https://pirsa.org/24100139

Abstract:

I present an overview of the work I have done over the last few years on the phase space structure of gauge theories in the presence of boundaries. Starting with primers on the covariant phase space and symplectic reduction, I then explain how their generalization when boundaries are present fits into the reduction-by-stages framework. This leads me to introduce the concept of (classical) superselection sectors, whose physical meaning is clarified by a gluing theorem. Applying the framework developed this far to a null hypersurface, I then discuss how the extension of the Ashtekar-Streubel symplectic structure by soft modes emerges naturally, and how electric memory ties to superselection. If time allows, and depending on the audience's interests, I will finally compare reduction-by-stages with the edge-mode formalism or discuss its relation to dressings and "gauge reference frames". An overarching theme will be the nonlocal nature of gauge theories. This seminar is based on work done with Gomes and Schiavina.

References:
The general framework: 2207.00568
Null Yang-Mills: 2303.03531
Gluing: 1910.04222
A pedagogical introduction: 2104.10182
Dressings and reference frames: 1808.02074, 2010.15894, 1608.08226

GAUGE THEORIES AND BOUNDARIES

FROM SUPERSELECTION TO SOFT MODES AND MEMORY

Aldo Riello

Perimeter Institute
Waterloo, Canada

Quantum Gravity Seminar

31 Oct 2024

based on work done in collaboartion with M. Schiavina (and H. Gomes)

∼ MENU ∼

Appetizer

Motivations in a nutshell, with a sprinkle of debate

Starter

Crispy review of covariant phase space and symplectic reduction

Main course

Symplectic reduction with boundaries:
a bulk of constraint reduction with a side of flux superselection

Second course

Soft extensions and electric memory from constraint reduction and flux superselection

Dessert

A choice between 'gluing' or 'gauge reference frames and dressings'

MOTIVATIONS – in a nutshell

Gauge (Gauss) constraints tie the value of the fields at spacelike separated points (e.g. Gauss's law)

$\rightsquigarrow$ Gauge theories are nonlocal $\rightsquigarrow$ How to define subsystems?

Idea:
Introduce boundaries and see what happens $\rightsquigarrow$ nontrivial interplay between gauge and boundaries

Remarks

- What do we mean by a 'subsystem'? What kind of boundaries? ☛ `I'll come back to this`
- A *null* hypersurface in a causal spacetime always has boundaries (maybe asymptotic) $\rightsquigarrow$ applications to horizons, scri, soft modes, memory etc.

GAUGE vs. BOUNDARIES

THE ISSUE

Hamiltonian gauge theory: gauge symmetries are 'generated' by canonical constraints (Noether 1+2).
E.g., in Maxwell theory:

$$\{\langle H,\xi\rangle, A_i(x)\} = \partial_i\xi(x) \qquad \text{where} \qquad \langle H,\xi\rangle = \int_\Sigma \xi \underbrace{\nabla_a E^a}_{\text{Gauss c.}} \approx 0 \qquad (\partial\Sigma = \emptyset)$$

Structural relationship: **Constraint = 0** $\leftrightsquigarrow$ **Gauge = 'unphysical'**.
☛ `I'll come back to this`

Boundaries
But, if $\partial\Sigma \neq \emptyset$, this relationship fails.
E.g., in Maxwell theory:

$$\{\langle H,\xi\rangle, A_i(x)\} = \partial_i\xi(x) \qquad \text{where} \qquad \langle H,\xi\rangle := -\int_\Sigma E^a\nabla_a\xi = \int_\Sigma \xi \underbrace{\nabla_a E^a}_{\text{Gauss c.}} - \oint_{\partial\Sigma} \xi \overbrace{n_a E^a}^{\text{el. flux}} \not\approx 0$$

How should we interpret this fact? What are its physical consequences?

GAUGE vs. BOUNDARIES

The debate

Traditionally, one *defines* 'true gauge' those transformations that are generated by the constraints, i.e., those that 'vanish at the boundary'.
[Regge, Teitelboim; Carlip; Giulini; ...]

$\rightsquigarrow$ What should one do with the residual 'boundary gauge transformations' ?

A common position: the presence of a boundary transmutes would-be-gauge symmetries into new physical symmetries, and would-be-gauge d.o.f. into new physical d.o.f. or 'edge modes'.
[Balachandran et al.; Carlip; Donnelly, Freidel; ...]

'Bulk' and 'bdry' gauge tr. are *mathematically* distinct. But their *physical* interpretation needs more. IMO, we should ask: what is *physically* meant by 'boundary', and how then should it be modelled. E.g., in Maxwell theory, how to model Casimir plates with their *own* 'boundary d.o.f.'.
[Susskind 2015; AR 2021]

But, if $\partial\Sigma$ is a 'fictitious boundary' (i.e., a mathematical construction meant to split a system into subsystems, but with no physical reality), IMO the only tenable position is that boundary-gauge is just gauge, i.e. redundancy. This will be my viewpoint in what follows.

DISCLAIMER

Regardless of whether you subscribe to a position or another, you can think of this seminar as an exploration of the consequences of quotienting-out *both* bulk & boundary gauge transformations. We will reassess the physics at the end.

Besides, the math is interesting in itself and will shed light on the 'edge modes', too.

~ MENU ~

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☛ **Starter**

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COVARIANT v. CANONICAL PHASE SPACE

☐ GAUGE, ☐ BOUNDARY

Consider e.g. a scalar field $\varphi \in \mathcal{F} \doteq C^\infty(M)$ over a spacetime $M \simeq \Sigma \times \mathbb{R}$, $\partial\Sigma = \emptyset$.
Consider a (local) Lagrangian density $\boldsymbol{L}(\varphi, \partial\varphi, x) \in \Omega^{\text{top},0}(M \times \mathcal{F})$.

$$\mathbb{d}\boldsymbol{L} = \boldsymbol{E}_I \mathbb{d}\varphi^I + \mathsf{d}\boldsymbol{\theta}$$

Covariant ph.sp. = on-shell histories $\overline{\mathcal{F}} \doteq \{\boldsymbol{E} = 0\}$, equipped with symplectic structure $\Omega \doteq \int_\Sigma \mathbb{d}\boldsymbol{\theta}$.

- thanks to e.o.m. Ω is independent of choice of Cauchy surface $\Sigma \hookrightarrow M$.
- this relies on $\overline{\mathcal{F}}$ being a *nonlocal* space of fields: a perturbation Σ_t 'propagates' to all other $\Sigma_{t'}$.

Is there a (time-)*local* description of the covariant ph.sp. $(\overline{\mathcal{F}}, \Omega)$?
Yes, the **canonical ph.sp.** $(\mathcal{P}, \omega)$!

Using the 1-to-1 relation between on-shell histories and initial conditions, typically one finds:

$$f_\Sigma : \quad \overline{\mathcal{F}} \xrightarrow{\simeq} \mathcal{P} = \mathsf{T}^*\mathcal{Q} \qquad \text{where} \qquad \phi \in \mathcal{Q} \doteq \Gamma(F|_\Sigma \to \Sigma)$$
$$\varphi \longmapsto (\phi, \pi) = (\varphi|_\Sigma, \dot\varphi|_\Sigma)$$

which allows us to define $\Omega = f_\Sigma^* \omega$.

COVARIANT v. CANONICAL PHASE SPACE

☑ GAUGE, ☐ BOUNDARY

Consider now (non-compact) Maxwell theory, in vacuum.
To have a *local* Lagrangian formulation, one must introduce gauge redundancy[1]

$$A \in \mathcal{F} \doteq \Omega^1(M) \circlearrowleft_\rho \mathcal{G} \doteq C^\infty(M, \mathbb{R}_+), \qquad (A, g) \mapsto A + g^{-1}\mathsf{d}g.$$

Then the e.o.m. of $\boldsymbol{L}(A, \partial A) = \frac{1}{4} F_A \wedge \star F_A$ define $(\overline{\mathcal{F}}, \Omega)$, with

$$\overline{\mathcal{F}} = \{A : \mathsf{d} \star F_A = 0\} \qquad \text{and} \qquad \Omega = \int_\Sigma \mathbb{d}\boldsymbol{\theta} = \int_\Sigma \star \mathbb{d}F_A \wedge \mathbb{d}A.$$

Problem : $(\overline{\mathcal{F}}, \Omega)$ is only *pre*-symplectic, i.e., $\mathsf{ker}(\Omega) \neq \{0\}$.
Remark : $\mathsf{ker}(\Omega) = \rho(\mathfrak{G})$, $\mathfrak{G} \doteq \mathsf{Lie}(\mathcal{G})$.

⇝ presymplectic reduction: modding out by the action of $\mathcal{G}$ one finds the **covariant ph. sp.** $(\underline{\overline{\mathcal{F}}}, \underline{\Omega})$

$$\text{presymplectic, gauge-variant} \quad (\overline{\mathcal{F}}, \Omega) \overset{\cdot/\mathcal{G}}{\rightsquigarrow} (\underline{\overline{\mathcal{F}}}, \underline{\Omega}) \quad \text{symplectic, gauge-}\textit{in}\text{variant}$$

[1] For a more general definition, consider a principal G-bundle P, then $\mathcal{A} \doteq J^1 P/G$ and $\mathcal{G} \doteq \Gamma(\mathsf{AD}P \to M)$.

COVARIANT v. CANONICAL PHASE SPACE

☑ GAUGE, ☐ BOUNDARY

What about the **canonical ph. sp.**?

Remark : The canonical ph.sp. cannot be local.
There are two sources of nonlocality. It is thus convenient to work in three steps:

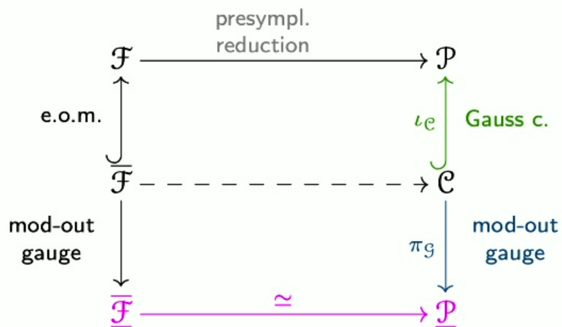
0. define **geometrical ph.sp.** (off-shell, kinematical), $\mathcal{P} \doteq \mathsf{T}^*\Omega^1(\Sigma) \ni (A, E) \rightsquigarrow$ local, symplectic;
1. impose the Gauss constraint, $\mathcal{C} \doteq \{(A, E) : \nabla_a E^a = 0\} \rightsquigarrow$ *non*local, *pre*symplectic;
2. modding-out the gauge, define the **reduced ph.sp.** (physical), $\underline{\mathcal{P}} \doteq \mathcal{C}/\mathcal{G} \rightsquigarrow$ *non*local, symplectic.

$$(\underline{\mathcal{P}}, \underline{\omega}) \simeq (\underline{\overline{\mathcal{F}}}, \underline{\Omega}).$$

symplectic, gauge-invariant, nonlocal

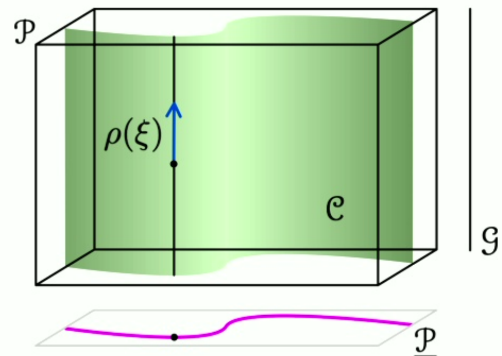
SUMMARY ☑ gauge, ☐ boundary

'covariant' 'canonical' @ Σ



reduced phase space

'canonical' @ Σ



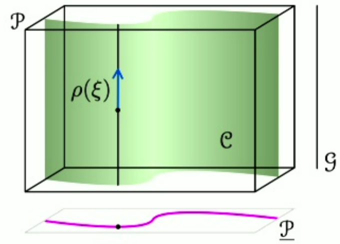
- Two step process: impose the constraint + mod-out gauge ☛ `structural relation`
- Maxwell in Coulomb gauge:

$$\mathcal{F} = \Omega^1(M), \quad \mathcal{P} = \mathsf{T}^*\mathcal{Q} \text{ with } \mathcal{Q} = \Omega^1(\Sigma), \quad \mathcal{C} = \{(A,E) \in \mathcal{P} : \nabla_a E^a = 0\}$$

$$\underline{\mathcal{P}} \doteq \mathcal{C}/\mathcal{G} \simeq \mathsf{T}^*(\mathcal{Q}/\mathcal{G}) \overset{\text{Coul.}}{\simeq} \{\text{transverse (i.e., divergence-free) pairs } (A,E)\} \rightsquigarrow \text{'photons'}$$

- The reduced phase space gives the *initial data* for the space of on-shell histories modulo gauge.

SYMPLECTIC REDUCTION OF MAXWELL Th. [Marsden-Weinstein 1974]



☑ GAUGE, ☐ BOUNDARY

- Geometrical phase space $(\mathcal{P}, \omega)$ with gauge flow $\rho : \mathfrak{G} \to \mathfrak{X}(\mathcal{P})$:

$$\omega = \int_\Sigma \mathbb{d}E^a \wedge \mathbb{d}A_a \quad \text{and} \quad \rho(\xi)\begin{pmatrix} A_a \\ E^a \end{pmatrix} = \begin{pmatrix} \nabla_a \xi \\ 0 \end{pmatrix}$$

- Hamiltonian flow: the Gauss constraint is the *momentum map* $H \in C^\infty(\mathcal{P}, \mathfrak{G}^*)$:

$$\mathbb{i}_{\rho(\xi)}\omega = -\mathbb{d}\langle H, \xi\rangle, \quad \langle H, \xi\rangle = -\int_\Sigma E^a \nabla_a \xi \overset{\text{i.b.p.}}{=} \int_\Sigma \xi \nabla_a E^a$$

- Constraint surface as *zero of the momentum map* H:

$$\mathcal{C} \doteq H^{-1}(0)$$

- The kernel of ω on-shell is given by gauge transformations:

$$\mathbb{i}_{\rho(\xi)}\iota_{\mathcal{C}}^*\omega = \iota_{\mathcal{C}}^*\mathbb{d}\langle H, \xi\rangle \equiv 0 \implies \rho(\mathfrak{G}) \subset \ker(\iota_{\mathcal{C}}^*\omega) \quad \text{in fact: } \ker(\iota_{\mathcal{C}}^*\omega) = \rho(\mathfrak{G}).$$

- Reduced symplectic space $(\underline{\mathcal{P}}, \underline{\omega})$:

$$\boxed{\underline{\mathcal{P}} \doteq \mathcal{C}/\ker(\iota_{\mathcal{C}}^*\omega) = \mathcal{C}/\mathcal{G} \quad \text{and} \quad \pi_{\mathcal{G}}^*\underline{\omega} = \iota_{\mathcal{C}}^*\omega.}$$

SYMPLECTIC REDUCTION OF MAXWELL Th. ?

☑ GAUGE, ☑ BOUNDARY

- Geometrical phase space $(\mathcal{P}, \omega)$ with gauge flow $\rho : \mathfrak{G} \to \mathfrak{X}(\mathcal{P})$:

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- Hamiltonian flow: the Gauss constraint is not quite the *momentum map* $H \in C^\infty(\mathcal{P}, \mathfrak{G}^*)$:

$$\mathbb{i}_{\rho(\xi)}\omega = -\mathbb{d}\langle H, \xi\rangle, \quad \langle H, \xi \rangle = -\int_\Sigma E^a \nabla_a \xi \overset{\text{i.b.p.}}{=} \int_\Sigma \xi \nabla_a E^a - \oint_{\partial\Sigma} \xi \underbrace{n_a E^a}_{\text{el.flux}}$$

- Constraint surface is not the *zero of the momentum map* H (unconstrained value):

$$\mathcal{C} \doteq \{(A, E) : \nabla_a E^a = 0\} \neq H^{-1}(0)$$

- The kernel of ω on-shell is given by 'bulk' gauge transformations $\mathcal{G}_\circ \doteq \{g : g|_{\partial\Sigma} = 1\}$:

$$\mathbb{i}_{\rho(\xi)}\iota_{\mathcal{C}}^*\omega = \iota_{\mathcal{C}}^*\mathbb{d}\langle H, \xi \rangle = -\oint_{\partial\Sigma} \xi\, n_a E^a \quad \rightsquigarrow \quad \mathsf{ker}(\iota_{\mathcal{C}}^*\omega) = \rho(\mathfrak{G}_\circ)$$

- Reduced phase space ?

$$\underline{\mathcal{C}} \doteq \mathcal{C}/\mathcal{G}_\circ \quad \text{vs.} \quad \underline{\underline{\mathcal{P}}} \doteq \mathcal{C}/\mathcal{G}$$

REDUCTION BY STAGES

☑ GAUGE, ☑ BOUNDARY

Proceed by **stages**
[reduction by stages originally devised to reduce semidirect product groups]:

1. Reduce 'bulk gauge transformations'[2]

$$\mathcal{G}_\circ \doteq \{g : g|_{\partial\Sigma} = 1\} \subset \mathcal{G}$$ ☚ normal subgroup

   Its momentum map is the Gauss constraint, $\langle H_\circ, \xi_\circ \rangle = \int_\Sigma \xi_\circ \nabla_a E^a$.

2. Take care of the residual group of 'corner gauge transformations'

$$\underline{\mathcal{G}} \doteq \mathcal{G}/\mathcal{G}_\circ \simeq C^\infty(\partial\Sigma, \mathbb{R}_+)$$

   Its 'residual momentum map' is the electric flux $\langle \underline{h}, \underline{\xi} \rangle = -\oint_{\partial\Sigma} \underline{\xi}\, n_a E^a$.

[2] I am glossing over the general definition, which is quite more technical, and also over some subtleties related to *constant* elements of $\mathcal{G}$.

First Stage: CONSTRAINT REDUCTION

☑ GAUGE, ☑ BOUNDARY

'Bulk gauge transformations':

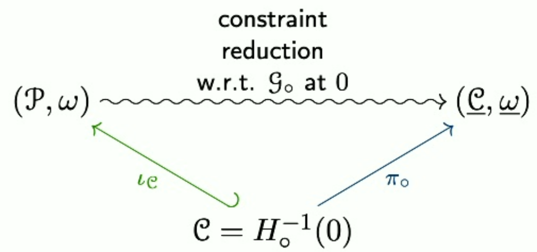
$$\mathcal{G}_\circ \doteq \{g : g|_{\partial\Sigma} = 1\} \subset \mathcal{G}$$ ☚ normal subgroup

Its momentum map is the Gauss constraint, $\langle H_\circ, \xi_\circ \rangle = \int_\Sigma \xi_\circ \nabla_a E^a$:

$$\mathcal{C} = H_\circ^{-1}(0)$$

First-stage, or, constraint reduction proceeds like in the boundary-less case – yielding $(\underline{\mathcal{C}}, \underline{\omega})$:

$$\underline{\mathcal{C}} \doteq \mathcal{C}/\mathcal{G}_\circ \quad \text{with} \quad \pi_\circ^* \underline{\omega} = \iota_{\mathcal{C}}^* \omega.$$



☛ see later for an interpretation

Second Stage: FLUX SUEPERSELECTION

☑ GAUGE, ☑ BOUNDARY

Residual, 'corner', gauge transformations :

$$\underline{\mathcal{G}} \doteq \mathcal{G}/\mathcal{G}_\circ \circlearrowleft (\underline{\mathcal{C}}, \underline{\omega})$$

Their momentum map is the electric flux, $\langle h, \underline{\xi} \rangle = \oint_{\partial\Sigma} \underline{\xi}\, n_a E^a$
$\rightsquigarrow$ it does *not* take one prescribed value

Therefore, the only reasonable space to study is:

$$\underline{\underline{\mathcal{P}}} \doteq \underline{\mathcal{C}}/\underline{\mathcal{G}} \simeq \mathcal{C}/\mathcal{G}.$$

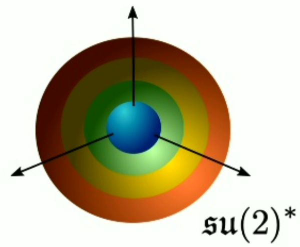
Toy example ($\sim$ 2d BF)

$$\underline{\mathcal{C}} = \mathsf{T}^*G \circlearrowleft \underline{\mathcal{G}} = G \quad \rightsquigarrow \quad \underline{\underline{\mathcal{P}}} \doteq \underline{\mathcal{C}}/\underline{\mathcal{G}} = \mathsf{T}^*G/G \simeq \mathfrak{g}^* \text{ with } \{z_\alpha, z_\beta\} = f_{\alpha\beta}{}^\gamma z_\gamma.$$

☛ Lesson: mom.map has no fixed value ⇒ *fully reduced phase space* $\underline{\underline{\mathcal{P}}}$ *is Poisson*, but not symplectic

Second Stage: FLUX SUEPERSELECTION

☑ GAUGE, ☑ BOUNDARY



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⇝ it does *not* take one prescribed value

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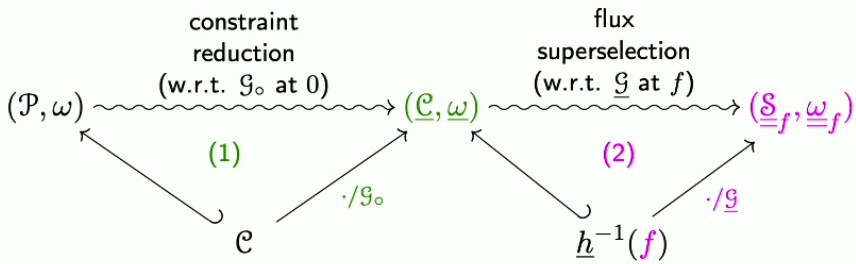
☛ Lesson: mom.map has no fixed value ⇒ *fully reduced phase space $\underline{\underline{\mathcal{P}}}$ is Poisson*, but not symplectic

☛ Lesson: $\underline{\underline{\mathcal{P}}}$ is foliated by symplectic leafs ⇝ 'flux' superselection sectors $(\underline{\underline{\mathcal{S}}}_f, \underline{\underline{\omega}}_f)$

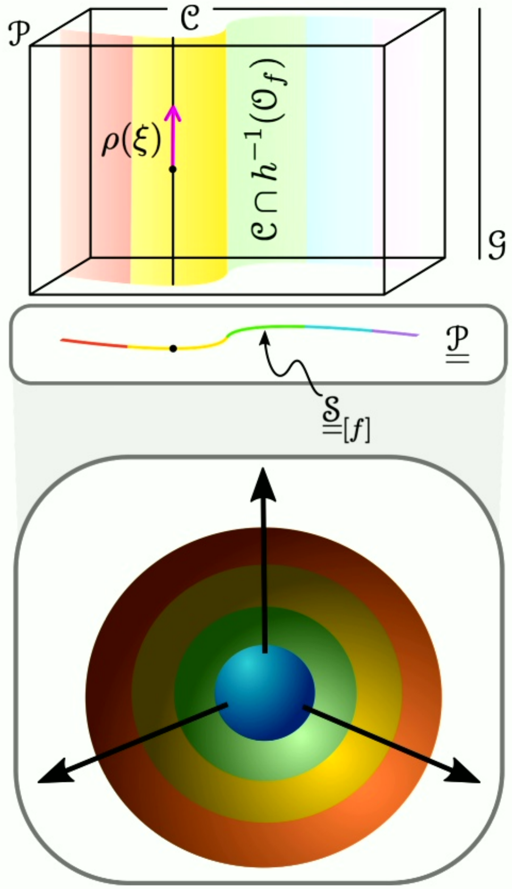
REDUCTION BY STAGES - Summary

☑ GAUGE, ☑ BOUNDARY

Denote by $f \in \underline{\mathfrak{G}}^*$ one *value* of the electric flux $n_a E^a$:



Remark : if $\underline{\mathcal{G}}$ is non-Abelian, then replace everywhere $f \rightsquigarrow \mathcal{O}_f \subset \underline{\mathfrak{G}}^*$ (coadjoint orbit)



$\mathcal{P}$
$\mathcal{C}$
$\rho(\xi)$
$\mathcal{C} \cap h^{-1}(\mathcal{O}_f)$
$\mathcal{G}$
$\underline{\underline{\mathcal{P}}}$
$\underline{\underline{\mathcal{S}}}_{[f]}$

PHYSICAL INTERPRETATION - Maxwell th.

☑ GAUGE, ☑ BOUNDARY

Say $\Sigma \simeq B_3$. Use Helmholtz-Hodge decomposition with boundaries to decompose:

$$A_a = A_a^{\mathsf{rad}} + \nabla_a \varsigma \quad \text{and} \quad E^a = E^a_{\mathsf{rad}} + \nabla^a \varphi$$

- (ς, φ) are free everywhere, including at $\partial\Sigma$ (no bdry conditions).
- $(A^{\mathsf{rad}}, E^{\mathsf{rad}})$ are transverse/divergence-free *and* flux-less ($n^a A_a^{\mathsf{rad}} = 0 = n_a E^a_{\mathsf{rad}}$).

On-shell of the Gauss constraint, $\varphi = \varphi(f)$ is fully determined by the value of the el. flux $f \in \mathfrak{F} \subset \underline{\mathfrak{G}}^*$:

$$\begin{cases} \Delta\varphi = \nabla_a E^a \approx 0 & \text{in } \Sigma \\ n^a \nabla_a \varphi = n_a E^a = f & \text{at } \partial\Sigma \end{cases}$$

In Maxwell theory, constraint reduction and flux superselection give:

$$\underline{\mathcal{C}} \simeq \underbrace{\mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathsf{T}^*\mathfrak{F}}_{\text{symplectic}} \quad \text{and} \quad \underline{\underline{\mathcal{P}}} \simeq \underbrace{\mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathfrak{F}}_{\text{Poisson}} \simeq \bigsqcup_{f\in\mathfrak{F}} \underline{\underline{\mathcal{S}}}_f, \quad \underline{\underline{\mathcal{S}}}_f \simeq \mathsf{T}^*\mathcal{A}^{\mathsf{rad}}$$

The fluxes are gauge invariant and central in $\underline{\underline{\mathcal{P}}} \implies$ superselected!

PHYSICAL INTERPRETATION - non-Abelian YM th.

☑ GAUGE, ☑ BOUNDARY

The non-Abelian theory is quite more complicated. In a nutshell:

$$\underline{\mathcal{C}} \simeq_{\mathsf{loc}} \mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathsf{T}^*\underline{\mathcal{G}} \quad ☚ \text{ symplectic}$$

with electric fluxes conjugate to residual ‘boundary’ gauge transformations;

$$\underline{\underline{\mathcal{P}}} \simeq_{\mathsf{loc}} \mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \underline{\mathfrak{G}}^* \simeq_{\mathsf{loc}} \bigsqcup_{\mathcal{O}_f} \underline{\underline{\mathcal{S}}}_{[f]}, \quad ☚ \text{ Poisson}$$

with

$$\underline{\underline{\mathcal{S}}}_{[f]} \simeq_{\mathsf{loc}} \mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathcal{O}_f \quad ☚ \text{ symplectic}$$

$\rightsquigarrow$ the flux superselection sectors are labelled by the Casimirs of the Noether charge algebra

Remark : flux rotations = change f within $\mathcal{O}_f$ with all other variables fixed $\sim$ ‘corner symmetries’
No canonical meaning: depends on choice of isomorphism $\simeq_{\mathsf{loc}}$ above,
i.e., on how f is chosen to *nonlocally* parametrize a ‘Coulombic’ electric field *inside* Σ.
Even most natural choice does *not* guarantee flux rotations preserve the energy content of Σ.
$\rightsquigarrow$ flux rotations/corner symmetries are neither canonically defined, nor ‘symmetries’, nor ‘corner’!

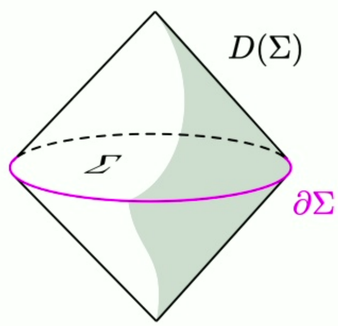
COVARIANT v. CANONICAL PHASE SPACE - scalar field

☐ GAUGE, ☑ BOUNDARY

Wish : preserve relation between covariant and canonical pictures in a subregion

⇝ 1-to-1 relation between 'canonical data' (ϕ, π) on Σ, $\partial\Sigma \neq \emptyset$, and solutions to the e.o.m. over ...

... over $D(\Sigma) \subset M$ – the causal domain of Σ



Remark : the finite spacetime domain $D(\Sigma)$ has *one*, distinguished, codim-2 surface $\partial\Sigma$, the corner

COVARIANT v. CANONICAL PHASE SPACE - scalar field

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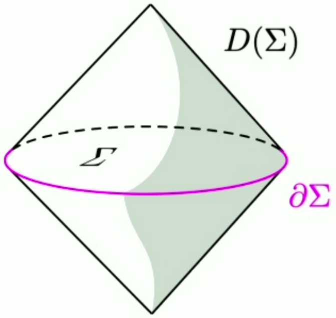
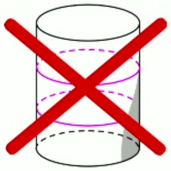




Remark : the finite spacetime domain $D(\Sigma)$ has *one*, distinguished, codim-2 surface $\partial\Sigma$, the corner

COVARIANT v. CANONICAL PICTURES - Maxwell

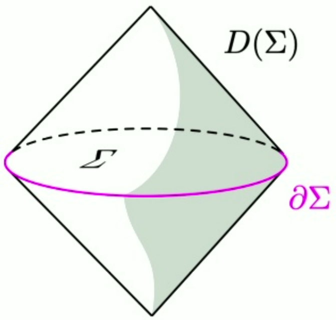
☑ GAUGE, ☑ BOUNDARY

Wish : preserve relation between covariant and canonical pictures in a subregion ☛ satisfied for $\underline{\underline{\mathcal{P}}}$!

$\rightsquigarrow$ 1-to-1 relation between

fully reduced 'canonical data' $(A^{\mathsf{rad}}, E_{\mathsf{rad}}, f) \in \underline{\underline{\mathcal{P}}}$ on $(\Sigma, \partial\Sigma \neq \emptyset)$, and

solutions to the e.o.m. over $D(\Sigma) \subset M$



Remark : $\underline{\underline{\mathcal{P}}} \simeq \mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathfrak{F}$

the 'photons' $(A^{\mathsf{rad}}, E_{\mathsf{rad}})$ are the 'canonical' radiative d.o.f.,

while f fixes the superselected Coulombic sector.

REDUCTION BY STAGES - remarks

I illustrated reduction by stages through Maxwell / YM theory at a spacelike hypersurface Σ.

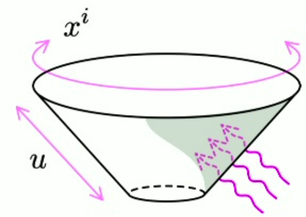
However, it has much broader applicability.

E.g., in Chern-Simons it nicely accounts for the central extension $\underline{\mathcal{G}} \rightsquigarrow \underline{\hat{\mathcal{G}}} = LG$ and related Hilbert space structure [Meinrenken-Woodward 1996].

In the next part of this talk, we turn instead to Maxwell / YM on a null hypersurface $\mathcal{I}$.

Remark
In its current form, our framework does not work for General Relativity, because diffeos don't have a Hamiltonian $\mathcal{G}$-action on $\mathcal{P}_{\mathsf{ADM}}$.
[Lee-Wald, Weinstein-Blohmann + Fernandes + Schiavina , ...]

NULL MAXWELL Th.



Let $\mathcal{I} \hookrightarrow M$ be a null hypersurface, $\mathcal{I} \simeq S \times [-1,1]$, $\partial\mathcal{I} \simeq S^{\mathsf{fin}} \sqcup S^{\mathsf{in}}$.

$$\mathcal{F} = \Omega^1(M), \quad \Omega_{\mathcal{I}} \doteq \int_{\mathcal{I}} \mathbb{d}(\star F_A) \wedge \mathbb{d}A$$

$\rightsquigarrow$ induces the following geometrical phase space $(\mathcal{P}, \omega) \circlearrowleft \mathcal{G}$ on $\mathcal{I}$ – now, $E \sim F_{ur}$:

$$\omega = \int_{\mathcal{I}} \mathbb{d}E \wedge \mathbb{d}A_u + \mathbb{d}(F_A)_u{}^i \wedge \mathbb{d}A_i \quad \rho(\xi)\begin{pmatrix} A_a \\ E^a \end{pmatrix} = \begin{pmatrix} \nabla_a \xi \\ 0 \end{pmatrix}$$

Constraint Reduction

Gauss $= \partial_u E + \nabla_i (F_A)_u{}^i \approx 0 \implies \mathcal{G}_\circ = \{g : g^{\mathsf{fin}} = g^{\mathsf{in}} = 1\}$.

$$\underline{\mathcal{C}} = \{\mathsf{Gauss} = 0\}/\mathcal{G}_\circ \simeq \mathcal{P}_{\mathsf{AS}} \times \mathsf{T}^*C^\infty(S)$$

$$\underline{\omega} = \int_{\mathcal{I}} \underbrace{\partial_u \mathbb{d}\mathsf{a}^i \wedge \mathbb{d}\mathsf{a}_i}_{\mathsf{Ash.-Str.}} + \oint_S \mathbb{d}\mathsf{e} \wedge \mathbb{d}\lambda$$

where

$$E^{\mathsf{in}} = \mathsf{e}, \quad E^{\mathsf{fin}} = \mathsf{e} - \underbrace{\nabla^i(\mathsf{a}_i^{\mathsf{fin}} - \mathsf{a}_i^{\mathsf{in}})}_{\mathsf{el.\ memory}}, \quad \lambda = \int_{\mathsf{in}}^{\mathsf{fin}} A_u \sim \text{‘null’ Wilson line}$$

$\rightsquigarrow$ Ashtekar-Streubel phase space is (partially) superselected at $\mathsf{e} = 0$ (or any other value).

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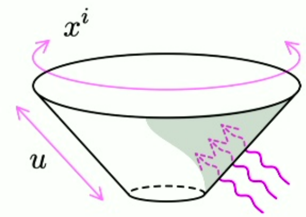

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Flux Superselection

$\partial\mathcal{I} \simeq S^{\mathsf{fin}} \sqcup S^{\mathsf{in}} \rightsquigarrow$ split superselection itself into two stages:

$$\underline{\mathcal{G}} \doteq \mathcal{G}/\mathcal{G}_\circ \simeq \mathcal{G}_S^{\mathsf{fin}} \times \mathcal{G}_S^{\mathsf{in}} \simeq \mathcal{G}_S^{\mathsf{soft}} \ltimes_{\mathsf{AD}} \mathcal{G}_S^{\mathsf{diff}} \quad : \quad (g_{\mathsf{fin}}, g_{\mathsf{in}}) = (g_{\mathsf{soft}}, g_{\mathsf{diff}}^{-1} g_{\mathsf{soft}})$$

Two-stage superselection

$$(1)\ \text{reduction by} \quad \mathcal{G}_S^{\mathsf{diff}} \rightsquigarrow \langle h_{\mathsf{diff}}, \xi_{\mathsf{diff}} \rangle = \oint_S \xi_{\mathsf{diff}}\, \mathsf{e} \quad \rightsquigarrow \quad \text{Ashtekar} - \text{Streubel } \mathcal{P}_{\mathsf{AS}}$$

$$(2)\ \text{reduction by} \quad \mathcal{G}_S^{\mathsf{soft}} \rightsquigarrow \langle h_{\mathsf{soft}}, \xi_{\mathsf{soft}} \rangle = \oint_S \xi_{\mathsf{soft}} \underbrace{\nabla^i(\mathsf{a}_i^{\mathsf{fin}} - \mathsf{a}_i^{\mathsf{in}})}_{\mathsf{el.\ memory}} \quad \rightsquigarrow \quad \underline{\underline{\mathcal{P}}} = \mathcal{P}_{\mathsf{AS}}/\mathcal{G}_S^{\mathsf{soft}}.$$

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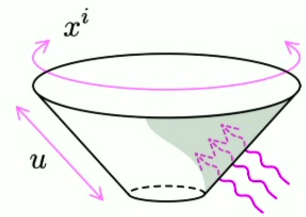

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$$(2)\ \text{reduction by} \quad \mathcal{G}_S^{\mathsf{soft}} \rightsquigarrow \langle h_{\mathsf{soft}}, \xi_{\mathsf{soft}} \rangle = \oint_S \xi_{\mathsf{soft}} \underbrace{\nabla^i(\mathsf{a}_i^{\mathsf{fin}} - \mathsf{a}_i^{\mathsf{in}})}_{\mathsf{el.\ memory}} \quad \rightsquigarrow \quad \underline{\underline{\mathcal{P}}} = \mathcal{P}_{\mathsf{AS}}/\mathcal{G}_S^{\mathsf{soft}}.$$

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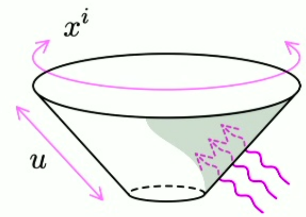

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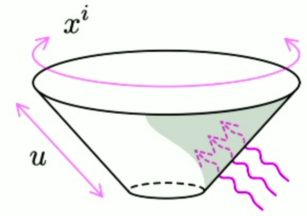
$$\text{(2) reduction by} \quad \mathcal{G}_S^{\mathsf{soft}} \rightsquigarrow \langle h_{\mathsf{soft}}, \xi_{\mathsf{soft}} \rangle = \oint_S \xi_{\mathsf{soft}} \underbrace{\nabla^i(\mathsf{a}_i^{\mathsf{fin}} - \mathsf{a}_i^{\mathsf{in}})}_{\mathsf{el.\ memory}} \quad \rightsquigarrow \quad \underline{\underline{\mathcal{P}}} = \mathcal{P}_{\mathsf{AS}}/\mathcal{G}_S^{\mathsf{soft}}.$$

NULL MAXWELL Th. - Remarks



- AS phase space is a partially superselected phase space, for initial electric flux $\mathsf{e} = fixed$
- 1-sphere extension of AS appears naturally, but not related to memory
- Memory / soft modes are already present in the AS phase space over compact $\mathfrak{I}$ (!)
- *No* need to restrict to 'electric vacua', like $\mathsf{a}_i^{\mathsf{in}} = \nabla_i \varsigma^{\mathsf{in}}$ and similarly for $\mathsf{a}_i^{\mathsf{fin}}$ (and anyway, these would have *nothing* to do with 'corner gauge')
- Non-Abelian YM : more involved, e.g., 'color memory' is not a momentum map / superselection.

WRAP UP

Symplectic reduction of a gauge theory on a hypersurface Σ with boundaries happens in *stages*.

First stage, $\mathcal{P} \rightsquigarrow \underline{\mathcal{C}} = \mathcal{C}/\mathcal{G}_\circ$: constraint reduction takes care of the constraint and 'bulk' gauge transformations.

Second stage, $\underline{\mathcal{C}} \rightsquigarrow \underline{\underline{\mathcal{P}}} = \underline{\mathcal{C}}/\underline{\mathcal{G}} \simeq \mathcal{C}/\mathcal{G}$: flux superselection yields a *Poisson* space that is not symplectic.
Its symplectic leaves, called flux superselection sectors, are labelled by electric fluxes,
viz. by the Casimirs of the corner Noether charge algebra. (Toy model: $\underline{\mathcal{C}} = \mathsf{T}^*G \circlearrowleft G \rightsquigarrow \underline{\underline{\mathcal{P}}} = \mathsf{T}^*G/G \simeq \mathfrak{g}^*$.)

In YM theory over a **spacelike** hypersurface Σ [non-canonical, non-local isomorphisms (!)] :

$$\underline{\mathcal{C}} \simeq_{\mathsf{loc}} \mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathsf{T}^*\underline{\mathcal{G}} \quad \text{and} \quad \underline{\underline{\mathcal{P}}} \simeq_{\mathsf{loc}} \bigsqcup_{\mathcal{O}_f} \underline{\underline{\mathcal{S}}}_{[f]} \quad \text{with} \quad \underline{\underline{\mathcal{S}}}_{[f]} \simeq_{\mathsf{loc}} T^*\mathcal{A}^{\mathsf{rad}} \times \mathcal{O}_f.$$

In Maxwell th., $\underline{\underline{\mathcal{P}}} \simeq \mathsf{T}^*\mathcal{A}^{\mathsf{rad}} \times \mathfrak{F} \simeq$ *space of initial data for gauge invariant on-shell histories in* $D(\Sigma)$.
The flux $f \in \mathfrak{F}$ fixes the Coulombic sector and has no symplectic partner $\rightsquigarrow$ superselected.

In YM theory over a **null** hypersurface $\mathcal{I}$:

$$\underline{\mathcal{C}} \simeq \mathcal{P}_{\mathsf{AS}} \times \mathsf{T}^*\mathcal{G}_S$$

and splitting $\underline{\mathcal{G}} = \mathcal{G}_S^{\mathsf{soft}} \ltimes \mathcal{G}_S^{\mathsf{diff}}$ one gets a two-stage superselection:
(1) superselecting wrt $\mathcal{G}_S^{\mathsf{diff}}$ at $E^{\mathsf{in}} = 0$ yields the Ashtekar-Streubel phase space $\mathcal{P}_{\mathsf{AS}}$
(2) superselecting wrt $\mathcal{G}_S^{\mathsf{soft}}$ requires fixing the electromagnetic memory $E^{\mathsf{fin}} - E^{\mathsf{in}} = -\nabla^i(A_i^{\mathsf{fin}} - A_i^{\mathsf{in}})$.

REFERENCES

For a pedagogical overview:
AR, 'Edge modes without edge modes' (2021), 2104.10182

For details on the abstract theory of stage reduction and applications to spacelike YM:
AR and Schiavina, '*Hamiltonian gauge theory with corners: constraint reduction and flux superselection*', ATMP (to appear), 2207.00568v3

For a summary of the abstract theory and applications to finite and asymptotic null YM:
AR and Schiavina, '*Null Hamiltonian Yang-Mills theory: Soft symmetries and memory as superselection*', Ann. Henri Poincaré (2024), doi : 10.1007/s00023-024-01428-z

For a comparison with edge modes and an approach to reduction based on 'reference frames':
AR, '*Symplectic reduction of Yang-Mills theory with boundaries: from superselection sectors to edge modes, and back*', SciPost Phys. (2021). doi : 10.21468/SciPostPhys.10.6.125

For a theorem on gluing of spacelike YM:
Gomes and AR, '*The quasilocal degrees of freedom of Yang-Mills theory*', SciPost Phys. (2021), doi : 10.21468/SciPostPhys.10.6.130

More on edge modes, dressings and reference frames:
Gomes and AR, *The observer's ghost: notes on a field space connection*, JHEP (2016)
Gomes, Hopfmüller, and AR, *Unified geometric framework for boundary charges and particle dressings*, Nuclear Phys B (2019)

GLUING [Gomes + AR 2021]

There is an explicit (nonlocal!) formula to glue two superselection sectors over $B_3^+ \sqcup_{S_2} B_3^- = S_3$.

It shows that superselection in B_3^+ is a consequence of tracing out the radiative d.o.f. in B_3^-, viz.

$$f = \left(\mathcal{R}_+^{-1} + \mathcal{R}_-^{-1}\right)^{-1} \Delta_{(S_2)}^{-1} \nabla^i_{(S_2)} \left(E^+_{\mathsf{rad}} - E^-_{\mathsf{rad}}\right)_i \qquad \mathcal{R}_\pm \doteq \text{Dirichlet-to-Neumann op.}$$

DRESSINGS AND REFERENCE FRAMES [Gomes + AR 2016-19]

'Reducing', i.e. quotienting gauge, is elegant but very abstract.

Alternatively, one can fix gauge or, more generally, use a functional connection on $\mathcal{C}$.

Gauge fixings and functional connections, are intimately related to dressings
(e.g. Dirac dressing = dressing for Coulomb gauge)

They also have a nice interpretations in terms of dynamical and intrinsic 'gauge reference frames'.

See also [Vilkovisky, DeWitt, Lavelle + McMullan] on dressings; [Rovelli's 'why gauge'] + [Gomes's PhD thesis] on gauge and relationalism; [Hoehn, Carrozza, +] on edge modes and reference frames; [Bartlett + Spekkens + al.] on QRF and superselection