Title: Resource dependence relations

Speakers: Yìlè Yīng

Collection/Series: Quantum Information

Subject: Quantum Information

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Abstract:

A resource theory imposes a preorder over states, with one state being above another if the first can be converted to the second by a free operation, and where the set of free operations defines the notion of resourcefulness under study. In general, the location of a state in the preorder of one resource theory can constrain its location in the preorder of a different resource theory. It follows that there can be nontrivial dependence relations between different notions of resourcefulness.

In this talk, we lay out the conceptual and formal groundwork for the study of resource dependence relations. In particular, we note that the relations holding among a set of monotones that includes a complete set for each resource theory provides a full characterization of resource dependence relations. As an example, we consider three resource theories concerning the about-face asymmetry properties of a qubit along three mutually orthogonal axes on the Bloch ball, where about-face symmetry refers to a representation of $\mathbb{Z}_2$, consisting of the identity map and a π rotation about the given axis. This example is sufficiently simple that we are able to derive a complete set of monotones for each resource theory and to determine all of the relations that hold among these monotones, thereby completely solving the problem of determining resource dependence relations. Nonetheless, we show that even in this simplest of examples, these relations are already quite nuanced.

At the end of the talk, we will briefly discuss how to witness nonclassicality in quantum resource dependence relations and demonstrate it with the about-face asymmetry example.

The talk is based on the preprint: arXiv:2407.00164 and ongoing work.

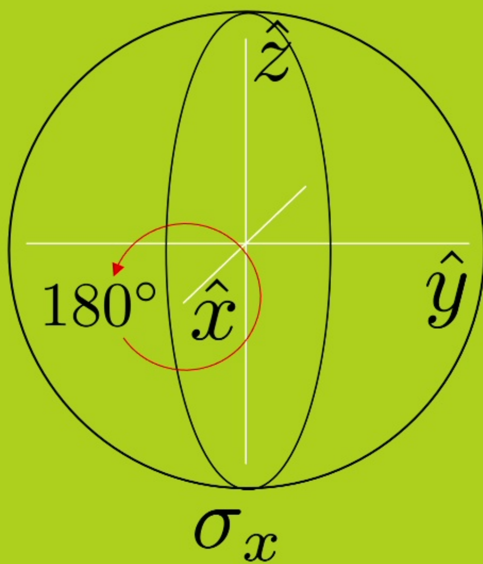
RESOURCE DEPENDENCE RELATIONS

Yìlè Yīng

arXiv:2407.00164 with Tomáš Gonda, Robert W. Spekkens
and ongoing work also with Iman Marvian

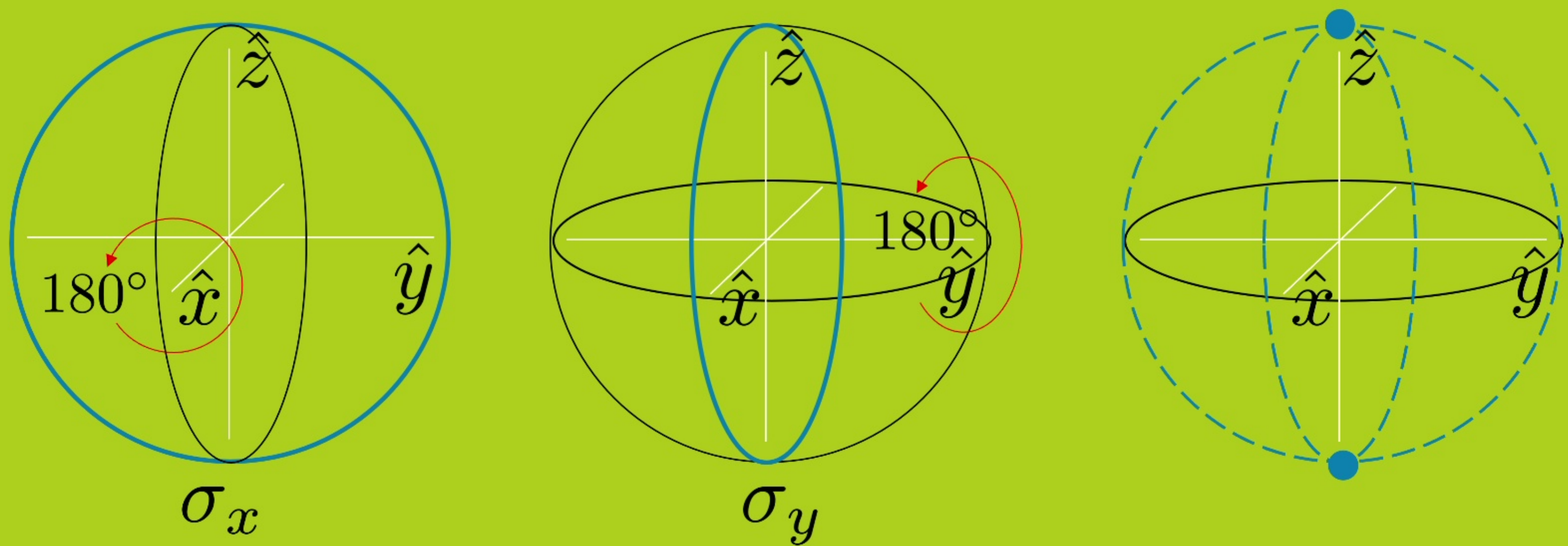
1. Can you find a state perfect for estimating

if a π rotation about the x axis happened or not with one measurement



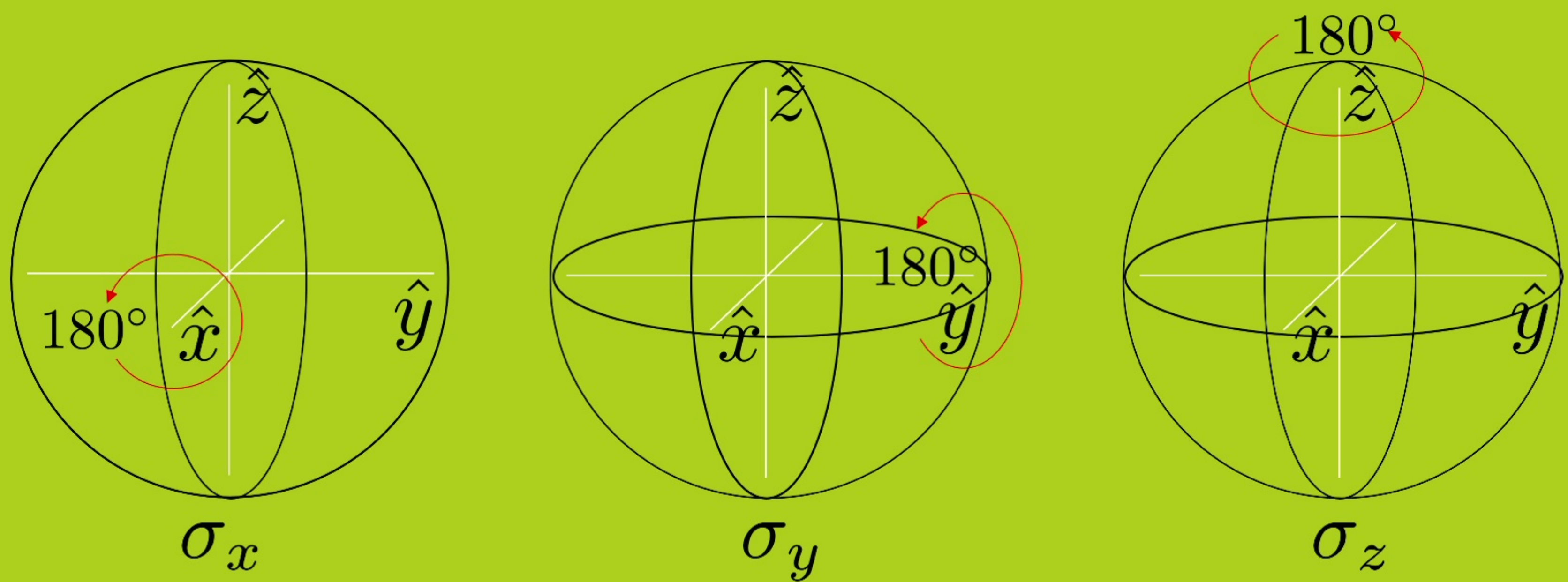
2. Can you find a state perfect for estimating

if a π rotation around ***x or y axis*** happened or not with a measurement

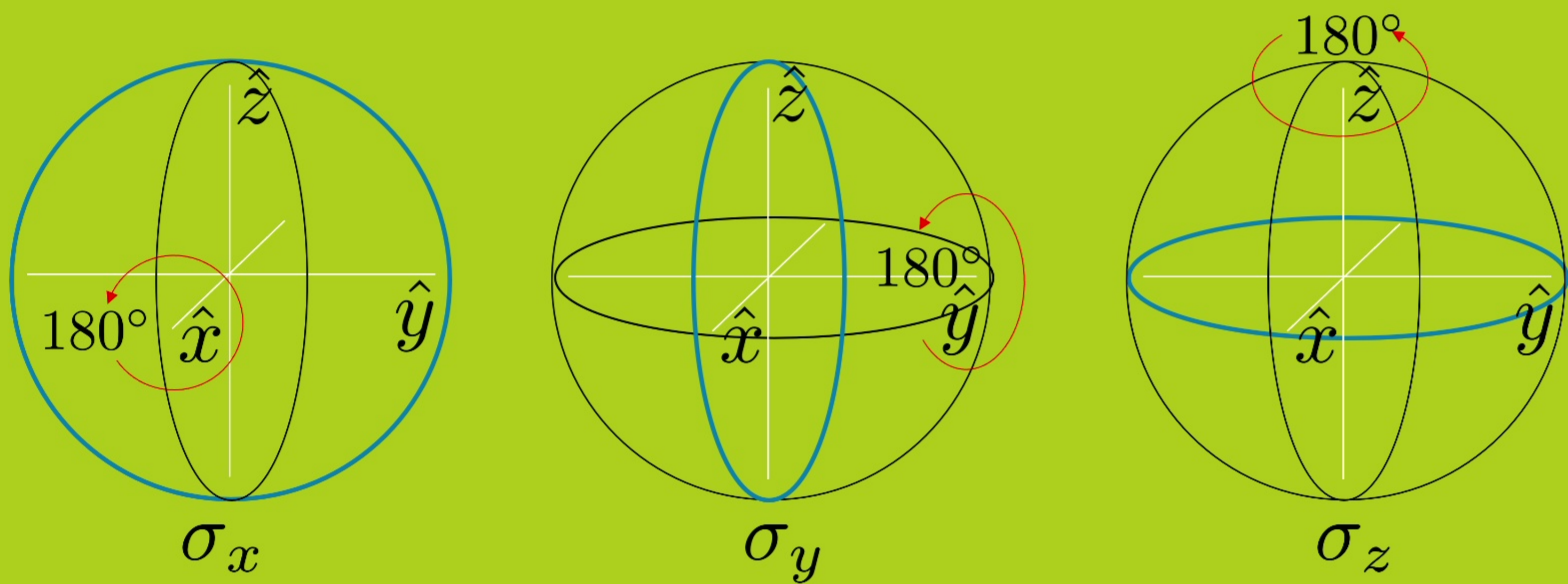


3. Can you find a state perfect for estimating

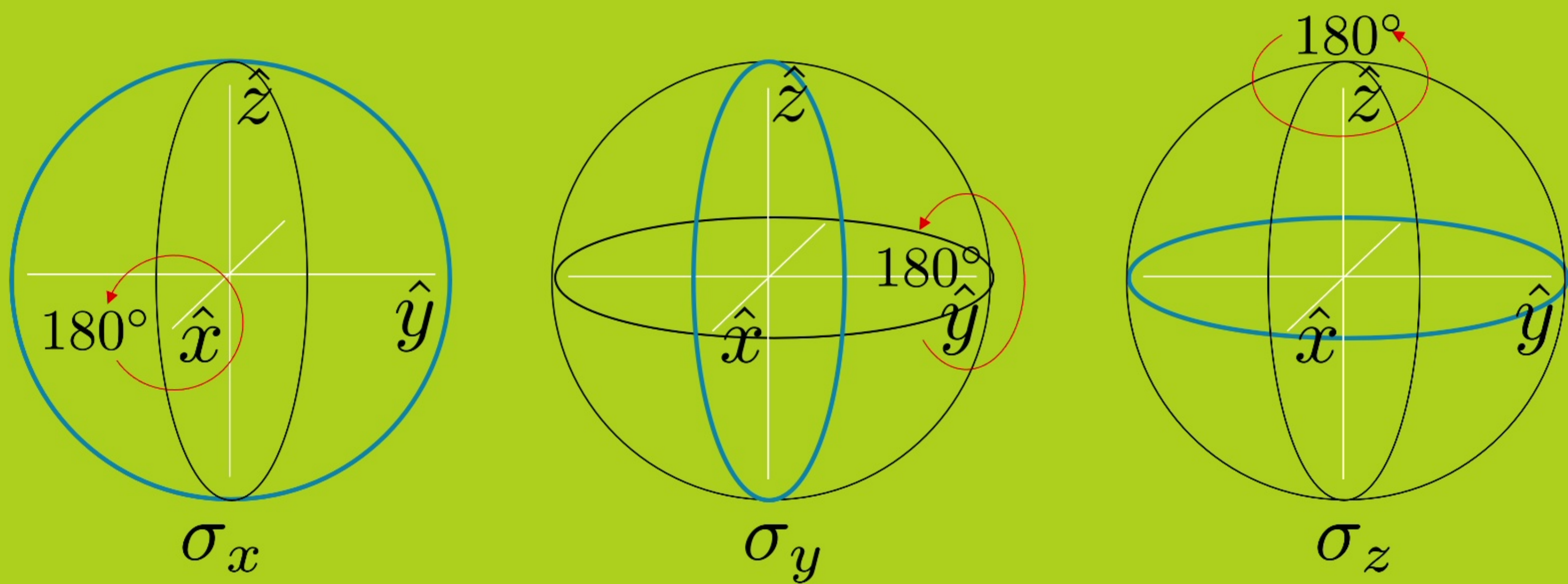
if a π rotation around ***x, y or z axis*** happened or not with a measurement



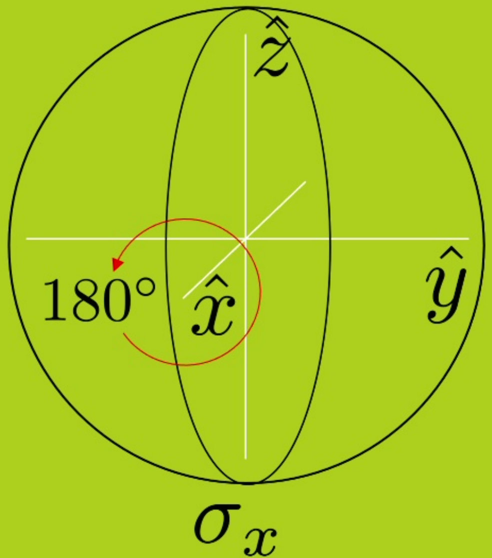
A trade-off among the usefulness for estimating different π rotations



A trade-off among the resourcefulness for estimating different π rotations

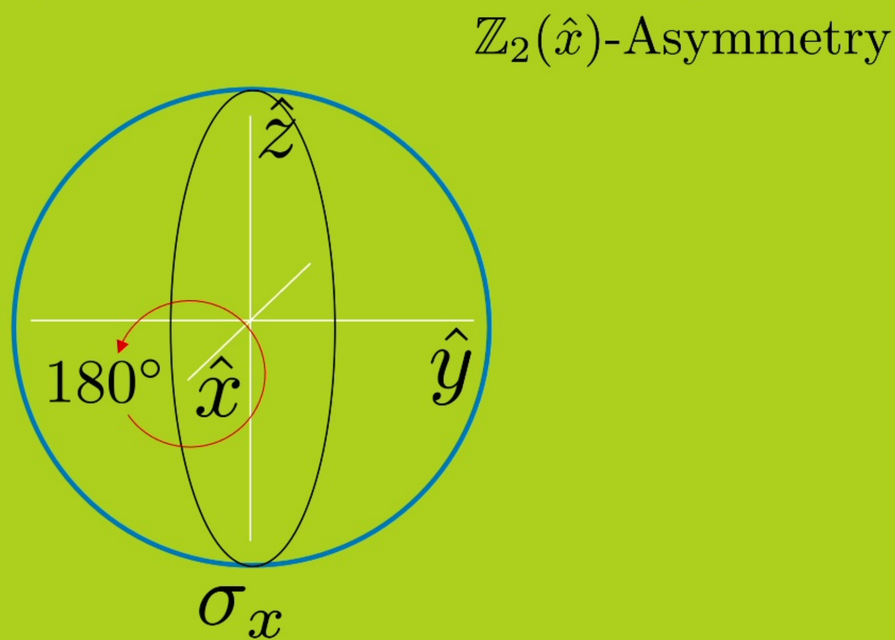


Resource theory of about-face asymmetry

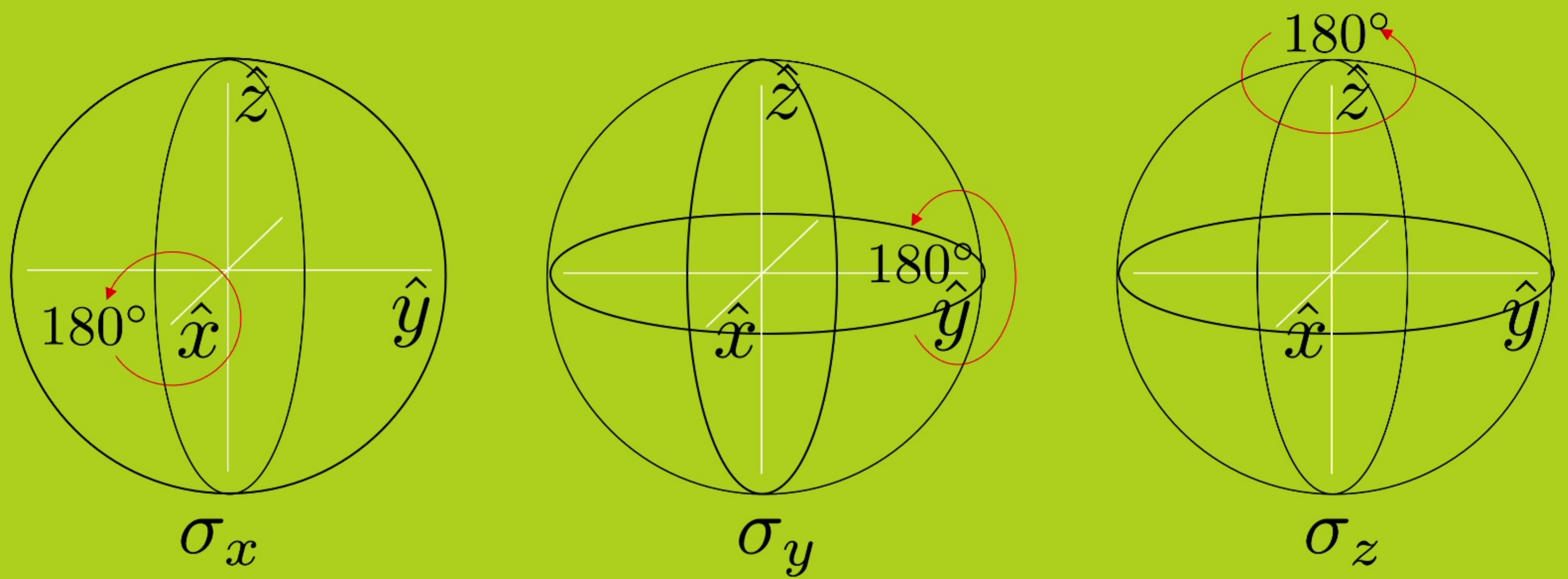


the Z_2 group represented by $\{I, \sigma_x\}$

Resource theory of about-face asymmetry



the Z_2 group represented by $\{I,\sigma_x\}$



$\mathbb{Z}_2(\hat{x})$-Asymmetry $\mathbb{Z}_2(\hat{y})$-Asymmetry $\mathbb{Z}_2(\hat{z})$-Asymmetry

- trade-off

Resource Dependence Relations

Examples

- Monogamy of entanglement
- Uncertainty relations for e.g., skew information
- Asymmetry/Coherence vs. Entanglement

arXiv > quant-ph > arXiv:2407.00164

Quantum Physics

[Submitted on 28 Jun 2024 (v1), last revised 12 Jul 2024 (this version, v2)]

Conceptual and formal groundwork for the study of resource dependence relations

Yìlè Yīng, Tomáš Gonda, Robert Spekkens

Resource Dependence Relations

Examples

- Monogamy of entanglement
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Conceptual and formal groundwork for the study of resource dependence relations

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A recipe for studying resource dependence relations

Completely characterize the resource dependence relations

$\mathbb{Z}_2(\hat{x})$**-Asymmetry,** $\mathbb{Z}_2(\hat{y})$**-Asymmetry,** $\mathbb{Z}_2(\hat{z})$**-Asymmetry**

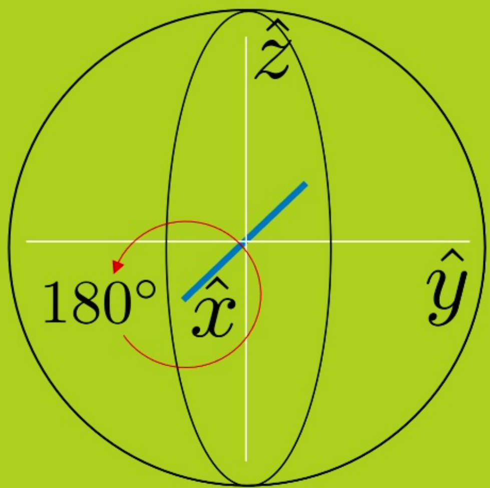
- what a complete solution looks like
- what technical steps are required and what difficulties one may encounter
- how to understand and interpret the results
- where to look for witness of nonclassicality

1. Understand each notion of resourcefulness individually

$\mathbb{Z}_2(\hat{x})$-**Asymmetry**, $\mathbb{Z}_2(\hat{y})$-**Asymmetry**, $\mathbb{Z}_2(\hat{z})$-**Asymmetry**

The Resource Theory of $\mathbb{Z}_2(\hat{x})$-Asymmetry

Free states:



σ_x

The Resource Theory of $\mathbb{Z}_2(\hat{x})$-Asymmetry

Free operation $\mathcal{T}$: $\sigma_x \mathcal{T}(\rho)\sigma_x = \mathcal{T}(\sigma_x \rho \sigma_x)$

$\mathbb{Z}_2(\hat{x})$-covariant operations

- an order relation among the resources

$$\rho \succeq \sigma \Leftrightarrow \rho \xmapsto{\text{free op.}} \sigma$$

$$\rho \sim \sigma \Leftrightarrow \rho \xleftrightarrow{\text{free op.}} \sigma$$

The Resource Theory of $\mathbb{Z}_2(\hat{x})$-Asymmetry

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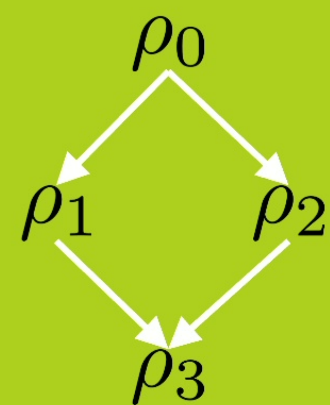


partial order

A complete set of monotones for the $\mathbb{Z}_2(\hat{x})$-asymmetry partial order

Monotone:

$$\rho \xmapsto{\text{free op.}} \sigma \implies f(\rho) \geq f(\sigma)$$

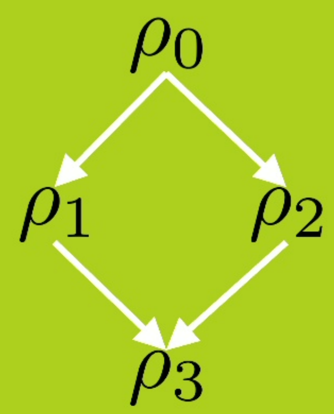
A complete set fully characterize the order

A complete set of monotones for the $\mathbb{Z}_2(\hat{x})$-asymmetry partial order

Monotone:

$$\rho \xmapsto{\text{free op.}} \sigma \implies f(\rho) \geq f(\sigma)$$

A complete set fully characterize the order

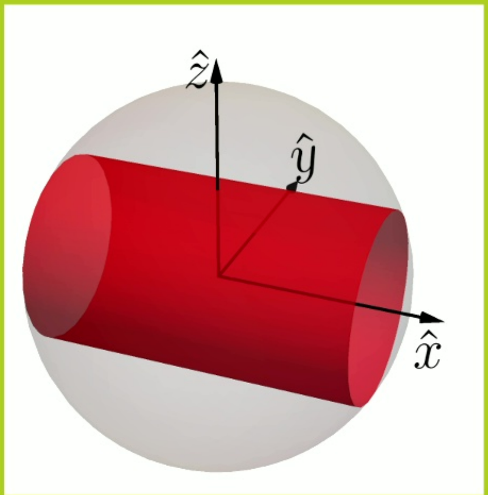


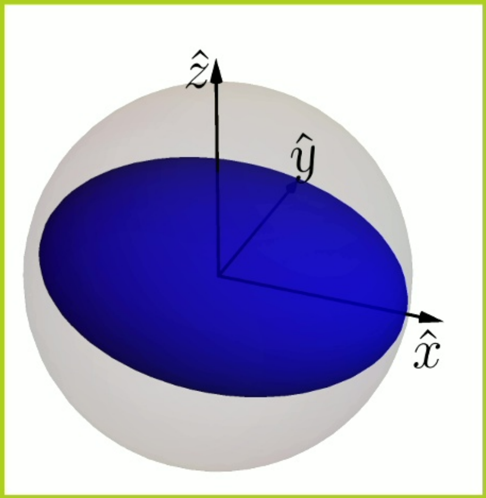
Partial order:

at least two monotones in a complete set

A complete set of monotones for the $\mathbb{Z}_2(\hat{x})$-asymmetry partial order

$$A_x(\rho) := \sqrt{r_y^2 + r_z^2} \qquad B_x(\rho) := \begin{cases} \sqrt{\frac{r_y^2+r_z^2}{1-r_x^2}} & \text{if } r_x^2 < 1 \\ 0 & \text{if } r_x^2 = 1 \end{cases}$$





A complete set of monotones for the $\mathbb{Z}_2(\hat{x})$-asymmetry partial order

$$A_x(\rho) := \sqrt{r_y^2 + r_z^2} \qquad B_x(\rho) := \begin{cases} \sqrt{\frac{r_y^2+r_z^2}{1-r_x^2}} & \text{if } r_x^2 < 1 \\ 0 & \text{if } r_x^2 = 1 \end{cases}$$

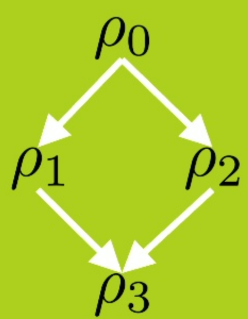
The $\mathbb{Z}_2(\hat{x})$-asymmetry properties of a qubit

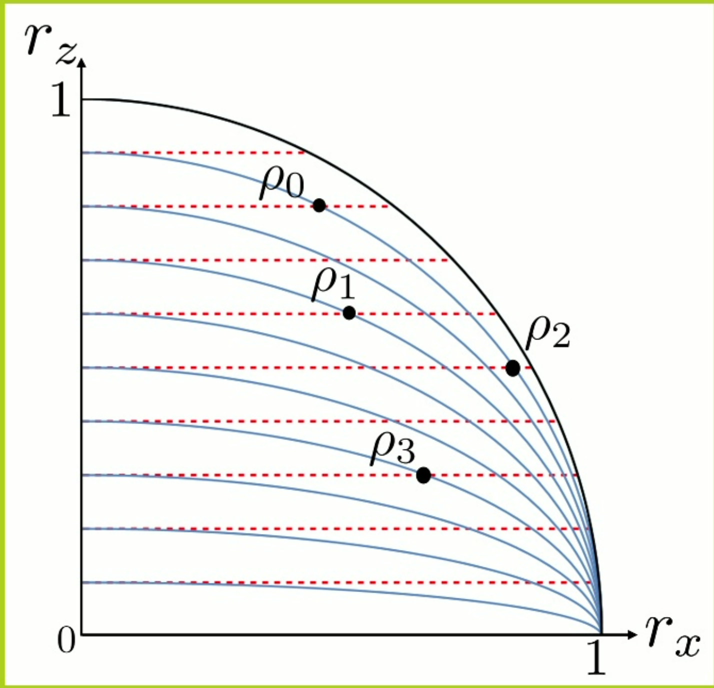
are completely specified by

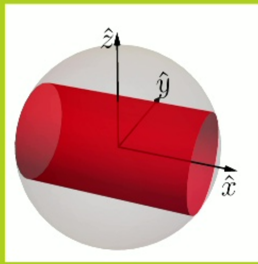
its values of A_x and B_x.

A complete set of monotones for the $\mathbb{Z}_2(\hat{x})$-asymmetry partial order

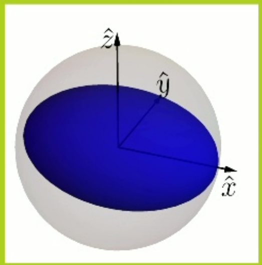
$$\rho \succeq \sigma \iff A_x(\rho) \geq A_x(\sigma) \text{ and } B_x(\rho) \geq B_x(\sigma)$$







A_x

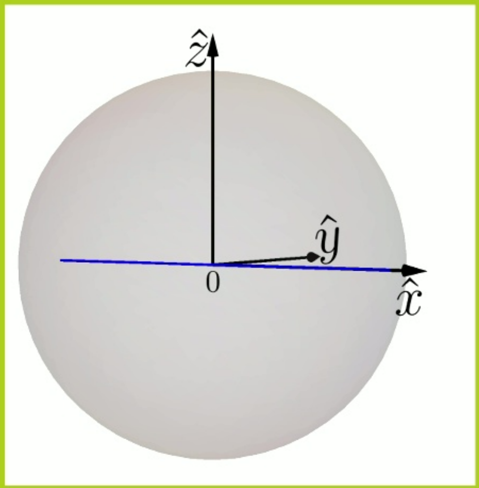


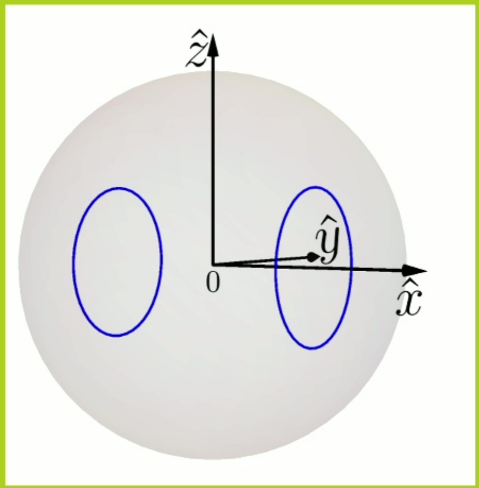
B_x

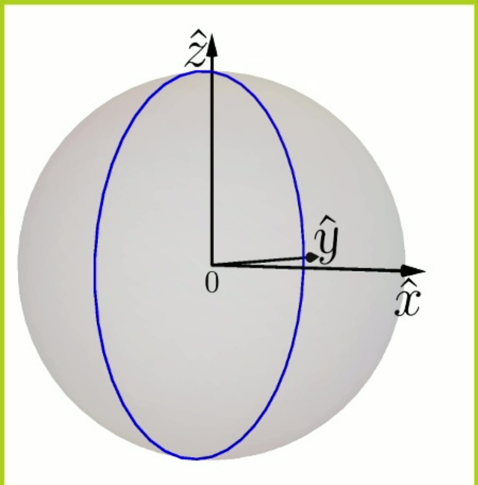
A complete set of monotones for the $\mathbb{Z}_2(\hat{x})$-asymmetry partial order

$$\rho \succeq \sigma \iff A_x(\rho) \geq A_x(\sigma) \text{ and } B_x(\rho) \geq B_x(\sigma)$$

$$\rho \sim \sigma \iff A_x(\rho) = A_x(\sigma) \text{ and } B_x(\rho) = B_x(\sigma)$$







The Recipe

1. Understand each resource theory individually

a) Characterize the partial order (find the complete set of monotones)
b) Find the full set of mathematical constraints on the monotones in the complete set

$$A_x(\rho), B_x(\rho) \in [0,1]$$
$$B_x(\rho) \geq A_x(\rho)$$
$$B_x(\rho) = 0 \text{ if } A_x(\rho) = 0$$

2. Derive dependence relations between resource theories

Derive all constraints on the collection of these six monotones

$\mathbb{Z}_2(\hat{x})$

$$A_x(\rho) := \sqrt{r_y^2 + r_z^2}$$

$$B_x(\rho) := \begin{cases} \sqrt{\frac{r_y^2+r_z^2}{1-r_x^2}} & \text{if } r_x^2 < 1 \\ 0 & \text{if } r_x^2 = 1 \end{cases}$$

$\mathbb{Z}_2(\hat{y})$

$$A_y(\rho) := \sqrt{r_x^2 + r_z^2}$$

$$B_y(\rho) := \begin{cases} \sqrt{\frac{r_x^2+r_z^2}{1-r_y^2}} & \text{if } r_y^2 < 1 \\ 0 & \text{if } r_y^2 = 1 \end{cases}$$

$\mathbb{Z}_2(\hat{z})$

$$A_z(\rho) := \sqrt{r_x^2 + r_y^2}$$

$$B_z(\rho) := \begin{cases} \sqrt{\frac{r_x^2+r_y^2}{1-r_z^2}} & \text{if } r_z^2 < 1 \\ 0 & \text{if } r_z^2 = 1 \end{cases}$$

Equality constraints

$$2\left[B_x^2(\rho) - A_x^2(\rho)\right] - B_x^2(\rho)\left[-A_x^2(\rho) + A_y^2(\rho) + A_z^2(\rho)\right] = 0,$$
$$2\left[B_y^2(\rho) - A_y^2(\rho)\right] - B_y^2(\rho)\left[A_x^2(\rho) - A_y^2(\rho) + A_z^2(\rho)\right] = 0,$$
$$2\left[B_z^2(\rho) - A_z^2(\rho)\right] - B_z^2(\rho)\left[A_x^2(\rho) + A_y^2(\rho) - A_z^2(\rho)\right] = 0.$$

$$B_x(\rho) = 0 \text{ if } A_x(\rho) = 0, \quad B_y(\rho) = 0 \text{ if } A_y(\rho) = 0, \quad B_z(\rho) = 0 \text{ if } A_z(\rho) = 0.$$

$$\mathbb{Z}_2(\hat{x})$$

$$A_x(\rho) := \sqrt{r_y^2 + r_z^2}$$

$$B_x(\rho) := \begin{cases} \sqrt{\frac{r_y^2+r_z^2}{1-r_x^2}} & \text{if } r_x^2 < 1 \\ 0 & \text{if } r_x^2 = 1 \end{cases}$$

$$\mathbb{Z}_2(\hat{y})$$

$$A_y(\rho) := \sqrt{r_x^2 + r_z^2}$$

$$B_y(\rho) := \begin{cases} \sqrt{\frac{r_x^2+r_z^2}{1-r_y^2}} & \text{if } r_y^2 < 1 \\ 0 & \text{if } r_y^2 = 1 \end{cases}$$

$$\mathbb{Z}_2(\hat{z})$$

$$A_z(\rho) := \sqrt{r_x^2 + r_y^2}$$

$$B_z(\rho) := \begin{cases} \sqrt{\frac{r_x^2+r_y^2}{1-r_z^2}} & \text{if } r_z^2 < 1 \\ 0 & \text{if } r_z^2 = 1 \end{cases}$$

Inequality Constraints

$$
\begin{aligned}
r_x^2 &\geq 0 \\
r_y^2 &\geq 0 \\
r_z^2 &\geq 0 \\
r_x^2 + r_y^2 + r_z^2 &\leq 1
\end{aligned}
$$

A Generating Set of Inequality Constraints

$$-A_x^2(\rho) + A_y^2(\rho) + A_z^2(\rho) \geq 0$$
$$A_x^2(\rho) - A_y^2(\rho) + A_z^2(\rho) \geq 0$$
$$A_x^2(\rho) + A_y^2(\rho) - A_z^2(\rho) \geq 0$$
$$A_x(\rho)^2 + A_y(\rho)^2 + A_z(\rho)^2 \leq 2$$

A Generating Set of Inequality Constraints

+

Equality Constraints

=>

All Constraints

The Recipe

1. Understand each resource theory individually

2. Derive dependence relations among monotones from different resource theories

3. Establish conceptual understandings of these relations

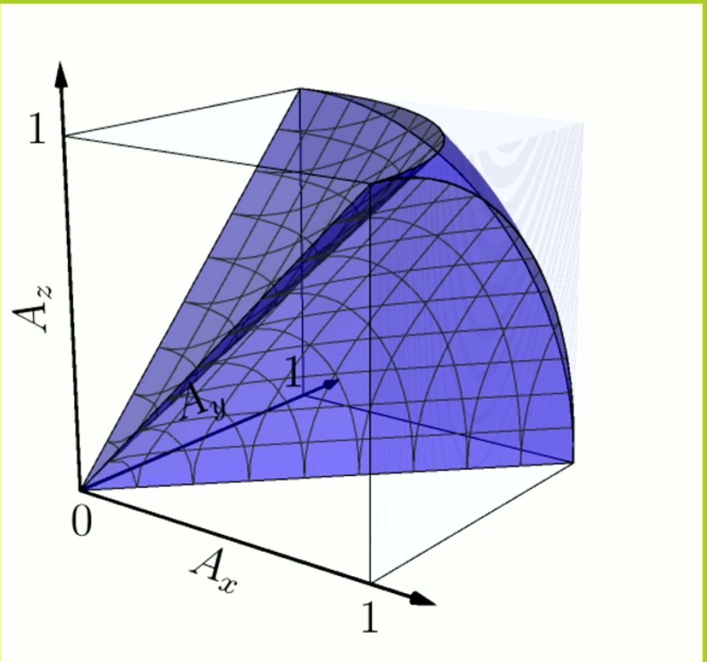
A Generating Set of Inequality Constraints

$$-A_x^2(\rho) + A_y^2(\rho) + A_z^2(\rho) \geq 0$$
$$A_x^2(\rho) - A_y^2(\rho) + A_z^2(\rho) \geq 0$$
$$A_x^2(\rho) + A_y^2(\rho) - A_z^2(\rho) \geq 0$$
$$A_x(\rho)^2 + A_y(\rho)^2 + A_z(\rho)^2 \leq 2$$

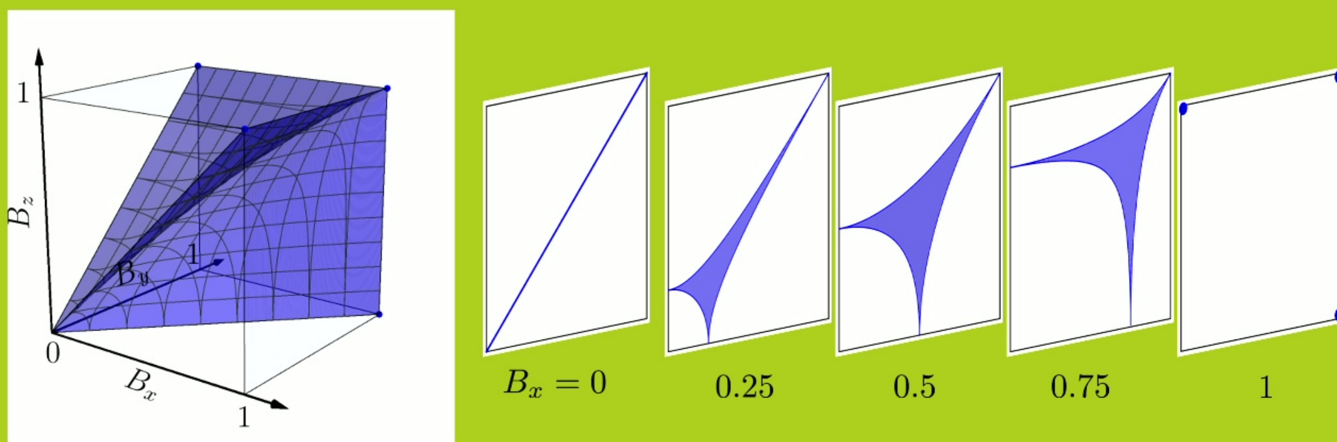
$$A_x(\rho), A_y(\rho), A_z(\rho) \leq 1$$

$$\boxed{A_x(\rho)^2 + A_y(\rho)^2 + A_z(\rho)^2 \leq 2}$$

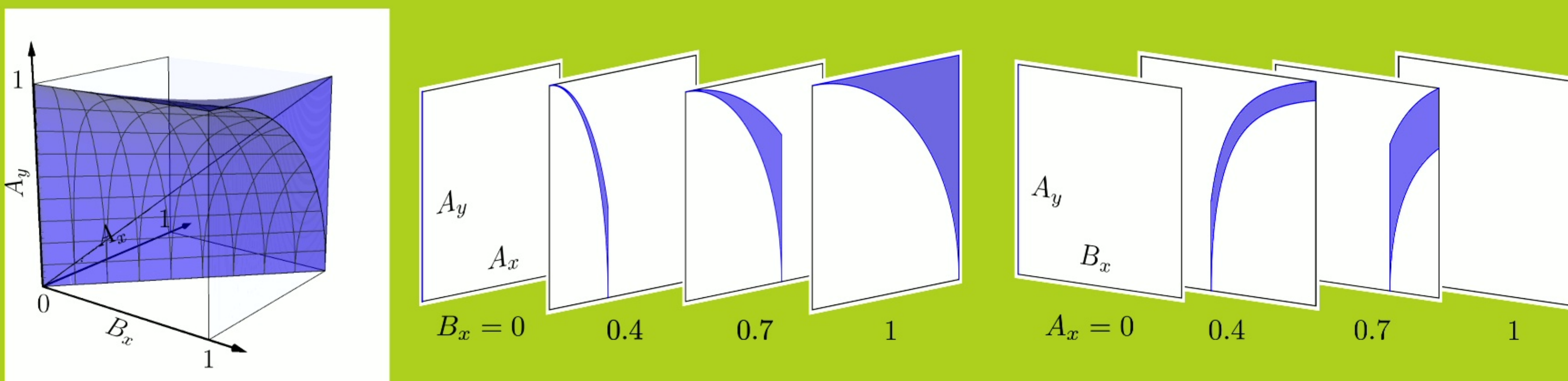
Joint-Realizable Region



Inequality constraints on B_x, B_y, B_z



Inequality constraints on A_x , A_y , B_x



The Recipe

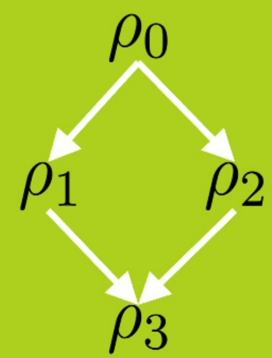
3. Establish conceptual understandings of the dependence relations

a) Characterize the dependence relations among monotones

b) Extract order-theoretic conclusions

(monotone-independent conclusions)

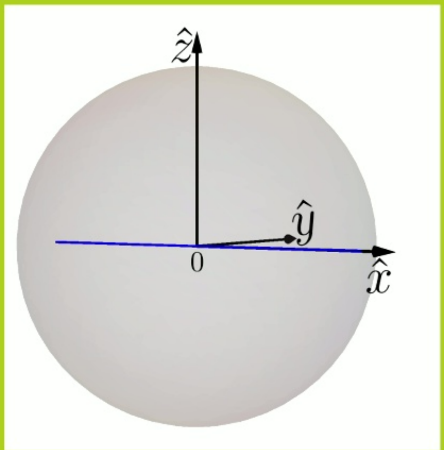
The fundamental object in a resource theory is the resource order, not monotones.

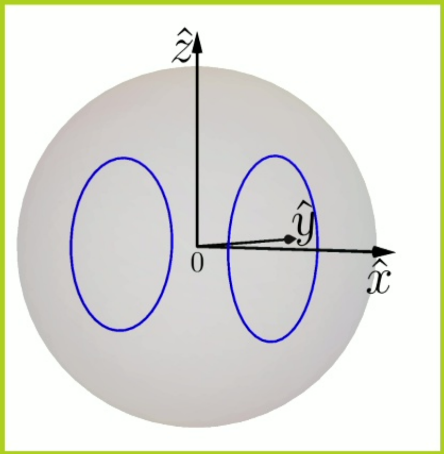


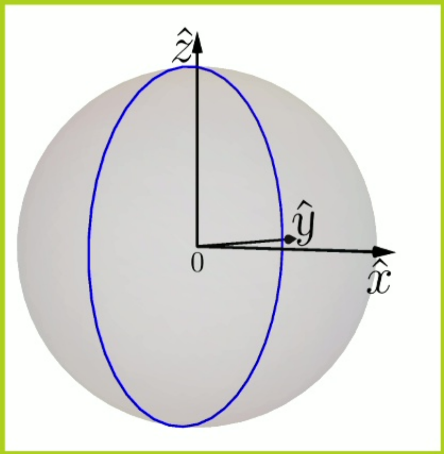
3.b) Extract order-theoretic conclusions

How the location in one resource order constrains the locations in the other

the corresponding equivalence class



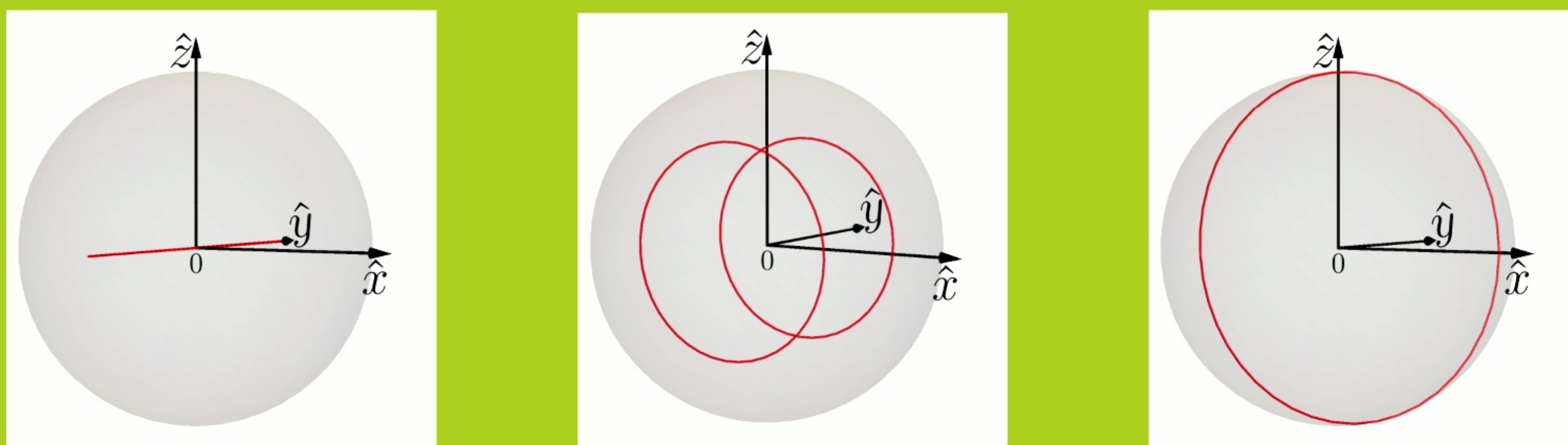




$\mathbb{Z}_2(\hat{x})$-Asymmetry

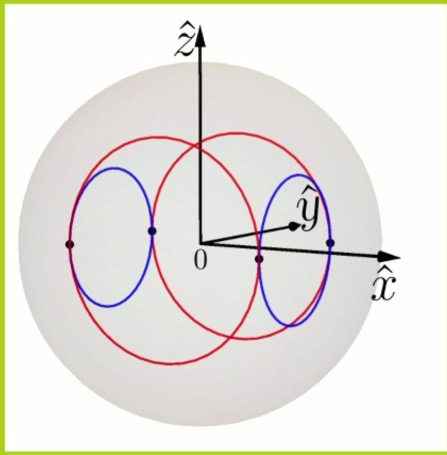
3.b) Extract order-theoretic conclusions

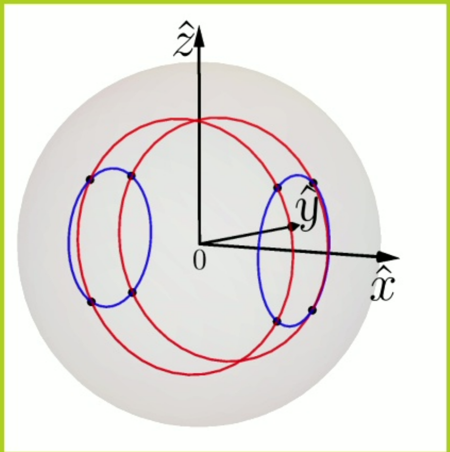
Location in the resource order:
the corresponding equivalence class

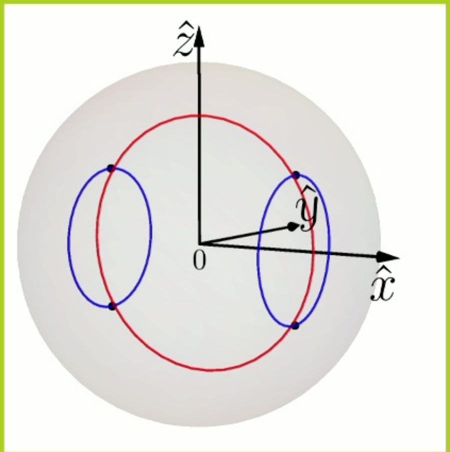


$\mathbb{Z}_2(\hat{y})$-Asymmetry

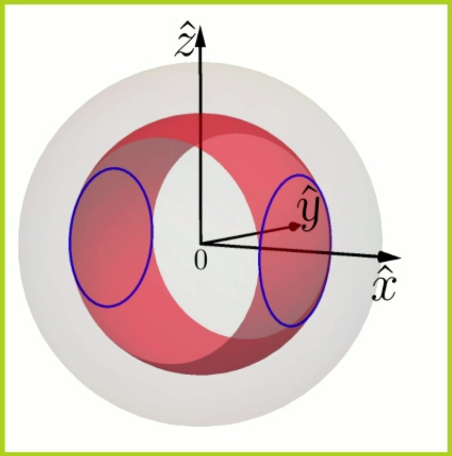
$\mathbb{Z}_2(\hat{x})$ vs $\mathbb{Z}_2(\hat{y})$







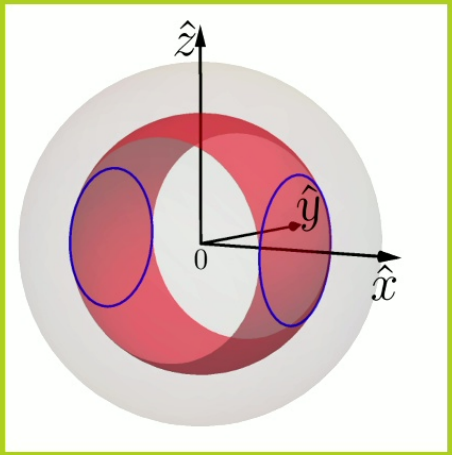
$\mathbb{Z}_2(\hat{x})$ vs $\mathbb{Z}_2(\hat{y})$



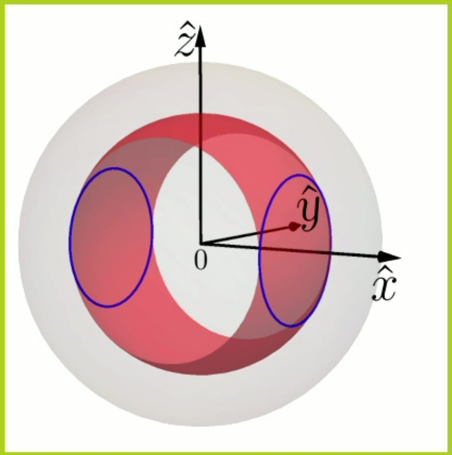
$\mathbb{Z}_2(\hat{x})$ vs $\mathbb{Z}_2(\hat{y})$

The equality constraints: fixing any 3 of the 6 monotones fixes the rest =>

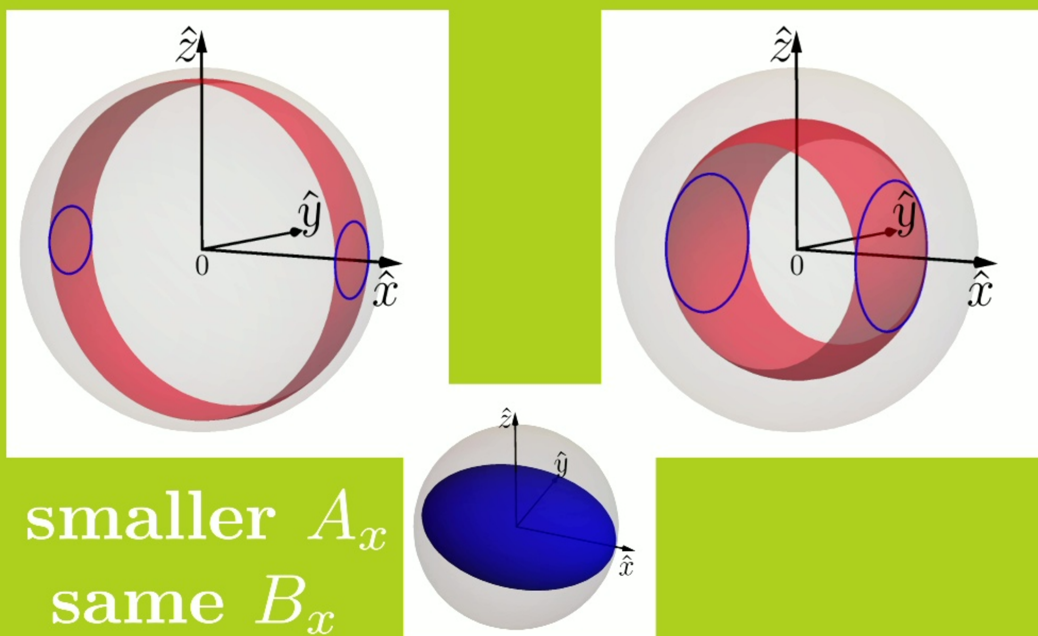
Fixing the location in one resource order reduces the other partial order to a total order



$\mathbb{Z}_2(\hat{x})$ vs $\mathbb{Z}_2(\hat{y})$



$\mathbb{Z}_2(\hat{x})$ vs $\mathbb{Z}_2(\hat{y})$

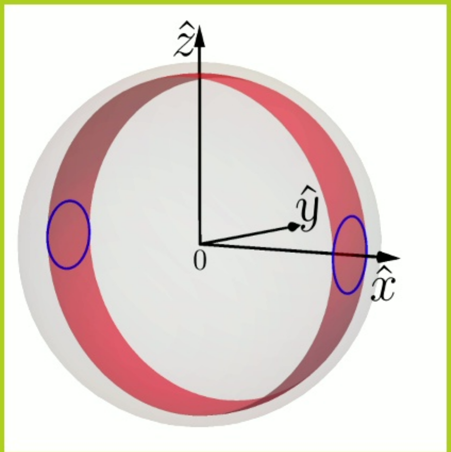


smaller A_x
same B_x

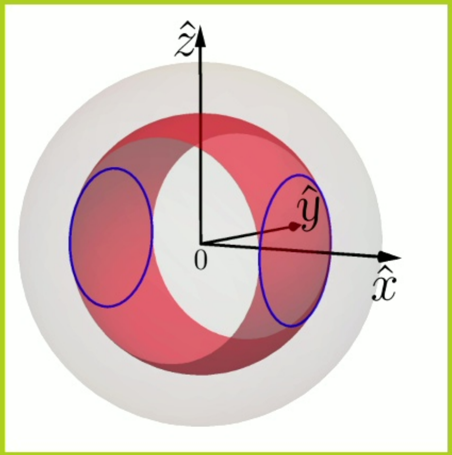
$\mathbb{Z}_2(\hat{x})\downarrow$ vs $\mathbb{Z}_2(\hat{y})$

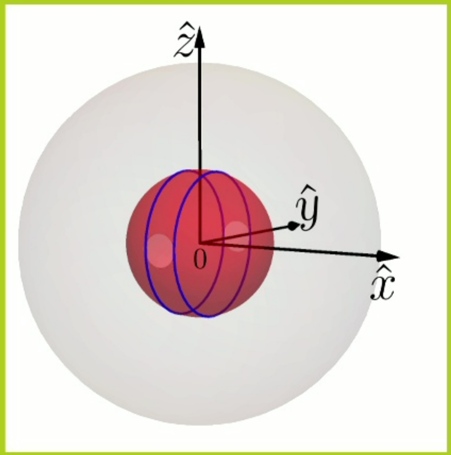
$\mathbb{Z}_2(\hat{y})\uparrow$

Trade-off



smaller A_x
same B_x





same A_x
smaller B_x

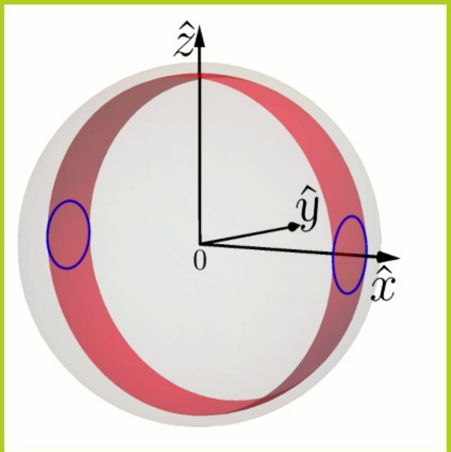
$\mathbb{Z}_2(\hat{x})\downarrow$ vs $\mathbb{Z}_2(\hat{y})$

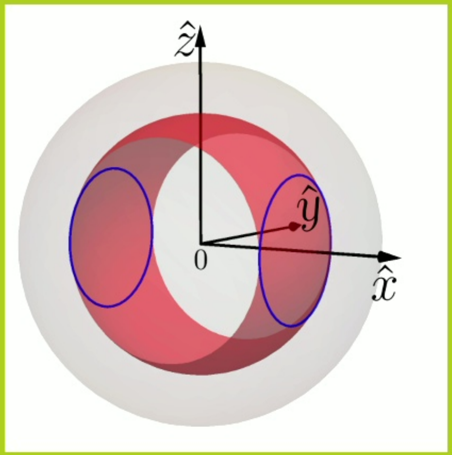
$\mathbb{Z}_2(\hat{y})\uparrow$

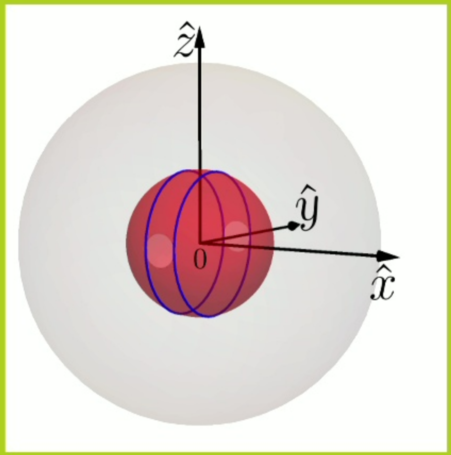
Trade-off

$\mathbb{Z}_2(\hat{y})\downarrow$

Synergy





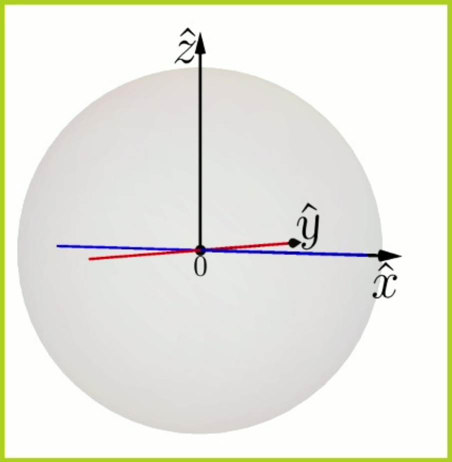


smaller A_x
same B_x

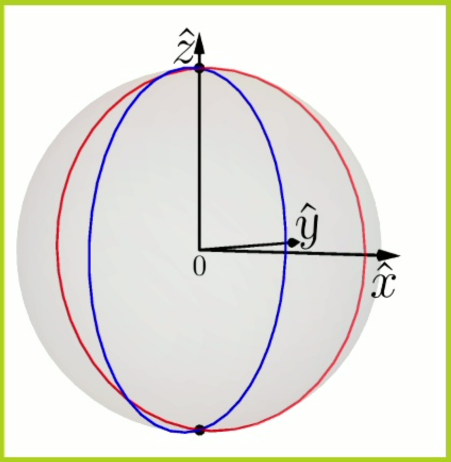
same A_x
smaller B_x

$\mathbb{Z}_2(\hat{x})$ vs $\mathbb{Z}_2(\hat{y})$

Simultaneously bottom

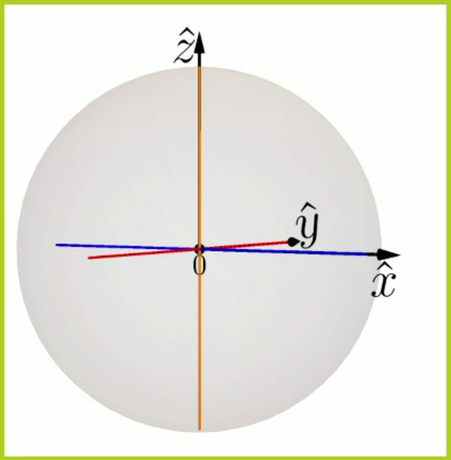


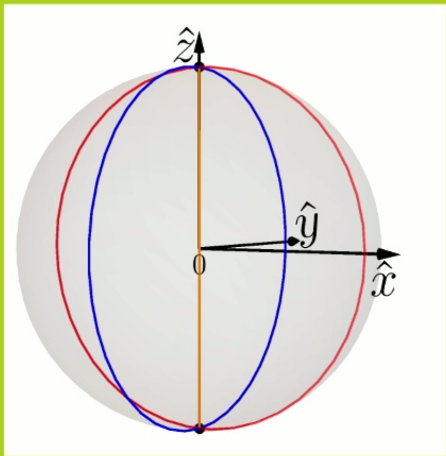
Simultaneously top



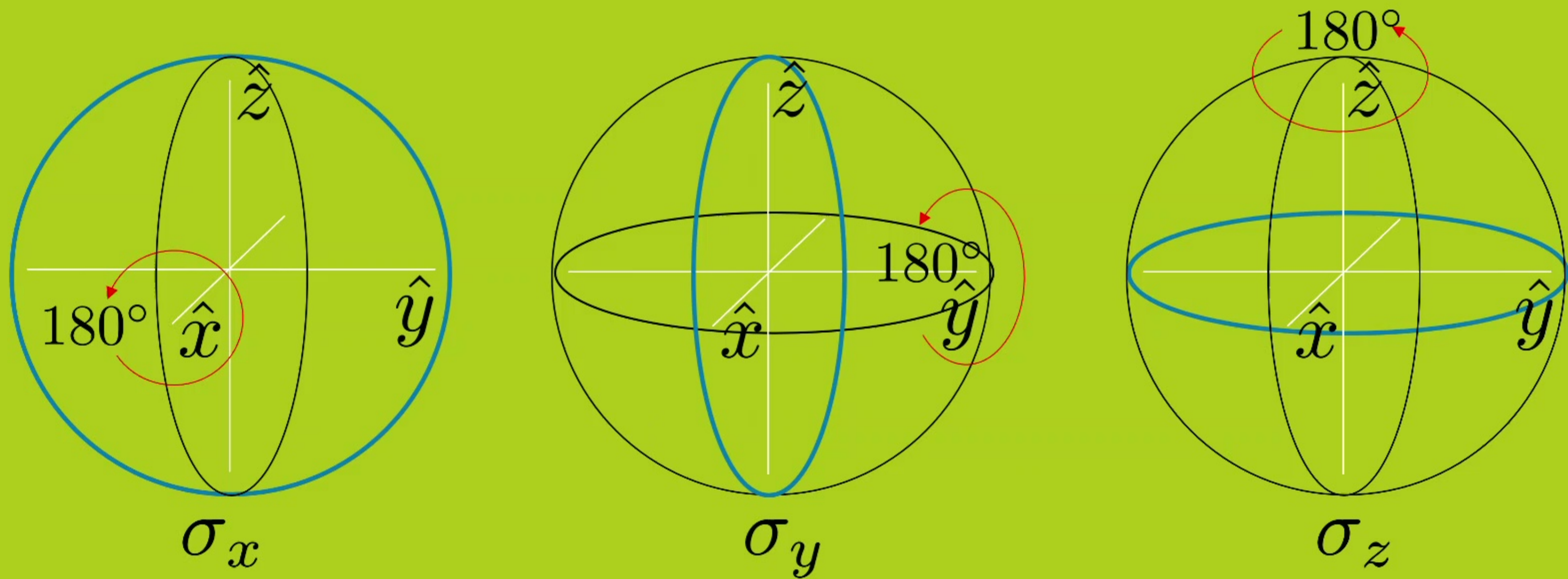
$$\mathbb{Z}_2(\hat{x}),\ \mathbb{Z}_2(\hat{y}),\ \mathbb{Z}_2(\hat{z})$$

Cannot **simultaneously be top-of-order**





A trade-off among the resourcefulness for estimating different π rotations



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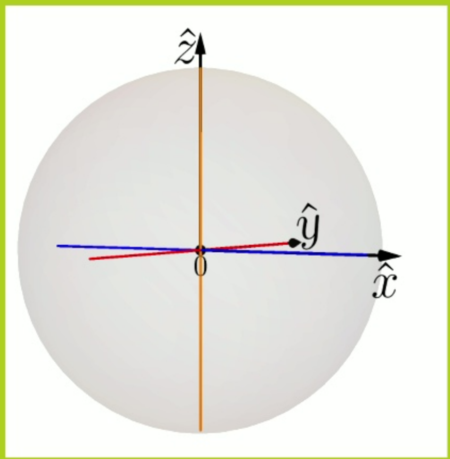

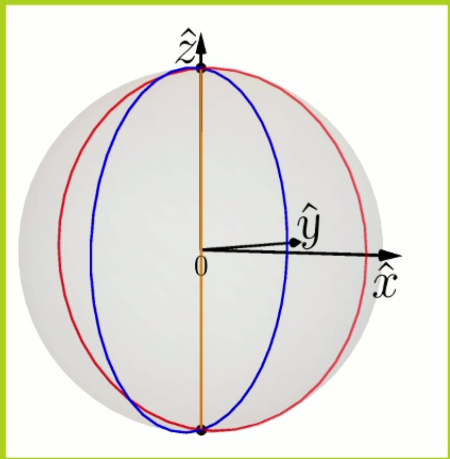

The Recipe

3. Establish conceptual understandings of the dependence relations

a) Characterize the dependence relations among monotones
b) Extract order-theoretic conclusions
(monotone-independent conclusions)

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a) Characterize the dependence relations among monotones
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c) Identify aspects that have operational significance

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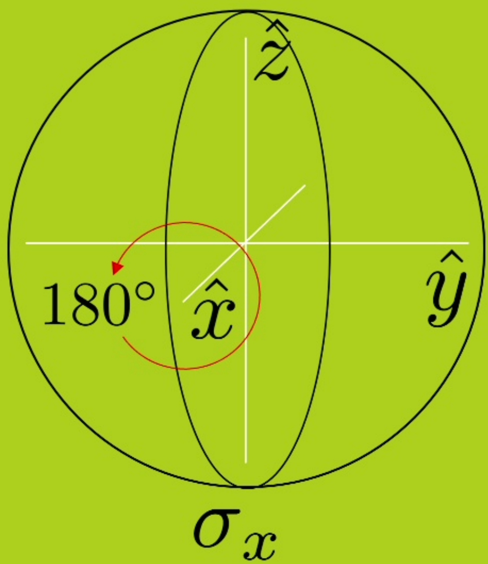
$A_x(\rho)$ the trace distance between ρ and $\sigma_x \rho \sigma_x$

$\Rightarrow$ the distinguishability between ρ and $\sigma_x \rho \sigma_x$



The Recipe

1. Understand each resource theory individually

a) Characterize the partial order (finding the complete set of monotones)
b) Find the full set of constraints on the monotones in the complete set (equalities and a generating set of inequalities)

2. Derive dependence relations among monotones

Technically difficult: marginal problem,

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Technically difficult: marginal problem, quantifier elimination

But it's okay. The recipe can still be followed.

Testing nonclassicality via resource dependence relations

Generalized Noncontextuality

- precise, experimentally testable, applies to any experiment/scenario

Subsumes:
- Kochen-Specker noncontextuality
- Bell local causality
- positive quasiprobability representation
- no anomalous week values
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Testing nonclassicality via resource dependence relations

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Contextuality fuels quantum advantage:
- computation
- communication
- cryptography
- state discrimination

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Robert W. Spekkens, 2005, *Contextuality for preparations, transformations, and unsharp measurements*
Robert W. Spekkens, 2019, *The ontological identity of empirical indiscernibles: Leibniz's methodological principle and its significance in the work of Einstein*

Testing nonclassicality via resource dependence relations

certain inequality constraints

$$r_x^2 \geq 0 \qquad -A_x^2(\rho) + A_y^2(\rho) + A_z^2(\rho) \geq 0$$

$$r_y^2 \geq 0 \qquad A_x^2(\rho) - A_y^2(\rho) + A_z^2(\rho) \geq 0$$

$$r_z^2 \geq 0 \qquad A_x^2(\rho) + A_y^2(\rho) - A_z^2(\rho) \geq 0$$

$$r_x^2 + r_y^2 + r_z^2 \leq 1 \qquad \boxed{A_x(\rho)^2 + A_y(\rho)^2 + A_z(\rho)^2 \leq 2}$$

~~Equality constraints~~

Testing nonclassicality w.r.t. fragments of quantum theory

Fragment:
Subsets of states, measurements and transformations

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Contextuality is necessary for quantum advantage:
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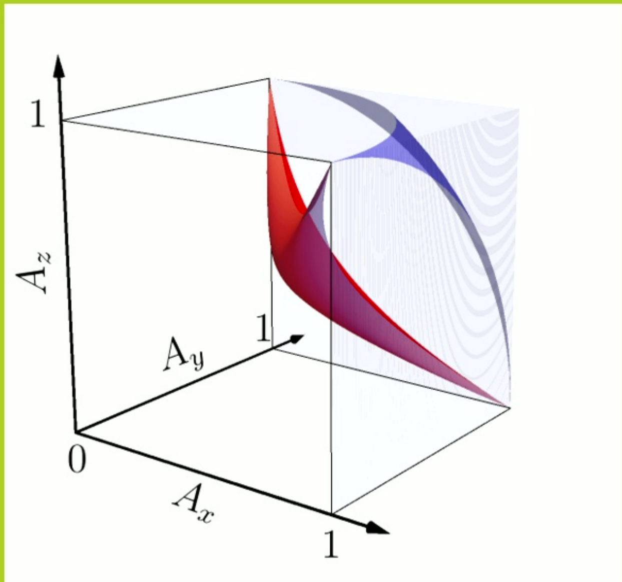
...

Have I accessed the nonclassical part of the quantum theory?

A Quantum-Classical Gap!

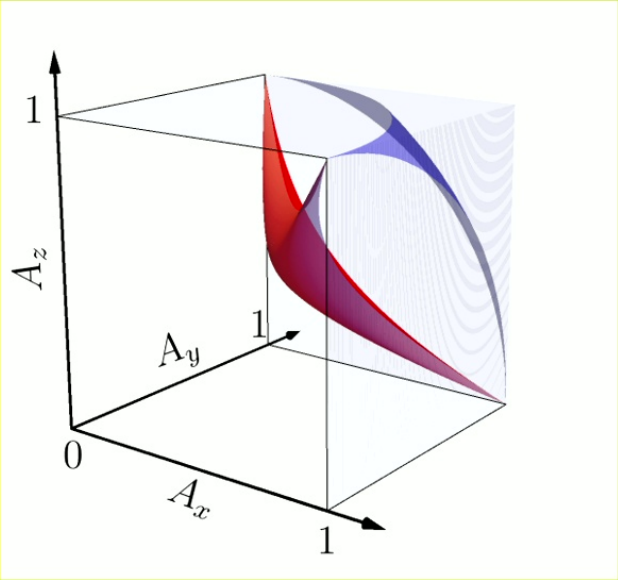


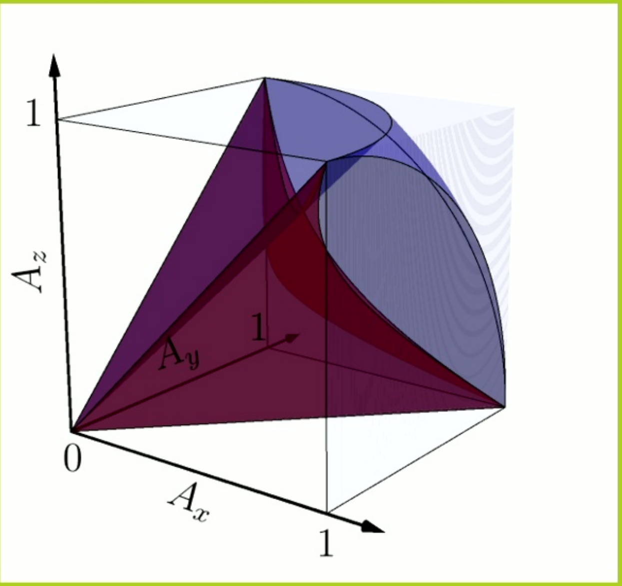
$$A_x(\rho)^2 + A_y(\rho)^2 + A_z(\rho)^2 \leq 2$$

fragments of quantum theory with certain properties and are ***noncontextual***:

$$\sqrt{A_x^2 + A_y^2 - A_z^2} + \sqrt{A_x^2 - A_y^2 + A_z^2} + \sqrt{-A_x^2 + A_y^2 + A_z^2} \leq \sqrt{2}$$

A Quantum-Classical Gap!





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A Recipe for Studying Resource Dependence Relations
(arXiv:2407.00164)

Witness Nonclassicality via Resource Dependence Relations
(ongoing work)

Thanks :)

yile.ying@gmail.com