Title: Observables, Hilbert Spaces and Entropies from the Gravitational Path Integral

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Abstract:

The Ryu-Takayanagi (RT) formula was originally introduced to compute the entropy of holographic boundary conformal field theories. In this talk, I will show how this formula can also be understood as the entropy of an algebra of bulk gravitational observables. Specifically, I will demonstrate that any Euclidean gravitational path integral, when it satisfies a simple set of properties, defines Hilbert spaces associated with closed codimension-2 asymptotic boundaries, along with type I von Neumann algebras of bulk observables acting on these spaces. I will further explain how the path integral naturally defines entropies on these algebras, and how an interesting quantization property leads to a standard state-counting interpretation. Finally, I will show that in the appropriate semiclassical limits, these entropies are computed via the RT formula, thereby providing a bulk Hilbert space interpretation of the RT entropy.

Observables, Hilbert Spaces and Entropies from the Gravitational Path Integral

Based on: arXiv:2310.02189 with Xi Dong, Donald Marolf and Zhencheng Wang

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October 24, 2024

Entropy of a region in quantum gravity?

Entropy of which algebra?

In quantum gravity, observables must be invariant under diffeomorphisms:

> **Relational approach:** localization of field operators with respect to some features of a state, or with respect to other field operators.
>
> Example: **gravitational dressing** (define operators creating a particle together with its gravitational field; location specified via geodesic distance from asymptotic boundary)

[Bergmann, DeWitt, Dittrich, Giddings, Giesel, Hartle, Höhn, Marolf, Rovelli, Thiemann, ...]

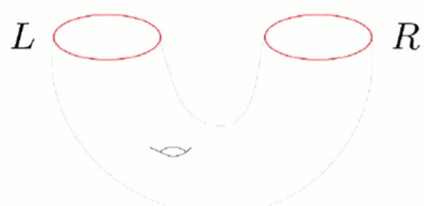
Diffeomorphism-invariant observables are non-local!

Entropy with a Hilbert space interpretation?

Because of the constraints, initial data in complementary (spatial) regions cannot be specified independently. This implies, at the quantum level, that the **Hilbert space in general does not factorize**.

[Donnelly, Freidel, Harlow, ...]

$$\mathcal{H}_{LR} \neq \mathcal{H}_L \otimes \mathcal{H}_R$$

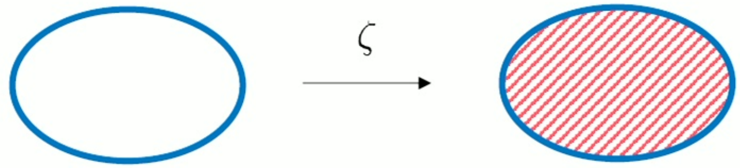


Entropy of a region in quantum gravity?

Strategy: Gravitational path integral as an “object” satisfying certain properties

- A UV-complete theory of quantum gravity should contain a map

boundary conditions M



number $\zeta(M)$ “transition amplitude”

- We might call this map a (Euclidean) gravitational path integral. It *might* look like $\zeta(M) = \int_{\mathrm{bc}:M} \mathcal{D}g e^{-S[g]}$
- We require this map to satisfy certain properties (imposed as “axioms”)

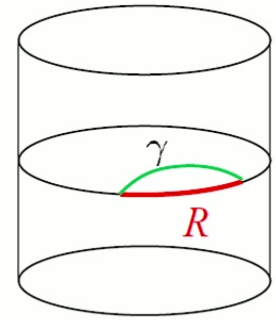
→ Observables and Entropies from the gravitational path integral so characterized

Advantages:

- Work at finite couplings
- Approach independent (UV bulk structure unconstrained)
  → compare results on gravitational entropy across different approaches!

Spin Foam | Causal Sets | String theory | Group Field Theory | AdS/CFT | ...

Holographic entanglement entropy



Ryu-Takayanagi formula

In AdS/CFT, the Ryu-Takayanagi (RT) formula[Ryu, Takayanagi 2006] assigns to a CFT subregion R the entropy

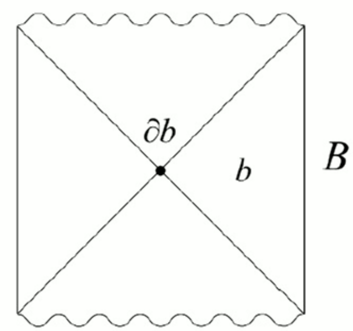
$$S(R) = \frac{A(\gamma)}{4G}$$

where γ is an extremal bulk surface (codimension-2) anchored to the boundary of R.

QES prescription

The QES prescription[Engelhardt, Wall 2014] for holographic entanglement entropy is a generalization of the Ryu-Takayanagi formula with quantum corrections. It states that the entanglement entropy of a CFT subregion B is equal to the **generalized entropy**

$$S_{\text{gen}}(b) = \frac{A(\partial b)}{4G} + S_{\text{bulk}}(b)$$



b = dual bulk quantum gravity region ("entanglement wedge")

∂b = "quantum extremal surface" (QES), codimension-2 surface bounding the wedge b

S_{bulk} = entanglement entropy of bulk quantum fields in b

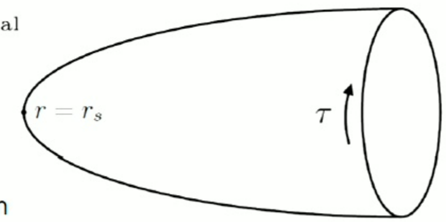
Lewkowycz-Maldacena construction

Gibbons-Hawking (1977)

$$Z(\beta) = \text{Path integral on the Euclidean black hole} \sim e^{I_{\text{classical}}}$$

$$S = (1 - \beta\partial\beta)\log Z(\beta) = \frac{A_{\text{horizon}}}{4G}$$

non-trivial part of variation near $r = r_s$

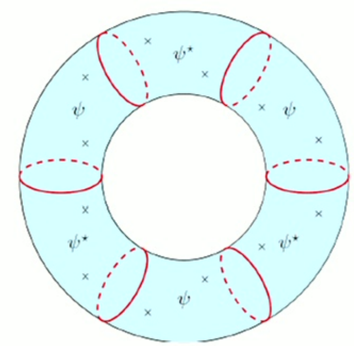


Lewkowycz-Maldacena (2013)

- Path integral prescription for the construction of the state
- Replica trick:
  - consider n copies of the system and compute $\mathrm{Tr}[\rho^n]$
  - analytically continue in n and compute the entropy as

$$S = (1 - n\partial n)\log \mathrm{Tr}[\rho^n]\big|_{n=1} = \frac{A(\gamma)}{4G}$$

non-trivial part of variation near γ

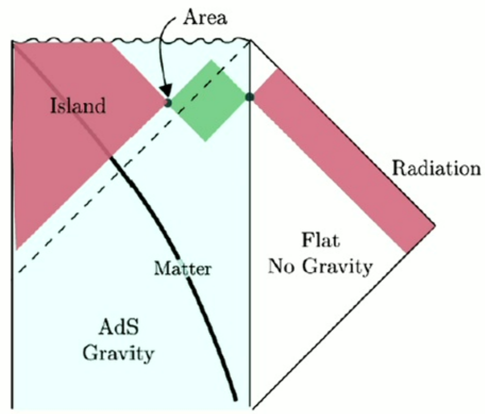


Holographic entanglement entropy from the bulk path integral?

- The Ryu-Takayanagi formula (and the more general QES prescription) can be obtained via the Euclidean gravity path integral using the **Lewkowycz-Maldacena construction**[Lewkowycz, Maldacena 2013; Dong, Lewkowycz 2017; Faulkner, Lewkowycz, Maldacena 2013]
  - AdS/CFT does NOT enter the derivation, but
  - **bulk interpretation problem**: AdS/CFT is needed to relate the boundary entanglement entropy to a bulk Euclidean gravity path integral on a replicated manifold! That is, to interpret the result as $S = -\mathrm{Tr}(\rho \log \rho)$

- QES discovered in the context of black hole evaporation[2019: Penington; Almheiri, Engelhardt, Marolf, Maxfield] and related to gravitational replicas calculations[2019: Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini; Penington, Shenker, Stanford, Yang]
  - In semiclassical limits, the von Neumann entropy of the bath can be studied using QES (**Island Formula**, special case of the quantum-corrected RT formula)
  - Inspired by AdS/CFT, but final versions of the arguments rely only on properties of the gravitational path integral!
  - Still no $S = -\mathrm{Tr}(\rho \log \rho)$, however:

    if the bulk theory allows baby-universe sectors[Coleman 1988; Giddings, Strominger 1988], the Island Formula gives the von Neumann entropy of the bath in a <u>typical baby-universe sector</u>[Marolf, Maxfield 2020]

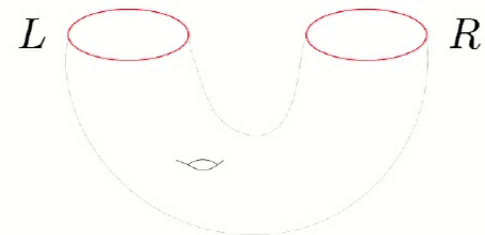


[Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini 2019]

Can this story be generalised? Can we understand these results beyond AdS/CFT?

Setting

- Consider a gravitational system with two asymptotic codimension-2 boundaries



- The Hilbert space $\mathcal{H}_{LR}$ a priori does not factorize!
  - reduced state on L/R?
  - entropy associated to L/R?

- Can we construct a Hilbert space $\mathcal{H}_L$ associated with L such that the corresponding Ryu–Takayanagi formula can be understood in terms of a standard trace on $\mathcal{H}_L$? (true with holography)

$$S_{\mathrm{vN}}(\rho_L) := -\mathrm{Tr}_L(\rho_L \ln \rho_L) = \textbf{RT formula}$$

$$\rho_L = \mathrm{Tr}_R(\rho) \quad \text{reduced state}$$

THIS TALK:

This type of structure is present in any UV-complete, asymptotically locally AdS theory of quantum gravity in which the Euclidean path integral satisfies a simple set of axioms.

Result 1

Related work

Axiomatic approach to TQFT and Gravity:

- Axiomatization of topological quantum field theories (TQFT) [Atiyah 1988]
- Attempt to define a gravitational partition function by generalising TQFT axioms [Rovelli, Barrett, Crane, Baez, Dolan, Freidel, Starodubtsev, Oeckl, ...]

Understanding holographic gravitational entropy from the bulk

Recent works have shown that, in various contexts, the **Ryu-Takayanagi** entropy can be derived (up to an infinite constant) as the entropy of a type II von Neumann algebra (observables gravitationally dressed to the energy of an observer in de Sitter or to the mass of a Schwarzschild-AdS black hole) [Chandrasekaran, Longo, Penington, Witten, Jensen, Sorce, Speranza, Satishchandran,...]

However:

The entropy of a standard quantum mechanical system is in terms of a Hilbert space trace $\mathrm{Tr}(\cdot) = \sum_i \langle i| \cdot |i\rangle$ which provides a "state-counting interpretation". A Hilbert space trace corresponds to a type I trace.

THIS TALK:

RT entropy as entropy of a type I von Neumann algebra, with a state-counting interpretation, without AdS/CFT.

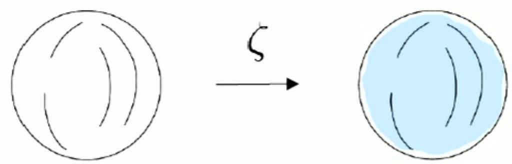
Result 2

Outline

1. Introduction ✓
2. Axioms for the path integral ←
3. Hilbert Space
4. Operator Algebras
5. Type I von Neumann Factors
6. Entropy (with state-counting interpretation)
7. Examples

Axioms

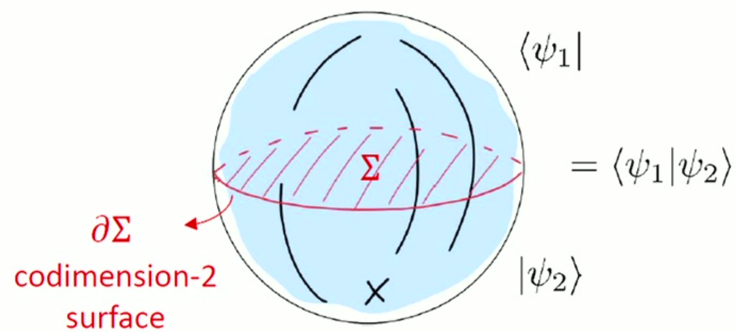
(Euclidean) Gravitational Path Integral



$M \supset g_M, \phi_M$

boundary conditions
"source-manifold"

$$\zeta(M) = \int_{\mathrm{bc}:M} \mathcal{D}g\mathcal{D}\phi e^{-S[g,\phi]}$$



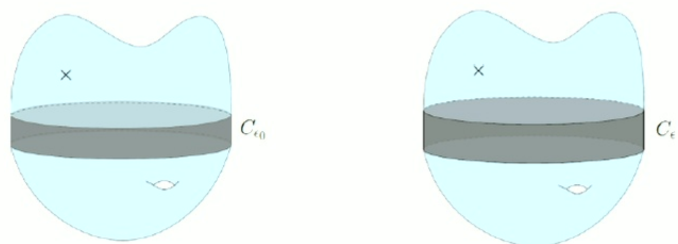
Axioms

1. **Finiteness:** The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

2. **Reality:** ζ is a real function of (possibly complex) boundary conditions, i.e. $[\zeta(M)]^* = \zeta(M^*)$

3. **Reflection Positivity**: ζ is reflection-positive

4. **Continuity:** if the boundary manifold contains a cylinder of size ε, ζ is continuous under changes of ε



5. **Factorization:** $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\zeta(M_2)$

If the path integral decomposes into baby universe sectors, the factorization holds sector-by-sector, and our analysis applies in that sense.

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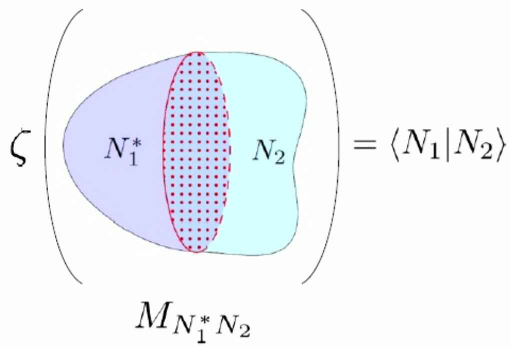
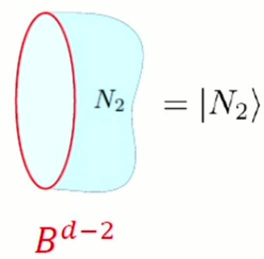
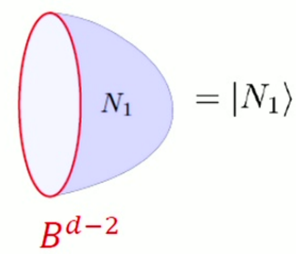
Hilbert Space







- The source-manifold $M_{N_1^* N_2}$ might not be smooth, and so $\zeta(M_{N_1^* N_2})$ might not be well defined
- We introduce rims: 
- $Y_B^{d-1} =$ set of rimmed source-manifolds with boundary B
- $\underline{Y}_B^{d-1} =$ linear combinations of rimmed source-manifolds with boundary B

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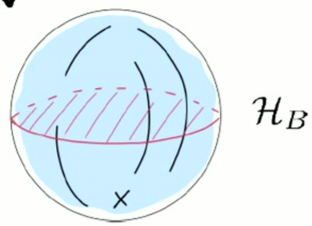
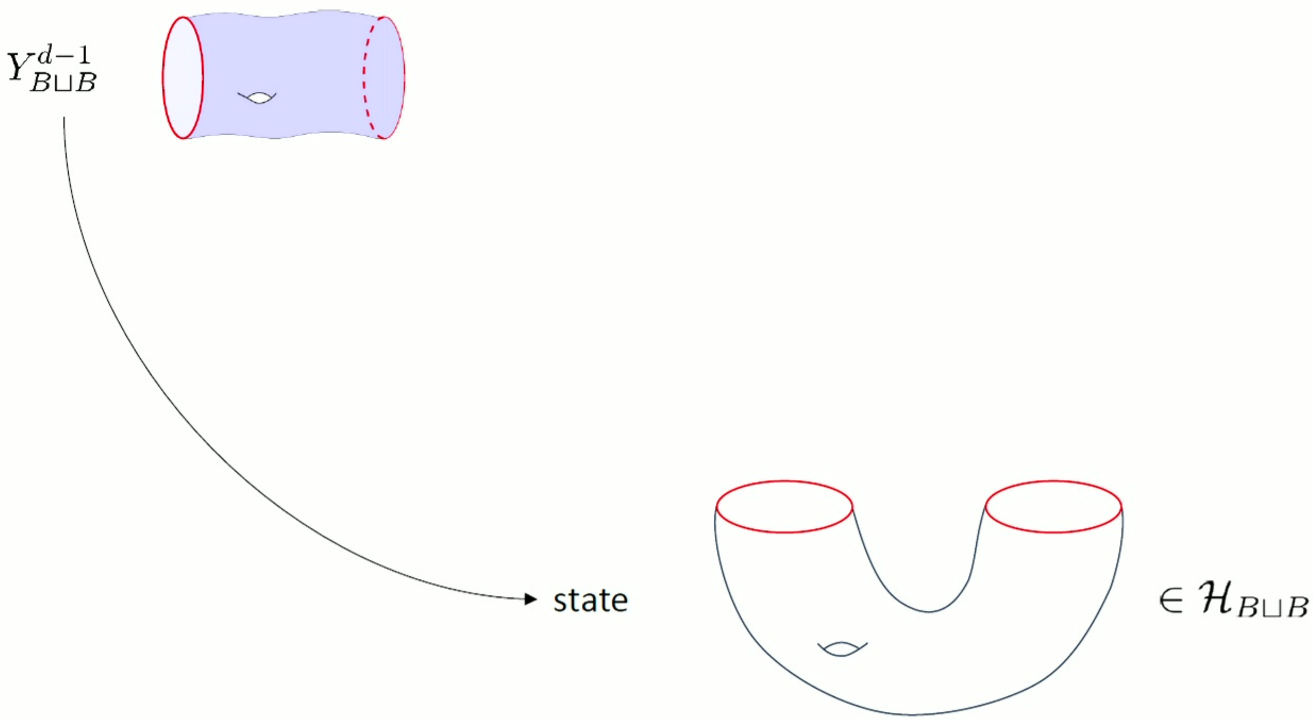
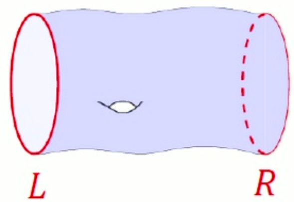


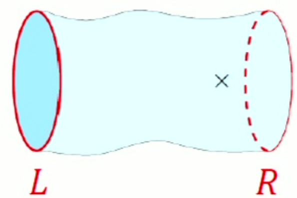
Algebra

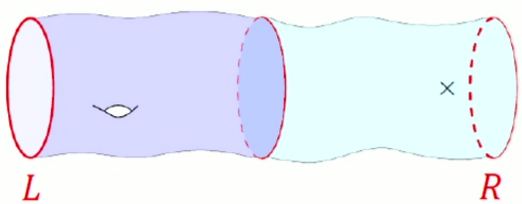


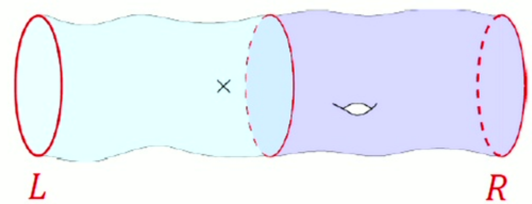
Algebra

- On the set $\underline{Y}^{d-1}_{B\sqcup B}$ we define a *left product* and a *right product*:

$a =$ 

$b =$ 

<u>left product:</u> $(\cdot_L)$ $a \cdot_L b =$ 

<u>right product:</u> $(\cdot_R)$ $a \cdot_R b =$ 

- For convenience $ab := a \cdot_L b = b \cdot_R a$

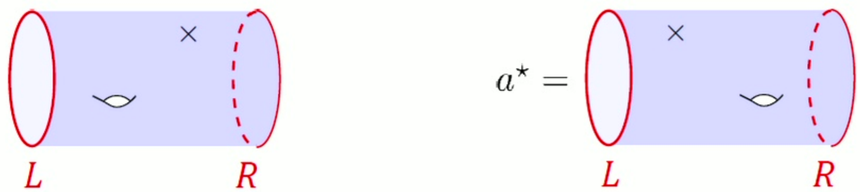
- The set $\underline{Y}^{d-1}_{B\sqcup B}$ equipped with the left (right) product defines a left (right) surface algebra A_L (A_R)

Algebra

- The algebras A_L and A_R are related by an antilinear isomorphism $\star$

$$a^\star := (a^*)^t$$

$a =$ $\quad$ $a^\star =$



- We will see that the left (right) algebra as a natural action on the left (right) B of $\mathcal{H}_{B\sqcup B}$

$\underline{Y}^{d-1}_{B\sqcup B}, (\cdot_L)$

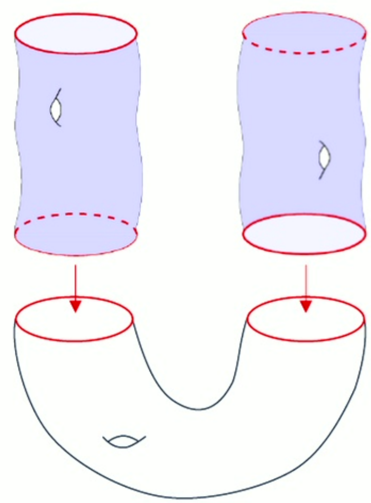
Left surface algebra

A_L

$\underline{Y}^{d-1}_{B\sqcup B}, (\cdot_R)$

Right surface algebra

A_R

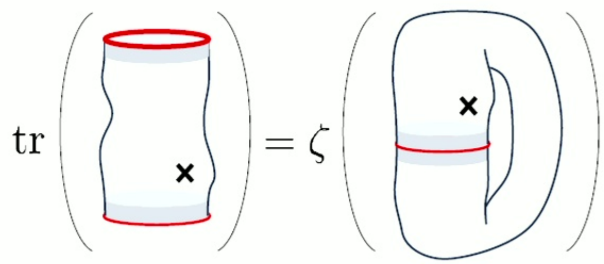


$\in \mathcal{H}_{B\sqcup B}$

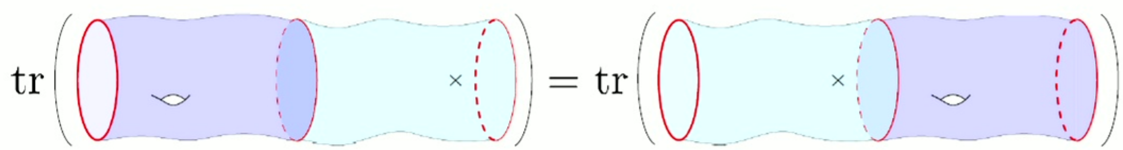
Trace

- The path integral defines a trace operation:

$$\mathrm{tr} : A_{L/R} \to \mathbb{C}$$



- It satisfies the cyclic property:



Trace

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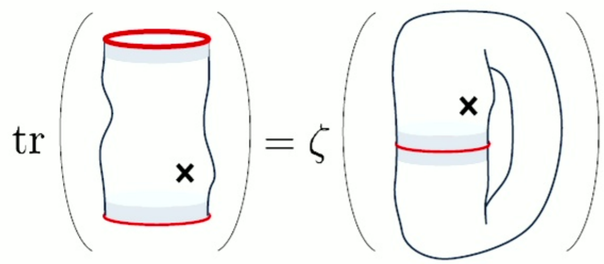
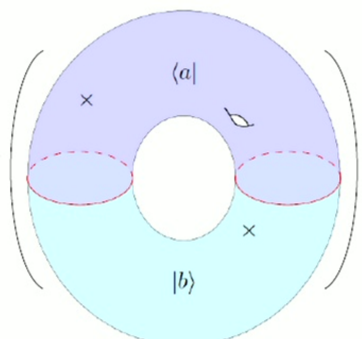


- The trace on A_L and A_R corresponds to the inner product on $\mathcal{H}_{B \sqcup B}$:

$$\langle a|b\rangle = \mathrm{tr}(a^\star b) = \zeta$$



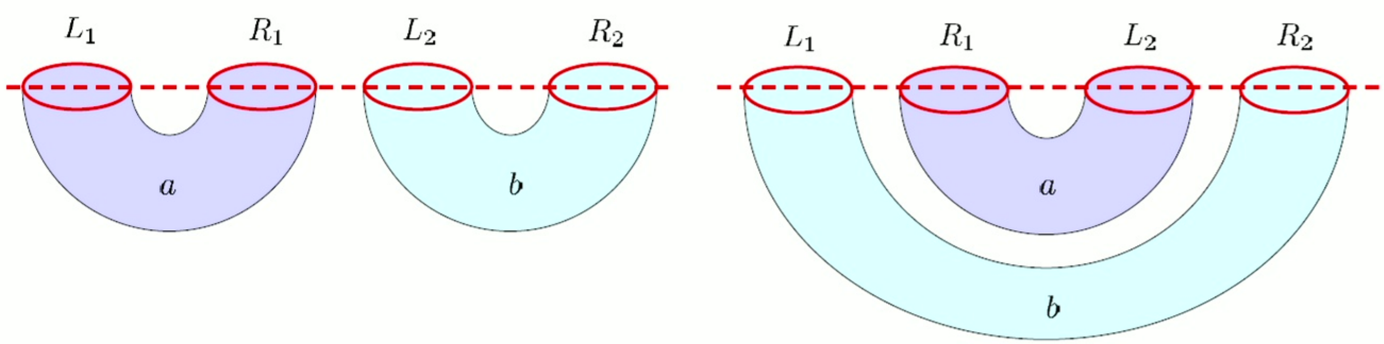
- It is positive-definite: $\mathrm{tr}(a^\star a) = \zeta\,(M(a^\star a)) = \langle a|a\rangle \geq 0$

Axiom 3

Trace Inequality

- We can prove the trace inequality $\mathrm{tr}(aa^{\star}bb^{\star}) \leq \mathrm{tr}(a^{\star}a)\mathrm{tr}(b^{\star}b)$

  Use $a, b \in Y^{d-1}_{B\sqcup B}$ to define elements of $Y^{d-1}_{(B\sqcup B)\sqcup(B\sqcup B)}$



$|\cup\cup\rangle := |a_{L_1R_1}, b_{L_2R_2}\rangle$ $\qquad$ $|\Cup\rangle := |a_{L_2R_1}, b_{L_1R_2}\rangle$

Trace Inequality

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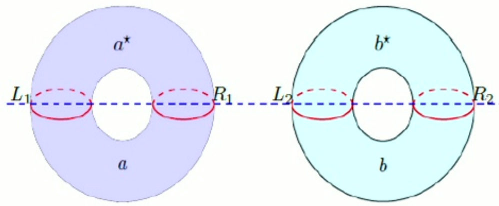
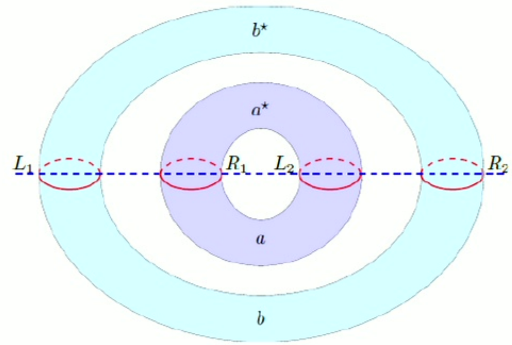
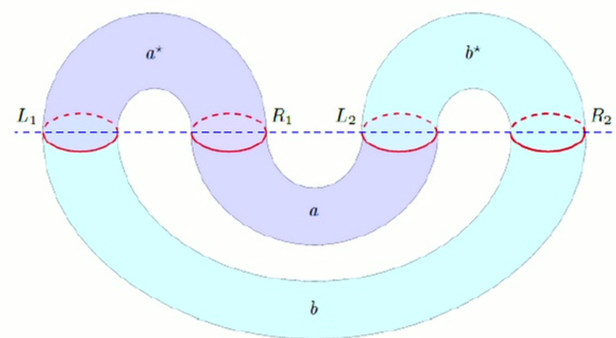




$$\langle \cup\,\cup \,|\, \cup\,\cup \rangle = \langle \Cup | \Cup \rangle = \langle a|a\rangle\,\langle b|b\rangle = \mathrm{tr}(a^\star a)\mathrm{tr}(b^\star b)$$

From the Cauchy-Schwarz inequality (consequence of positivity of the inner product on $\mathcal{H}_{B\sqcup B\sqcup B\sqcup B}$):

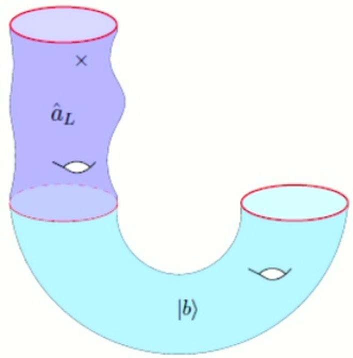
$$\Big|\langle \Cup | \cup\,\cup \rangle\Big| \leq \Big|\,|\cup\cup\rangle\Big|\,\Big|\,|\Cup\rangle\Big|$$



Operator algebras

- We define a representation of the left surface algebra on the Hilbert space: given $a \in A_L$ there is an associated operator $\hat{a}_L \in \hat{A}_L$ such that

$$\hat{a}_L \, |b\rangle = |a \cdot_L b\rangle = |ab\rangle$$



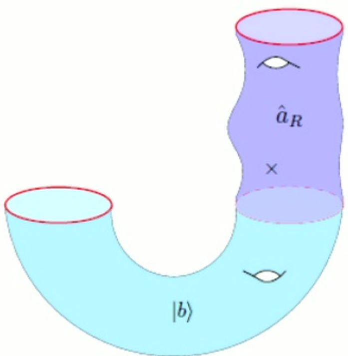
- These operators are **bounded**:

$$|\hat{a}_L|b\rangle|^2 = \langle ab|ab\rangle = \mathrm{tr}(a^\star abb^\star) \leq \mathrm{tr}(a^\star a)\mathrm{tr}(bb^\star) = \mathrm{tr}(a^\star a)\langle b|b\rangle$$

trace inequality

- We can similarly define a representation $\hat{A}_R$ of A_R :

$$\hat{a}_R \, |b\rangle = |a \cdot_R b\rangle = |b \cdot_L a\rangle = |ba\rangle$$



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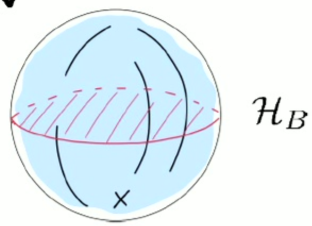
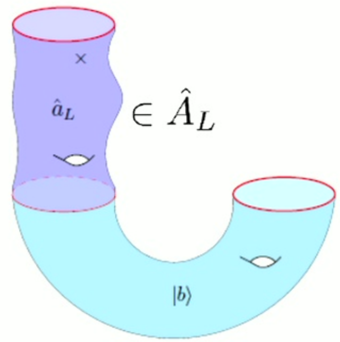
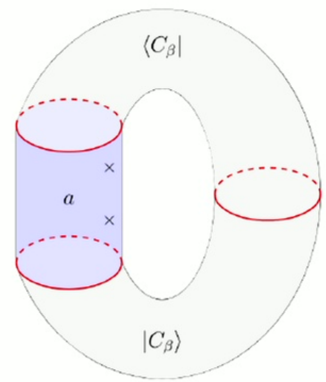




Type I von Neumann algebras

- We constructed $\hat{A}_L, \hat{A}_R =$ commuting algebras of bounded operators on $\mathcal{H}_{B \sqcup B}$
- We can complete $\hat{A}_L, \hat{A}_R$ to von Neumann algebras $\mathcal{A}_L, \mathcal{A}_R$ by taking the closure in the weak (or strong) operator topology (or taking the double commutant of $\hat{A}_{L/R}$)
- We show that the trace defined on $\hat{A}_{L/R}$ can be extended to (all positive elements of) the von Neumann algebra:

$$\mathrm{tr}(a) = \lim_{\beta \downarrow 0} \langle C_\beta | a | C_\beta \rangle$$



- We can study the structure of the von Neumann algebras via properties of the trace!

Type I von Neumann algebras

- We can prove that the trace is
  1) Faithful $\mathrm{tr}(a) = 0 \text{ iff } a = 0$
  2) Normal for any bounded increasing sequence a_n, $\mathrm{tr}\ sup\ a_n = sup\ \mathrm{tr}\ a_n$
  3) Semifinite $\forall a \in \mathcal{A}^+$, $\exists b < a$ such that $\mathrm{tr}(b) < \infty$

- It also satisfies the trace inequality (an extension of the 4-boundaries argument applies)

- Applying the trace inequality to $a = b = P \in \mathcal{A}_L$ gives $\mathrm{tr}(P) \geq 1$

Some known results on von Neumann algebras:

- Every von Neumann algebra is a direct sum or integral of factors (algebras with trivial center)
- These factors can be type I, II or III
- There is *no faithful, normal and semifinite trace on type III* ⇒ we cannot have type III
- on type II, for any faithful, normal and semifinite trace there are *nonzero projections with arbitrarily small trace* ⇒ we cannot have type II

Type I von Neumann algebras

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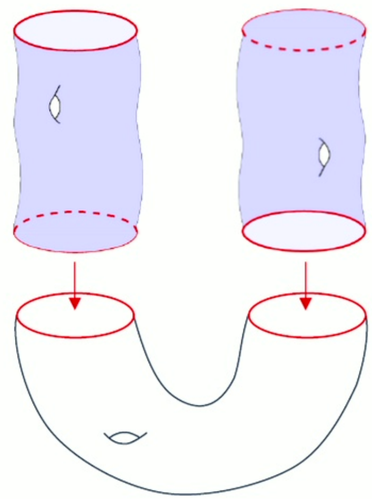
  1) Faithful $\mathrm{tr}(a) = 0 \text{ iff } a = 0$
  2) Normal for any bounded increasing sequence a_n, $\mathrm{tr}\ sup\ a_n = sup\ \mathrm{tr}\ a_n$
  3) Semifinite $\forall a \in \mathcal{A}^+, \exists b < a$ such that $\mathrm{tr}(b) < \infty$
- It also satisfies the trace inequality (an extension of the 4-boundaries argument applies)
- Applying the trace inequality to $a = b = P \in \mathcal{A}_L$ gives $\mathrm{tr}(P) \geq 1$
- Therefore, $\mathcal{A}_{L/R}$ is a direct sum/integral of type I factors!
- The spectrum of $z \in \mathcal{Z}_L$ (center of $\mathcal{A}_L$) is discrete

$$\mathcal{A}_L = \bigoplus_{\mu} \mathcal{A}_{L,\mu}$$

Type I von Neumann algebras

- $\mathcal{A}_L, \mathcal{A}_R$ are each other commutants on $\mathcal{H}_{B\sqcup B}$, and so they have the same center $\mathcal{Z}$

$$\mathcal{A}_L = \bigoplus_\mu \mathcal{A}_{L,\mu} \qquad \mathcal{A}_R = \bigoplus_\mu \mathcal{A}_{R,\mu}$$



- $\mathcal{H}_{B\sqcup B}$ can be decomposed into eigenspaces of $\mathcal{Z}$

$$\mathcal{H}_{B\sqcup B} = \bigoplus_\mu \mathcal{H}^\mu_{B\sqcup B}$$

with $\mathcal{H}^\mu_{B\sqcup B} = \mathcal{H}^\mu_{B\sqcup B,L} \otimes \mathcal{H}^\mu_{B\sqcup B,R}$

Trace Normalization

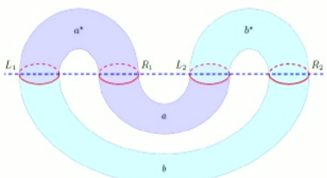
- Faithful, normal, semifinite traces on type I algebras are unique up to an overall normalization constant. Therefore, on a given μ-sector

$$\mathrm{tr}(a) = n_\mu \mathrm{Tr}_\mu(a) = n_\mu \sum_i {}_L\langle i|a|i\rangle_L,$$

- For $a = P$ one-dimensional projection onto a state in $\mathcal{H}^\mu_{B\sqcup B,L}$ we have $\mathrm{Tr}_\mu(P) = 1$

$$1 \leq \mathrm{tr}(P) = n_\mu$$

trace inequality



positivity of the inner product on $\mathcal{H}_{B\sqcup B\sqcup B\sqcup B}$

$\mathrm{tr}(P) \geq 1$ $\qquad$ $\mathrm{tr}(P) = 0$

positivity of the inner product on $\mathcal{H}_{\sqcup_{i=1}^n (B\sqcup B)}$

$\mathrm{tr}(P) \geq n-1$ $\qquad$ $\mathrm{tr}(P) = 0, 1, \cdots, n-2$

$\Rightarrow n_\mu$ is a positive integer!

Trace Normalization

- Therefore on a given μ-sector $\mathrm{tr} = n_\mu \mathrm{Tr}_\mu$ with $n_\mu \in \mathbb{Z}^+$

- We define the extended Hilbert space factors:

$$\widetilde{\mathcal{H}}^{\mu}_{B\sqcup B,L/R} := \mathcal{H}^{\mu}_{B\sqcup B,L/R} \otimes \mathcal{H}_{n_\mu}$$

"hidden sector"

where $\mathrm{tr} = \tilde{\mathrm{Tr}}_\mu$!

- The full extended Hilbert space:

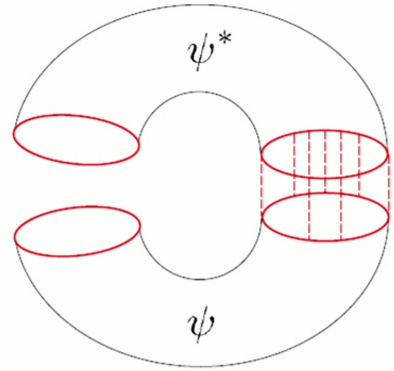
$$\widetilde{\mathcal{H}}_{B\sqcup B} := \bigoplus_{\mu\in\mathcal{I}} \left(\widetilde{\mathcal{H}}^{\mu}_{B\sqcup B,L} \otimes \widetilde{\mathcal{H}}^{\mu}_{B\sqcup B,R}\right)$$

⇒ The hidden sectors allow to interpret the path integral trace as a Hilbert space trace

Entropy

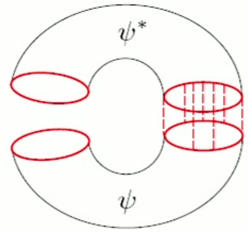
✓ The trace tr defines an entropy on the left/right B

- Given a state $|\psi\rangle \in \mathcal{H}_{B\sqcup B}$ we can define a reduced density operator $\rho_\psi \in \mathcal{A}_L$



- The von Neumann entropy is $S^L_{vN}(\psi) = \mathrm{tr}(-\rho_\psi \ln \rho_\psi)$

Entropy



✓ The trace tr defines an entropy on the left/right B

$$S^L_{vN}(\psi) = \mathrm{tr}(-\rho_\psi \ln \rho_\psi)$$

✓ Thanks to the relation $\mathrm{tr} = \tilde{\mathrm{Tr}}_\mu$ this entropy has a state-counting interpretation as left entropy on the extended Hilbert space $\widetilde{\mathcal{H}}_{B\sqcup B}$

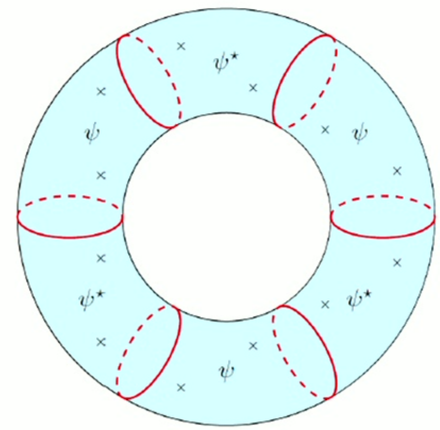
✓ We can compute this entropy via the replica trick:

$$\mathrm{tr}(\rho^n_\psi) = \zeta\left(M([\psi\psi^\star]^n)\right)$$

$$S^L_{vN}(\psi) = (1 - n\partial n) \log \mathrm{tr}(\rho^n_\psi)\big|_{n=1}$$

$$= \frac{A(\gamma)}{4G}$$

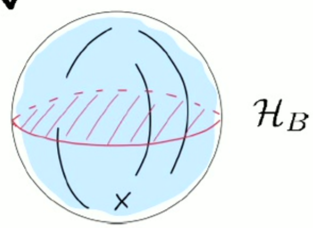
RT

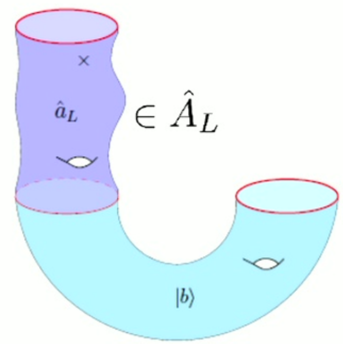


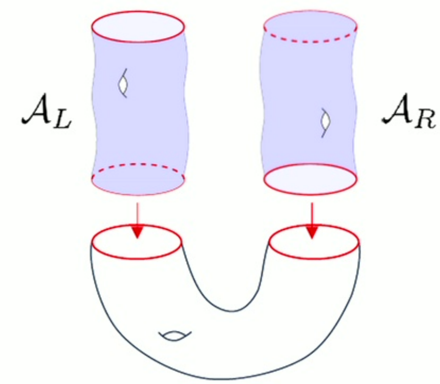
✓ If the theory admits a semiclassical limit described by Einstein-Hilbert or JT gravity, we can argue (by following Lewkowycz-Maldacena) that in such a limit the entropy is given by the Ryu-Takayanagi entropy

Outline

1. Introduction
2. Axioms for the path integral ✓
3. Hilbert Space ✓
4. Operator Algebras ✓
5. Type I von Neumann Factors ✓
6. Entropy (with state-counting interpretation) ✓
7. Examples







Example: a simple topological theory

- Consider a theory of purely topological 2-dimensional gravity
  - spacetime = $2d$ manifold $\mathcal{M}$ (with orientation)
  - histories = set of oriented topological surfaces (classified by genus and number of circular boundaries)
  - Z = boundary condition on any circular boundary (no extra sources)
- Path integral: $\sum_{\mathcal{M}} \mu(\mathcal{M}) e^{-S(\mathcal{M})}$
- Most general action allowed by the degrees of freedom: $S(\mathcal{M}) = -S_0 \chi(\mathcal{M}) - S_\partial \underbrace{|\partial \mathcal{M}|}_{\#\text{ circular boundaries}}$

$S_\partial = S_0$ → cancels boundary contributions to χ in the action

$$\implies \langle Z^n \rangle = \sum_{\mathcal{M}: |\partial \mathcal{M}| = n} \mu(\mathcal{M}) e^{S_0 \tilde{\chi}(\mathcal{M})}$$

$$\tilde{\chi} = \sum_{\text{connected components}} (2 - 2g)$$ (modified Euler characteristic that does not count boundaries)

$$\langle Z^n | Z^m \rangle = \langle Z^{n+m} \rangle = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^{n+m} \qquad \lambda = \frac{e^{2S_0}}{1 - e^{-2S_0}}$$

states in $\mathcal{H}_{BU}$

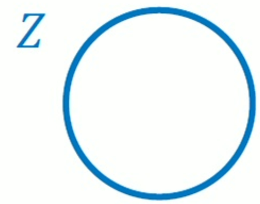
The path integral depends on the number of boundary circles but is independent of their lengths (topological model).

[Marolf, Maxfield 2020]

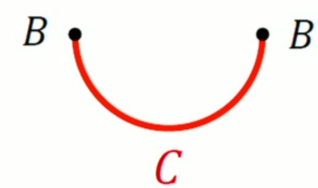
Example: a simple topological theory

$$\langle Z^n | Z^m \rangle = \langle Z^{n+m} \rangle = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} d^{n+m} \qquad \lambda = \frac{e^{2S_0}}{1 - e^{-2S_0}}$$

states in $\mathcal{H}_{BU}$



- closed source manifolds = disjoint unions of circles
- source-manifolds-with-boundary = unions of line segments



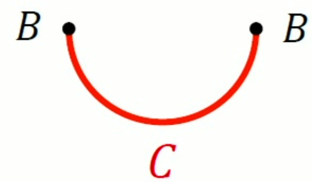
even number of boundary points

$\Longrightarrow$ m-boundary sectors with m odd are all empty!

$$\mathcal{H}_B = 0$$

- for any α-sector, there is exactly one state in $\mathcal{H}_{B \sqcup B}$: $|C\rangle$
- The Hilbert space $\mathcal{H}_{B \sqcup B} \cong \mathbb{C}$ factorizes in a trivial way: $\mathcal{H}_{B \sqcup B} \cong \mathbb{C} = \mathbb{C} \otimes \mathbb{C}$
- $\mathcal{A}_L^B$ and $\mathcal{A}_R^B$ are both isomorphic to the algebra of all bounded operators on $\mathbb{C}$

Example: a simple topological theory



- The operator C is a projection so it must have trace $\mathrm{tr}(C) = n \in \mathbb{Z}^+$

- The normalized state $n^{-1/2}|C\rangle$ has left density matrix $\tilde{\rho} = C/n = c$ for which the entropy is

$$S_{vN} = \mathrm{tr}(-c \ln c) = \ln n$$

- This entropy can be reproduced by embedding the one-dimensional Hilbert space in $\mathcal{H}_n \otimes \mathcal{H}_n$ by mapping

$$n^{-1/2}|C\rangle \;\rightarrow\; n^{-1/2}|C\rangle \otimes |\chi\rangle$$

for some normalized maximally entangled state $|\chi\rangle$

→ This model provides an example where the hidden sectors are required to give a Hilbert space interpretation of what might here be called the Ryu-Takayanagi entropy.

Example: pure asymptotically-AdS JT gravity

Pure JT: 2d toy model of gravitational systems containing only a metric g and a dilaton ϕ.
Action on asymptotically AdS_2 spacetime:

$$I = -\phi_0 \left[\int_{\mathcal{M}} \sqrt{g} R + 2 \int_{\partial \mathcal{M}} \sqrt{h} K \right] - \int_{\mathcal{M}} \sqrt{g} \phi (R+2) + 2 \int_{\partial \mathcal{M}} \sqrt{h} \phi (K-1)$$

JT path integral specified by

- constant ϕ_0
- length of boundary circle β
- boundary conditions for the dilaton: positive constant $\bar{\phi}_b$

Example: pure asymptotically-AdS JT gravity

Pure JT: 2d toy model of gravitational systems containing only a metric g and a dilaton ϕ.
Action on asymptotically AdS_2 spacetime:

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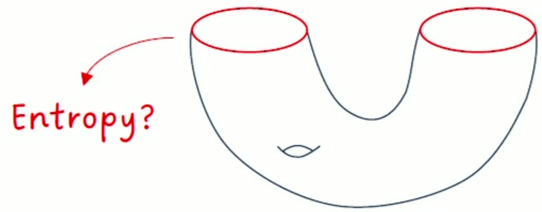
- closed source manifolds = disjoint unions of circles (but the path integral does depend on their lengths!)
- no one-boundary states
- no local degrees of freedom → basis of 2-boundary states of the form $|E, E\rangle$
- all operators are defined by line segments; distinct operator $e^{-\beta H}$ for every possible length of the boundary β
- The algebra is $\mathcal{A}_L^B$ is the abelian algebra defined by bounded functions of the Hamiltonian H.

$$\mathcal{H}_{B \sqcup B} = \bigoplus_E \mathcal{H}_{B \sqcup B}^E$$

- $\mathcal{H}_{B \sqcup B}^E$ is the one-dimensional Hilbert space of states proportional to $|E, E\rangle$

Conclusions



$$\mathcal{H}_{B \sqcup B} \overset{?}{\neq} \mathcal{H}_B \otimes \mathcal{H}_B$$

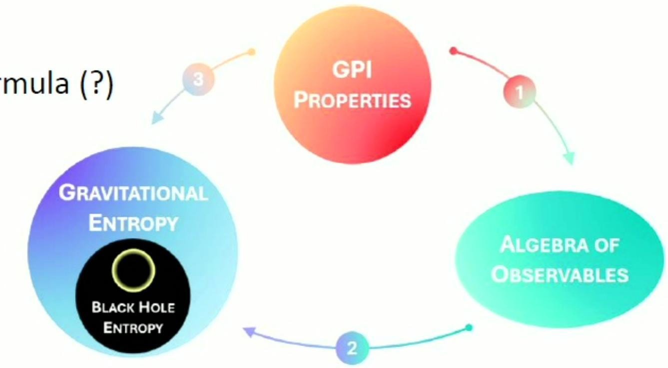
- A gravitational path integral satisfying a simple and familiar set of axioms defines type I von Neumann algebras of observables associated with codimension-2 boundaries.
- The path integral also defines a trace and entropy on these algebras.
- The Hilbert space on which the algebras act decomposes as

$$\mathcal{H}_{B \sqcup B} = \bigoplus_{\mu} \mathcal{H}^{\mu}_{B \sqcup B, L} \otimes \mathcal{H}^{\mu}_{B \sqcup B, R}$$

- The path integral trace is equivalent to a standard trace on an extended Hilbert space: $\mathrm{tr} = \tilde{\mathrm{Tr}}_{\mu}$.
- This provides a state-counting interpretation of the entropy, even when the gravitational theory is not known to have a holographic dual.
- In the semiclassical limit, the entropy is given by the Ryu-Takayanagi formula.

Outlook

Island Formula (?)



1 + 2

Spin Foam?
Group Field Theory?

- **Axiomatic structure for Spin Foam and Group Field Theories** (in collaboration with M. Bruno, F. Mele; ongoing discussion with C. Rovelli, L. Freidel, F. Girelli, D. Oriti)
  - Inspiration from Atiyah's axiomatization of TQFT, to be generalised to accommodate gravity in $d > 3$
    - Include **spacetime corners** [Baez, Dolan 1995; Donnelly, Freidel 2016; Freidel, Geiller, Pranzetti 2020; …]
    - Include **topology change** [Banerjee, Moore 2020; …]
  - Axioms A.1 – A.5 for Spin Foam/Group Field Theories?
    - Check compatibility with **Marolf-Maxfield construction** (under "reasonable" assumptions, the theory decomposes into baby-universe sectors where axiom factorization holds).
- Observables and Entropy from the Spin Foam/Group Field Theories path integral
- Black Hole Evaporation
  - Boundary regions **not necessarily closed or disjoint** [Freidel, Oliveri, Pranzetti, Speziale 2021; …]
  - Incorporate **quantum corrections** [Faulkner, Lewkowycz, Maldacena 2013; …]
  - From the Euclidean to the **Lorentzian path integral** [Marolf 2022; Dittrich, Jacobson, Padua-Argüelles 2024; …]