Title: Condensation in topological orders and topological holography

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Collection/Series: Quantum Matter

Subject: Condensed Matter

Date: October 22, 2024 - 3:30 PM

URL: https://pirsa.org/24100131

Abstract:

Condensation of topological defects is the foundation of the modern theory of bulk-boundary correspondence, also known as topological holography. In this talk, I discuss string condensation in 3+1D topological orders, which plays a role analogous to anyon condensation in 2+1D topological orders. I will demonstrate through examples how they correspond to 2+1D symmetry enrichd phases, including both gapped and gapless phases. Then I give a detailed analysis of string condensaiton in 3+1D discrete gauge theories. I compute the outcome of the condensation, namely the category of excitations surviving the condensation. The results suggest that a complete topological holography for 2+1D phases can only be established by taking into account all possible ways of condensing strings in the bulk 3+1D topological order.

Condensation in topological orders and topological holography

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Quantum Matter Seminar, Perimeter Institute, October 22, 2024

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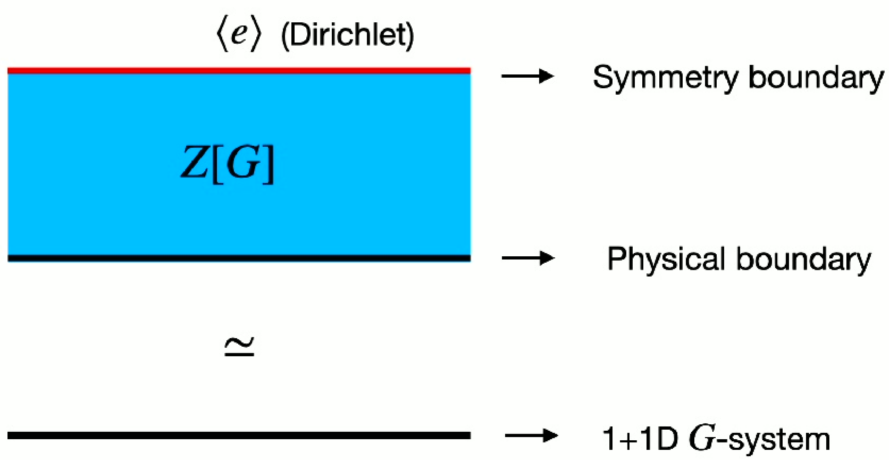
Topological holography from bulk-boundary relation

Let $\mathcal{S}$ be a 1+1D G-system, then it can be coupled to a G-gauge field(in the case of SSB, we may Higgs the gauge field to a subgroup), therefore it defines a boundary condition of the 2+1D G-gauge theory.

Conversely, any boundary condition of the 2+1D G-gauge theory defines a 1+1D system with G-symmetry(the sandwich construction):



Topological holography from bulk-boundary relation

Therefore we have:

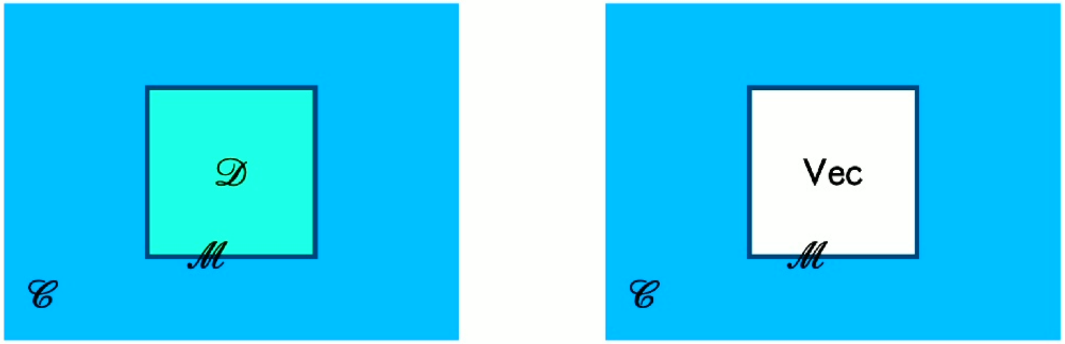
1+1D gapped G-phases $\Leftrightarrow$ Gapped boundaries of 2+1D G-gauge theory

For finite G, 2+1D G-gauge theory is a topological order(quantum double model).

There is a beautiful theory about gapped boundaries of 2+1D topological orders, known as the bulk-boundary relation. [Kong 13,Kong-Wen-Zheng 15, Kong-Wen-Zheng 17]

Bulk-boundary relation in 2+1D

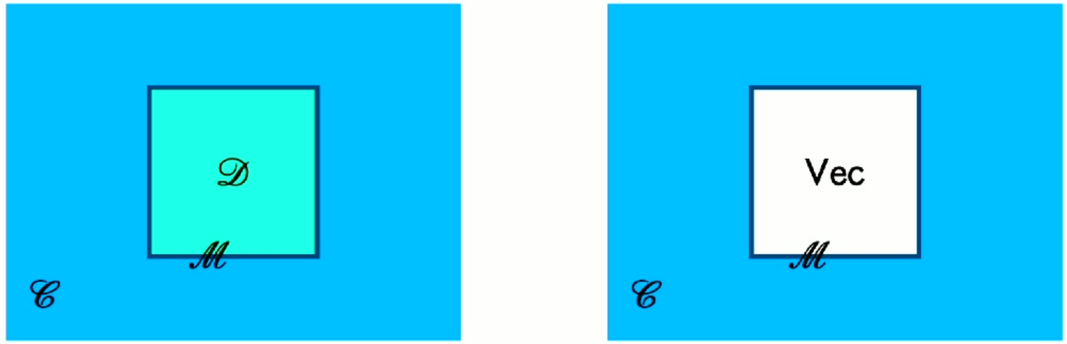
The general setup of an anyon condensation is shown below:



Left: A condensation happens in some region inside the phase $\mathcal{C}$, resulting in a new phase $\mathcal{D}$, and a gapped wall $\mathcal{M}$ is formed between $\mathcal{C}$ and $\mathcal{D}$.
Right: When the condensation is maximum, $\mathcal{D}$ is trivial and $\mathcal{M}$ becomes a gapped boundary of $\mathcal{C}$. Conversely, all gapped boundaries arise in this way.
Furthermore, the bulk is the center of the (gapped)boundary:

$$\mathcal{C} = \mathcal{Z}[\mathcal{M}] \tag{1}$$

Bulk-boundary relation in 2+1D



The region $\mathcal{D}$, when shrunk to a point, can be viewed as an anyon(often composite) of the original phase $\mathcal{C}$. It carries some additional structure that makes it a connected commutative algebra $\mathcal{A}$ in $\mathcal{C}$.
The new order $\mathcal{D}$ is formed by local modules over $\mathcal{A}$:

$$\mathcal{D} \cong \mathrm{Mod}^0_{\mathcal{C}}(\mathcal{A}). \quad (2)$$

When the new phase is trivial, $\mathcal{D} = \mathrm{Vec}$, we say the condensation is Lagrangian.
Therefore: Gapped boundaries $\Leftrightarrow$ Lagrangian algebras.

Bulk-boundary relation in 2+1D

Recall

$$1{+}1\text{D gapped } G\text{-phases} \Leftrightarrow \text{Gapped boundaries of } Z[G]$$

Now by the bulk-boundary relation, we conclude

$$1{+}1\text{D gapped } G\text{-phases} \Leftrightarrow \text{Lagrangian algebras in } Z[G]$$

Quick check: Classification of 1+1D gapped G-phases: SSB+SPT

$$(K,\omega),\ K < G, \omega \in H^2[K, U(1)] \tag{3}$$

Classification of Lagrangian algebras in 2+1D G-gauge theory[Davydov 09]:

$$(K,\omega),\ K < G, \omega \in H^2[K, U(1)] \tag{4}$$

Bulk-boundary relation in 2+1D

Example: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, there are 6 gapped phases:

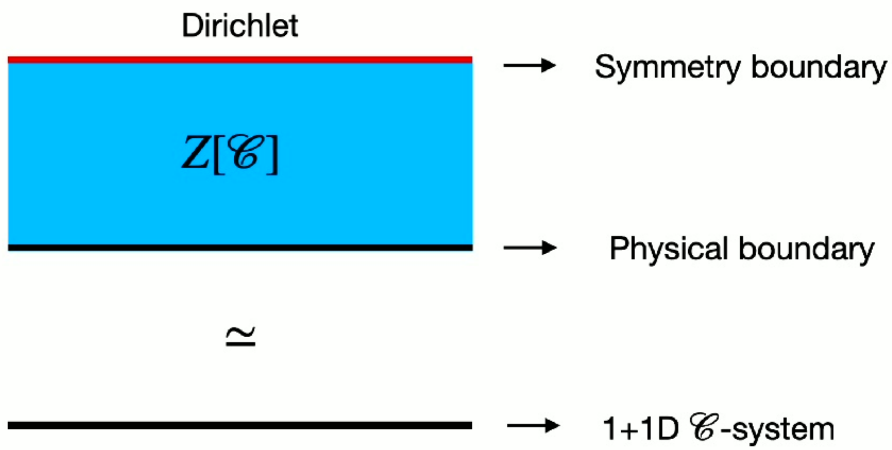
Unbroken subgroup	# of SPTs
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2
$\mathbb{Z}_2^A$	1
$\mathbb{Z}_2^B$	1
$\mathbb{Z}_2^d$	1
0	1

There are also 6 Lagrangian algebras in the 2+1D $\mathbb{Z}_2^2$-gauge theory:

Lagrangian subgroups
$\mathcal{A}^1 = 1 + m_1 + m_2 + m_1 m_2$
$\mathcal{A}^2 = 1 + e_1 m_2 + e_2 m_1 + m_1 m_2$
$\mathcal{A}^3 = 1 + m_1 + e_2 + m_1 e_2$
$\mathcal{A}^4 = 1 + e_1 + m_2 + e_1 m_2$
$\mathcal{A}^5 = 1 + e_1 e_2 + m_1 m_2 + f_1 f_2$
$\mathcal{A}^6 = 1 + e_1 + e_2 + e_1 e_2$

Topological holography for 1+1D gapped phases: the general theory

A general finite unitary symmetry in 1+1D is described by a fusion category $\mathcal{C}$. The symmetry topological order is the center $\mathcal{Z}[\mathcal{C}]$. Any $\mathcal{C}$-phase can be obtained via a sandwich construction:



In particular, gapped $\mathcal{C}$-phases are in 1-1 correspondence with Lagrangian algebras in $\mathcal{Z}[\mathcal{C}]$.

Poweful tool for analyzing non-invertible symmetries. See e.g. [Zhang-Cordova 23],[Kaidi-Ohmori-Zheng 22],[Bhardwaj-Nameki 23]

Topological holography for gapless SPTs

To answer this, let me start by introducing gapless SPT phases.

Definition(gSPT)

A Γ-symmetric system is called a gapless SPT, if there is a gap Δ_c, such that all charges transforming nontrivially under a normal subgroup $N \lhd G$ only appear above Δ_c, and all charges transforming trivially under N are gapless.

Below the gap Δ_c, we don't see any N-charges, therefore the symmetry reduces to $G := \Gamma/N$ in the IR, called the IR symmetry.
The IR symmetry could be anomalous, called the emergent anomaly of the gSPT.

Topological holography for gapless SPTs

Can we say anything about gapless phases?

Since gapped boundaries must have Lagrangian condensation, a boundary with a non-Lagrangian condensation must be gapless[Ji-Wen 19],[Chatterjee-Wen 22],[Chatterjee-Ji-Wen 22].

Example: A gapped boundary of the toric code must have either e condensed or m condensed. If neither is condensed, the boundary must be critical, which is equivalent to the Ising CFT.

What can the non-Lagrangian condensation tell us about the gapless boundary?

Some generalities of gSPTs

Definition(gSPT)

A Γ-symmetric system is called a gapless SPT, if there is a gap Δ_c, such that all charges transforming nontrivially under a normal subgroup $N \triangleleft \Gamma$ only appear above Δ_c, and all charges transforming trivially under N are gapless.

We call N the gapped symmetry, which can have topological behaviours such as string orders. These topological behaviours are stable as long as Δ_c is not closed.

Sometimes these topological properties can not be observed in any gapped system with the same symmetry, in which case we say the gapless SPT is an intrinsically gapless SPT(igSPT).

Topological holography for 1+1D gSPTs

The simplest 1+1D igSPT is the so-called $\mathbb{Z}_4$-igSPT, it is realized by the following lattice model [RW-Potter 22]:

$$
\begin{aligned}
H_{\mathbb{Z}_4-\mathrm{igSPT}} &= H_0 + H_\Delta \\
H_0 &= -g \sum_i \left(\tau^z_{i-1/2} \sigma^x_i \tau^z_{i+1/2} + \tau^y_{i-1/2} \sigma^x_i \tau^y_{i+1/2} \right) \\
H_\Delta &= \Delta \sum_i \sigma^z_{i-1} \tau^x_{i-1/2} \sigma^z_i
\end{aligned} \tag{5}
$$

The system has a $\mathbb{Z}_4$ symmetry generated by

$$
U_s = \prod_i \sigma^x_i e^{\frac{i\pi}{4}(1-\tau^x_{i+1/2})} \tag{6}
$$

Below the energy scale Δ, $\tau^x_{i-1/2}$ is locked to $\sigma^z_{i-1}\sigma^z_i$, and the symmetry effectively reduces to an *anomalous* $G = \mathbb{Z}_2$ symmetry:

$$
U_s \approx \prod_i \sigma^x_i e^{\frac{i\pi}{4}(1-\sigma^z_i \sigma^z_{i+1})} \tag{7}
$$

Topological holography for 1+1D gSPTs

At low energies the model reduces to the critical spin-chain:

$$H_0 \approx -\sum_i \sigma_i^x - \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z. \tag{8}$$

which is exactly the gapless boundary of the Levin-Gu phase.

There is also a nontrivial string order parameter:

$$S_{i,j} := \sigma_i^z \prod_{i \leq k \leq j} e^{\frac{i\pi}{2}(1-\tau_{k+1/2}^x)} \sigma_j^z \tag{9}$$

which is the $2 \in \mathbb{Z}_4$-symmetry defect decorated with double $\mathbb{Z}_4$-charges on the edges. In the IR we have

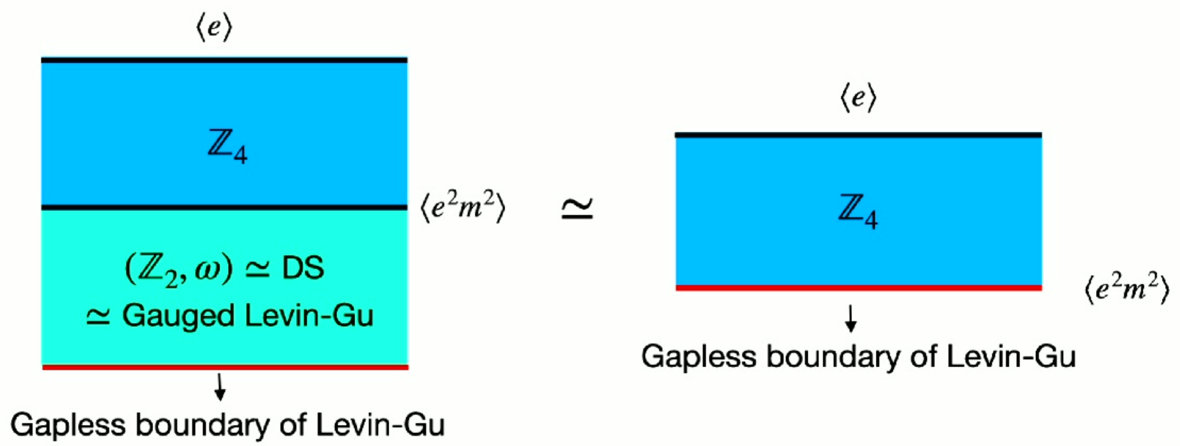
$$S_{i,j} \sim \sigma_i^z \sigma_i^z \sigma_j^z \sigma_j^z = 1 \tag{10}$$

which leads to a 2-fold GSD on an open chain. This GSD can not be lifted unless Δ_c goes to zero.
Notice $H^2[\mathbb{Z}_4, U(1)] = 0$, thus the $\mathbb{Z}_4$-gSPT is an igSPT.

Topological holography for gapless SPTs

Is there a holographic dual for the $\mathbb{Z}_4$-igSPT?

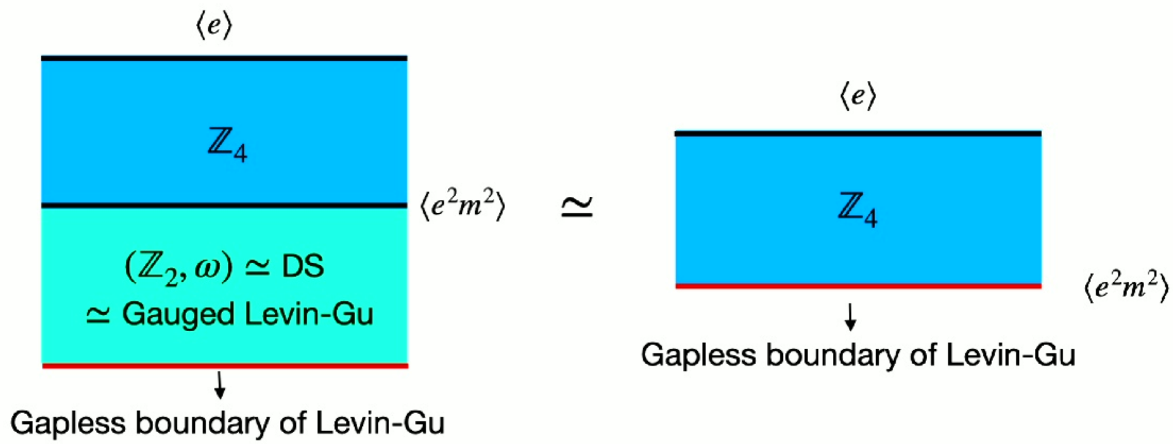
Consider the following sandwich construction:[RW-Potter 23],[Huang-Cheng 23]



It recovers all essential features of the $\mathbb{Z}_4$-gSPT:

1. The full symmetry is $\mathbb{Z}_4$, implemented by defects at the top boundary.
2. In the IR the system has anomalous $\mathbb{Z}_2$ symmetry.
3. The condensation of $e^2 m^2$ implies the $2 \in \mathbb{Z}_4$-domain wall is decorated with a double charge.
4. One can see the 2-fold GSD by considering open boundaries.

Topological holography for gapless SPTs



In conclusion, the non-Lagrangian condensation $e^2 m^2$ encodes all topological properties of the $\mathbb{Z}_4$-igSPT: emergent anomaly, string order, edge mode.

The condensation does not say much about the critical behaviour(e.g. scaling dimension) of the igSPT, but determines all topological properties.

Topological holography for gapless SPTs

A complete holographic duality for 1+1D gSPTs:

In [RW-Potter 23], we obtained a classification of 1+1D Γ-gSPTs, which turns out to be identical to the classification of condensable algebras in the 2+1D Γ-gauge theory.
Furthermore, since the condensation is not Lagrangian, it generically leaves behind a twisted $G = \Gamma/N$-gauge theory. This twist turns out to be exactly the same as the emergent anomaly of the corresponding 1+1D gSPT[RW-Potter 23].

The complete dictionary

non-Lagrangian condensation of $Z(\Gamma)$ ⟷ **1d Γ-gSPT**

$Z(\Gamma)$
↓
Non-Lagrangian condensation
↓
Twisted gauge theory: $Z(\Gamma/N, \omega)$
twist: $\omega \in H^3[\Gamma/N, U(1)]$

Full symmetry: Γ

Symmetry extension:
$1 \to N \to \Gamma \to \Gamma/N \to 1$

Emergent anomaly
$\omega \in H^3[\Gamma/N, U(1)]$

=

Generalizations

Topological holography for 1+1D gSPTs with fermionic and/or non-invertible symmetries: [RW-Ye-Potter 24],[Huang 24],[Bhardwaj-Inamura-Tiwari 24],[Bhardwaj-Bottini-Pajer-Nameki 24]

3+1D gSPTs from topological holography: [Antinucci-Copetti-Nameki 24]

Topological holography in 2+1D/3+1D

As we have seen, topological holography really comes from the bulk-boundary relation of topological orders.

Based on the bulk-boundary relation for 3+1D topological orders proposed in [Kong-Zhang-Zhao-Zheng 24], I will show that topological holography indeed holds in 2+1D/3+1D, for both gapped phases and gapless SPTs.

The bulk-boundary relation is again intimately related to condensation, therefore I will start by discussing the condensation theory in 3+1D, developed in [Kong-Zhang-Zhao-Zheng 24].

Condensation theory in 3+1D topological order

In a 3+1D topological order, there are two types of fundamental topological defects: strings and particles.

However, particle condensation can be reduced to string condensation: we first condense the particle along a string, obtaining the so-called condensation descendant of the particle, then condensation of the particle is equivalent to the condensation of the descendant string.

Example: in the 4D $\mathbb{Z}_2$-toric code, condensing the charge is the same as condensing the Cheshire string.

$\Rightarrow$ String condensation is all you need.

Condensation theory in 3+1D topological orders

The theory of string condensation is similar to that of anyon condensation in 2+1D:[Kong-Zhang-Zhao-Zheng 24]

Theory of string condensation

A string condensation is defined by a condensable algebra $\mathcal{A}$. The residual phase is given by local modules over $\mathcal{A}$:

$$\mathrm{Mod}^0_{\mathcal{B}}(\mathcal{A}) \tag{11}$$

We say a condensation is Lagrangian if the residual phase is trivial.

Bulk-boundary relation

All gapped boundaries of a 4D topological order are obtained by Lagrangian condensations. Furthermore, let $\mathcal{M}$ be the category of excitations on a gapped boundary, we have

$$\mathcal{Z}(\mathcal{M}) = \mathcal{B} \tag{12}$$

where $\mathcal{Z}$ is the Drinfeld center. I.e. "bulk=center".

Testing topological holography in 2+1D/3+1D: gapped phases

By the bulk-boundary relation and the sandwich construction, we should see that gapped 2+1D G-phases are in 1-1 correspondence with Lagrangian algebras in the 3+1D G-gauge theory. Is this true?

Classification of gapped 2+1D G-phases: $(H < G, \mathcal{B})$, where H is the unbroken subgroup, and $\mathcal{B}$ is a non-anomalous H-SET.

And we have:

Theorem

[Decoppet-Xu 23] A Lagrangian algebra in $Z[2\mathrm{Vec}_G]$ is determined by a subgroup $H < G$ and a modular tensor category containing Rep(H).

What about gapless SPT? Does the gSPT/non-Lagrangian condensation correspondence hold in 2+1D/3+1D?

Warmup: the minimal 2+1D igSPT

The simplest 2+1D igSPT has symmetry $\mathbb{Z}_2 \times \mathbb{Z}_4$[RW-Potter 22], with a symmetry extension structure

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1. \quad (13)$$

and a nontrivial emergent anomaly in $H^4[\mathbb{Z}_2^2, U(1)] = \mathbb{Z}_2^2$.

The domain wall of the gapped symmetry is decorated with a nontrivial 1+1D SPT of $\mathbb{Z}_2 \times \mathbb{Z}_4$.

If topological holography holds, we should find a string condensation in the 3+1D $\mathbb{Z}_2 \times \mathbb{Z}_4$-gauge theory that leaves behind a twisted $\mathbb{Z}_2 \times \mathbb{Z}_2$-gauge theory.

The decorated domain wall structure of the 2+1D igSPT suggests looking at the string $m_2^2 S$, where m_2 is the flux for $\mathbb{Z}_4$, and S is a defect string corresponding to the nontrivial 1+1D $\mathbb{Z}_2 \times \mathbb{Z}_4$-SPT.

To establish a holographic description for the $\mathbb{Z}_2 \times \mathbb{Z}_4$-igSPT, we just need to show condensing $m_2^2 S$ in the $\mathbb{Z}_2 \times \mathbb{Z}_4$-gauge theory results in a twisted $\mathbb{Z}_2^2$-gauge theory.

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Analyzing the $m_2^2 S$ condensation, method 1

We can simply try to identify all the deconfined strings and particles[RW 2024].

1. Deconfined particles: clearly, e_1, e_2^2 are deconfined. They form $\mathrm{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, suggesting the residual phase is an $\mathbb{Z}_2^2$-gauge theory.
2. m_1 is confined: its braiding with $m_2^2 S$ is e_2^2: the SPT-string S dimension reduces to e_2^2 when threaded with m_1-flux. This leads to nontrivial braiding

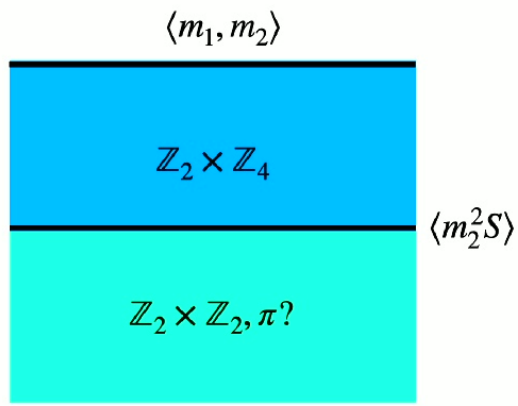
$$\beta_{m_1, m_2^2 S} = e_2^2. \tag{14}$$

To fix this, we can condense e_2^2 on m_1, obtaining a Cheshire string $m_1^{e_2^2}$. This Cheshire string is then deconfined. This is a characteristic of twisted gauge theories: flux loops are Cheshire[Else-Nayak 17].

3. One can analyze the Cheshire charges on other deconfined loops similarly, the result matches exactly with the Cheshire charge structure of one of the twisted-$\mathbb{Z}_2^2$ gauge theory[RW 2024].

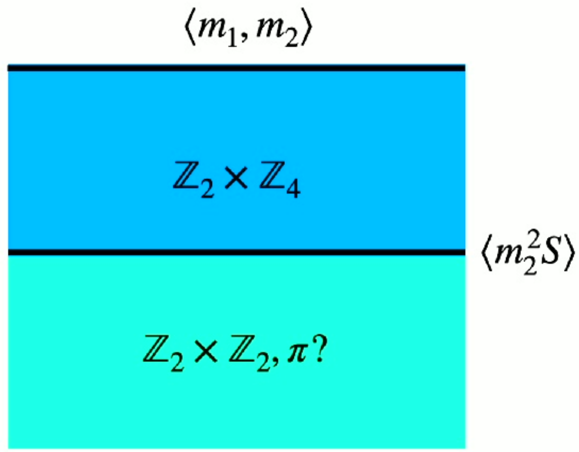
Analyzing the $m_2^2 S$ condensation, method 2

There is a more systematic approach, which not only reveals the structure of the residual phase, but eventually completely solves the condensation problem.
Consider the following configuration:



The top half is the initial $\mathbb{Z}_2 \times \mathbb{Z}_4$-order, the condensation of $m_2^2 S$ takes place in the bottom half of the system.
We add a top boundary for the top half, with m_1, m_2-condensed.

Analyzing the $m_2^2 S$ condensation, method 2

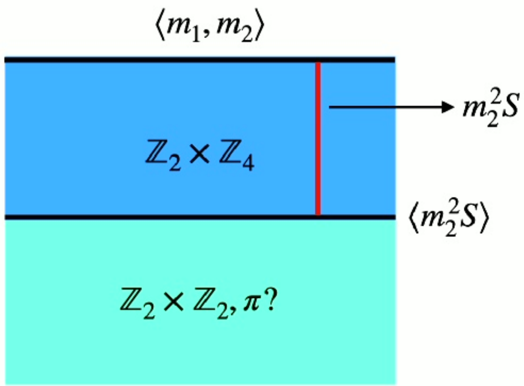


Focusing on the top layer. If we shrink this layer, we may view it as a 2+1D system, then the bottom layer is simply the bulk of the 2+1D system. The twist π is simply the anomaly of this 2+1D system.

Claim: The top layer is equivalent to an anomalous 2+1D SET, namely the toric code enriched by $\mathbb{Z}_2^2$ and symmetry fractionalization pattern $e_C m_P$, where C stands for half charge of $\mathbb{Z}_2$, and P stands for projective representation of $\mathbb{Z}_2^2$.

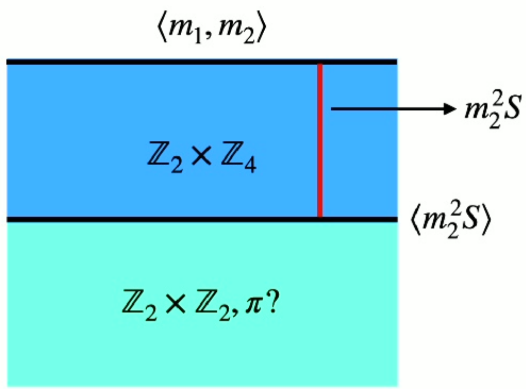
Analyzing the $m_2^2 S$ condensation, method 2

Top layer$=e_C m_P$, proof: The charges in the top layer are: e_1, e_2, with $e_1^2 = e_2^4 = 1$, which all become anyons after shrinking. A vertical string of $m_2^2 S$ can end on both the top boundary and the middle domain wall:



It can end on the middle domain wall because $m_2^2 S$ is condensed there, it can end on the top boundary because m_2 is condensed there and S **can be open**. After shrinking the top layer, this vertical string becomes an anyon as well, denoted as $\widetilde{m}$.

Analyzing the $m_2^2 S$ condensation, method 2



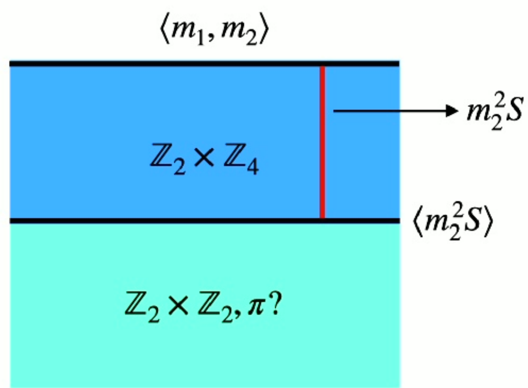
In conclusion, the top layer, viewed as a 2+1D system, contains anyons $e_1, e_2, \widetilde{m}$, where e_1, e_2 have $\mathbb{Z}_2 \times \mathbb{Z}_4$ fusion rule, and we have

$$\widetilde{m}^2 = (1 + e_1)(1 + e_2^2) \qquad (14)$$

due to the projective edge mode of S.
This topological order is **degenerate**, as e_1, e_2^2 are transparent. The only nontrivial braiding is $\theta_{e_2,\widetilde{m}} = -1$. Therefore e_1, e_2^2 should be viewed as symmetry charges of a $\mathbb{Z}_2^2$ symmetry. The remaining anyons are $e_2, \widetilde{m}$.

Clearly e_2 is a half charge of e_2^2, and $\widetilde{m}$ carries projective representation of $\mathbb{Z}_2^2$. □

Analyzing the m_2^2S condensation, method 2

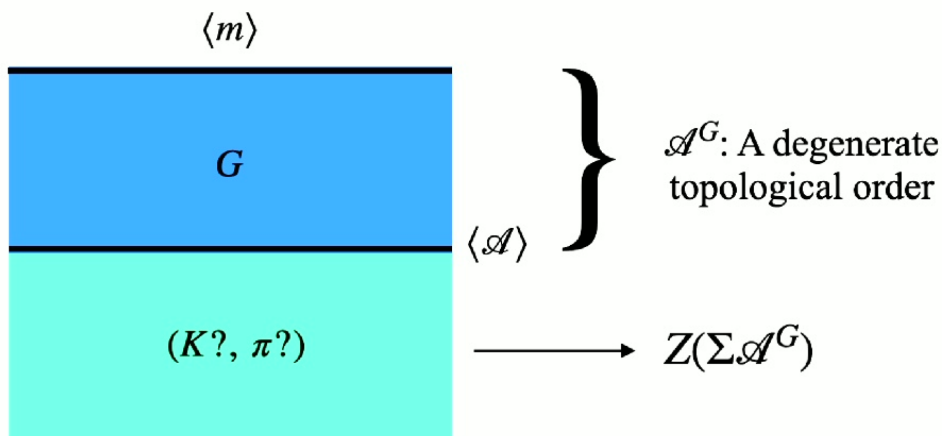


Since the top layer is the $e_C m_P$ SET, and it is well-known that this SET is anomalous, we conclude the bottom layer, being the bulk of the top layer, must be a twisted $\mathbb{Z}_2^2$-gauge theory, with the twist being exactly the H^4-anomaly of the $e_C m_P$ SET.

From either method, we conclude condensing m_2^2S results in a twisted $\mathbb{Z}_2^2$-gauge theory, agreeing with the structure of the 2+1D gSPT.

Some systematical results on condensation in 3+1D

The second method is in fact quite general.



Consider a condensation $\mathcal{A}$ happening the bottom layer. By shrinking the top layer, we obtain a **degenerate** topological order $\mathcal{A}^G$, called the equivariantization of $\mathcal{A}$. Then the bottom layer is simply the bulk of $\mathcal{A}^G$: $\mathcal{Z}[\Sigma\mathcal{A}^G]$.

The question becomes: what is the bulk of a degenerate topological order?

Some systematical results on condensation in 3+1D

The question becomes: what is the bulk of a degenerate topological order?

Example: In the $m_2^2 S$ condensation example, we viewed the degenerate topological order generated by $e_1, e_2, \widetilde{m}$ as an anomalous $\mathbb{Z}_2^2$-enriched SET, with the anyons being $e_2, \widetilde{m}$, and the symmetry charges being e_1, e_2^2. Then we claimed its bulk is a $\mathbb{Z}_2^2$-gauge theory, with the twist given by the anomaly of the SET.

This arguement turns out to be true in general.

Some systematical results on condensation in 3+1D

Definition(Degenerate topological order)

A degenerate 2+1D topological order is a braided fusion category with nontrivial transparent anyons: $\mathcal{Z}_{(2)}(\mathcal{B}) \neq \mathrm{Vec}$.

The transparent anyons $\mathcal{Z}_{(2)}(\mathcal{B})$ either form $\mathrm{Rep}(G)$ or $\mathrm{Rep}(G^f, (-1)^F)$. In either case, the degenerate topological order can be viewed as bosonic or fermionic SET orders, potentially anomalous.

Degenerate topological orders have gravitational anomaly: they can be viewed as excitations in the symmetric subspace of an SET, but the symmetric subspace does not have a tensor product structure. Therefore degenerate topological orders have nontrivial bulk.

Gravitational anomaly of a degenerate topological order

Question: What is the bulk/gravitational anomaly of a degenerate topological order?
Answer[RW 24(to appear)]:

Theorem(Gravitational anomaly of a degenerate topological order)

Let $\mathcal{B}$ be a degenerate topological order with $\mathcal{Z}_{(2)}(\mathcal{B}) = \mathrm{Rep}(G)$, then $\mathcal{Z}[\Sigma\mathcal{B}] \cong \mathcal{Z}[2\mathrm{Vec}_G^{\pi}]$, where π is the H^4-anomaly of $\mathcal{B}_G$.

Gravitational anomaly of a degenerate topological order

Theorem(Gravitational anomaly of a degenerate topological order)

Let $\mathcal{B}$ be a degenerate topological order with $\mathcal{Z}_{(2)}(\mathcal{B}) = \text{Rep}(G)$, then $\mathcal{Z}[\Sigma\mathcal{B}] \cong \mathcal{Z}[2\text{Vec}_G^\pi]$, where π is the H^4-anomaly of $\mathcal{B}_G$.

In[JF-Reutter 21], it was shown that $\mathcal{Z}[\Sigma\mathcal{B}] \simeq \mathcal{Z}[2\text{Vec}_G]$ if and only the H^4-anomaly of $\mathcal{B}$ vanishes. This result is a natural generalization.
Outline of the proof:
1. We prove a Morita equivalence:

$$\Sigma\mathcal{B} \simeq^M \Sigma\mathcal{B}_G \boxtimes 2\text{Vec}_G^\pi, \tag{15}$$

where π is the H^4-anomaly of $\mathcal{B}_G$. Physically, this simply comes from condensing the G-charges in $\mathcal{B}$: the condensation breaks the G-symmetry, and in the symmetry broken phase the G-domain walls carry anomaly π.
2. We then take the bulk of the above relation, noticing Morita equivalence preserves bulk, and bulk of a non-degenerate topological order is trivial, we arrive at the theorem.

Gravitational anomaly of a degenerate topological order

Theorem(Gravitational anomaly of a degenerate topological order)

Let $\mathcal{B}$ be a degenerate topological order with $\mathcal{Z}_{(2)}(\mathcal{B}) = \text{Rep}(G)$, then $\mathcal{Z}[\Sigma\mathcal{B}] \cong \mathcal{Z}[2\text{Vec}_G^\pi]$, where π is the H^4-anomaly of $\mathcal{B}_G$.

Example 1: When $\mathcal{B}$ is non-degenerate, we have $\mathcal{Z}[\Sigma\mathcal{B}] = 2\text{Vec}(\text{vacuum})$.
Example 2: When $\mathcal{B} = \text{Rep}(G)$($G$-charges), we have $\mathcal{Z}[\Sigma\text{Rep}(G)] = \mathcal{Z}[2\text{Vec}_G]$.

This also completely solves string condensation in 3+1D G-gauge theory:

Theorem

Let $\mathcal{A}$ be a string condensation in $\mathcal{Z}[2\text{Vec}_G]$, then the residual order is

$$\text{Mod}^0_{\mathcal{Z}[2\text{Vec}_G]}(\mathcal{A}) \cong \mathcal{Z}[\Sigma\mathcal{A}^G] \cong \mathcal{Z}[2\text{Vec}_K^\pi] \tag{16}$$

where $\mathcal{Z}_{(2)}(\mathcal{A}^G) = \text{Rep}(K)$, and π is the anomaly of $\mathcal{A}^G$, viewed as a K-SET.

Applications

Now we can freely compute any string condensation in a 3+1D G-gauge theory.

Example 1

$G = \mathbb{Z}_N \times \mathbb{Z}_{N^2}$, condensing $m_2^N S$, where S is the minimal 1+1D SPT of G, then the residual phase is a $\mathbb{Z}_N^2$-gauged theory with twist

$$\pi(\vec{i}, \vec{j}, \vec{k}, \vec{l}) = \exp\left(\frac{2\pi i}{N^2} i_1 j_2 (k_2 + l_2 - [k_2 + l_2]_N)\right) \tag{17}$$

Example 2

$G = D_4$, let r^2 be the nontrivial central element of D_4. There is a way to condense the flux m_{r^2}, such that the residual phase is a twisted $\mathbb{Z}_2^2$-gauge theory.

Remark: The above examples are all dual to known 2+1D gSPTs.

Summary

1. Topological properties of d-D phases $\Leftrightarrow$ Condensations in the $d+1$-D bulk:

Gapped phases $\Leftrightarrow$ Lagrangian condensations.
Gapless phases $\Leftrightarrow$ non-Lagrangian condensations.

2. Bulk of a degenerate (2+1D)topological order:

Bulk of $\mathcal{B}$ is a 3+1D twisted gauge theory, with the twist being the H^4-anomaly of an SET associated with $\mathcal{B}$.

3. Computing condensation in 3+1D gauge theory:

Condensable algebra $\Rightarrow$ Degenerate 2+1D topological order
$\Rightarrow$ Residual phase=Bulk of the degenerate topological order