Title: The Moore-Tachikawa conjecture via shifted symplectic geometry

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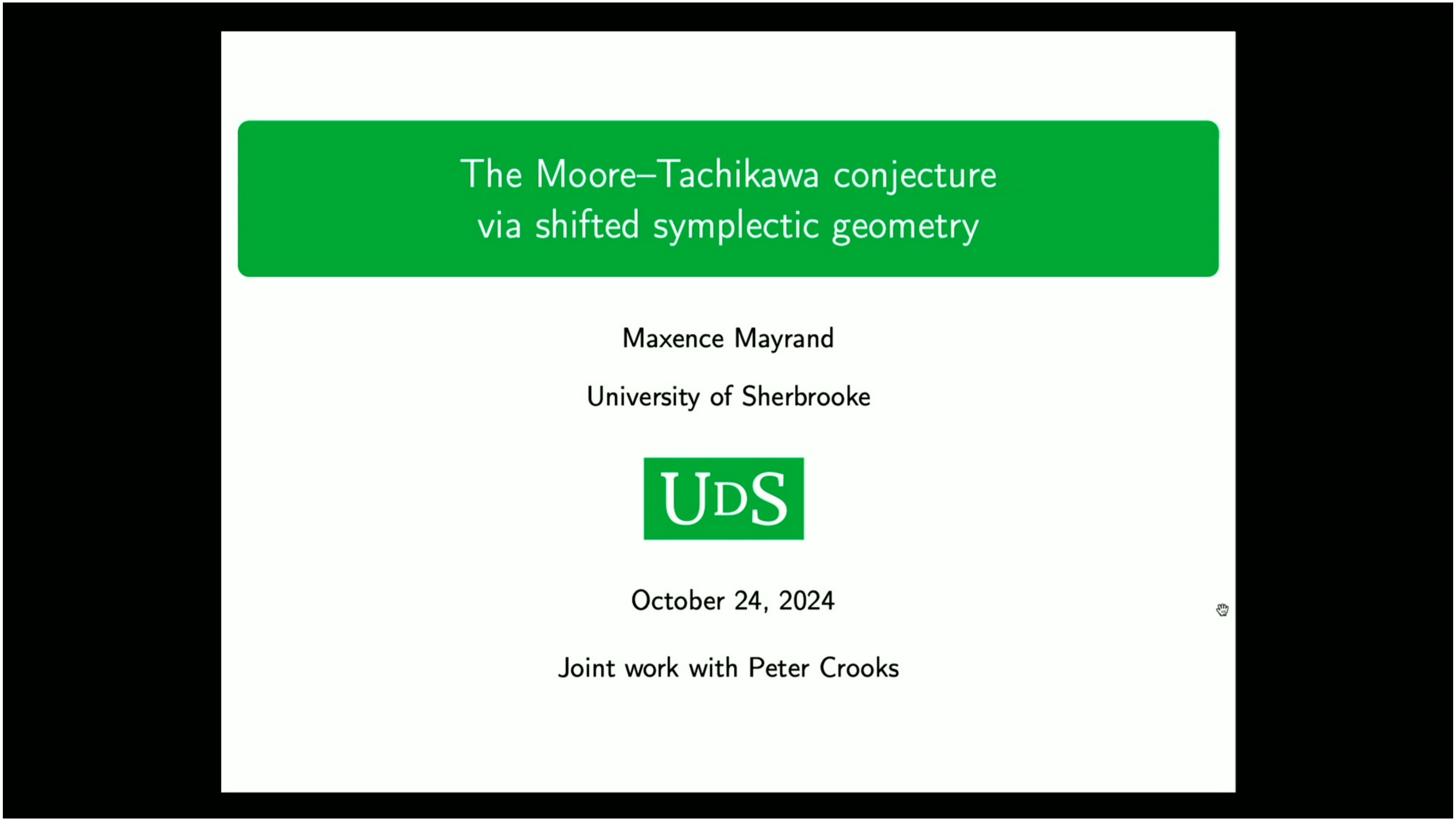
Abstract:

The Moore-Tachikawa conjecture posits the existence of certain 2-dimensional topological quantum field theories (TQFTs) valued in a category of complex Hamiltonian varieties. Previous work by Ginzburg-Kazhdan and Braverman-Nakajima-Finkelberg has made significant progress toward proving this conjecture. In this talk, I will introduce a new approach to constructing these TQFTs using the framework of shifted symplectic geometry. This higher version of symplectic geometry, initially developed in derived algebraic geometry, also admits a concrete differential-geometric interpretation via Lie groupoids and differential forms, which plays a central role in our results. It provides an algebraic explanation for the existence of these TQFTs, showing that their structure comes naturally from three ingredients: Morita equivalence, as well as multiplication and identity bisections in abelian symplectic groupoids. It also allows us to generalize the Moore-Tachikawa TQFTs in various directions, raising interesting questions in Lie theory and Poisson geometry. This is joint work with Peter Crooks.



The Moore–Tachikawa conjecture via shifted symplectic geometry

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October 24, 2024

Joint work with Peter Crooks

Overview

Theorem (Kostant 1963). Let G be a complex semisimple group, $\mathfrak{g} := \mathrm{Lie}(G)$.

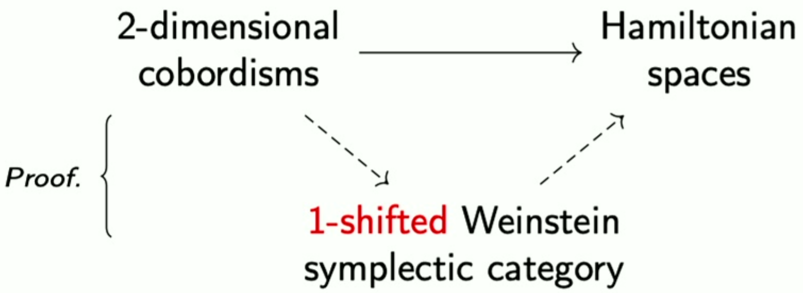
(1) $\exists$ *global slice* $\mathcal{S} \subset \mathfrak{g}^*_{\mathrm{reg}}$ for the coadjoint action

(2) The stabilizers G_ξ are *abelian* for all $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$

(3) $\mathfrak{g}^*_{\mathrm{reg}}$ is *Hartogs* : holomorphic functions on $\mathfrak{g}^*_{\mathrm{reg}}$ extend to $\mathfrak{g}^*$

Upshot of the talk.

Any (not necessarily semisimple) Lie algebra satisfying (1)–(3), or, more generally, Poisson affine variety satisfying analogues of (1)–(3), defines a 2-dimensional Topological Quantum Field Theory valued in Hamiltonian spaces.



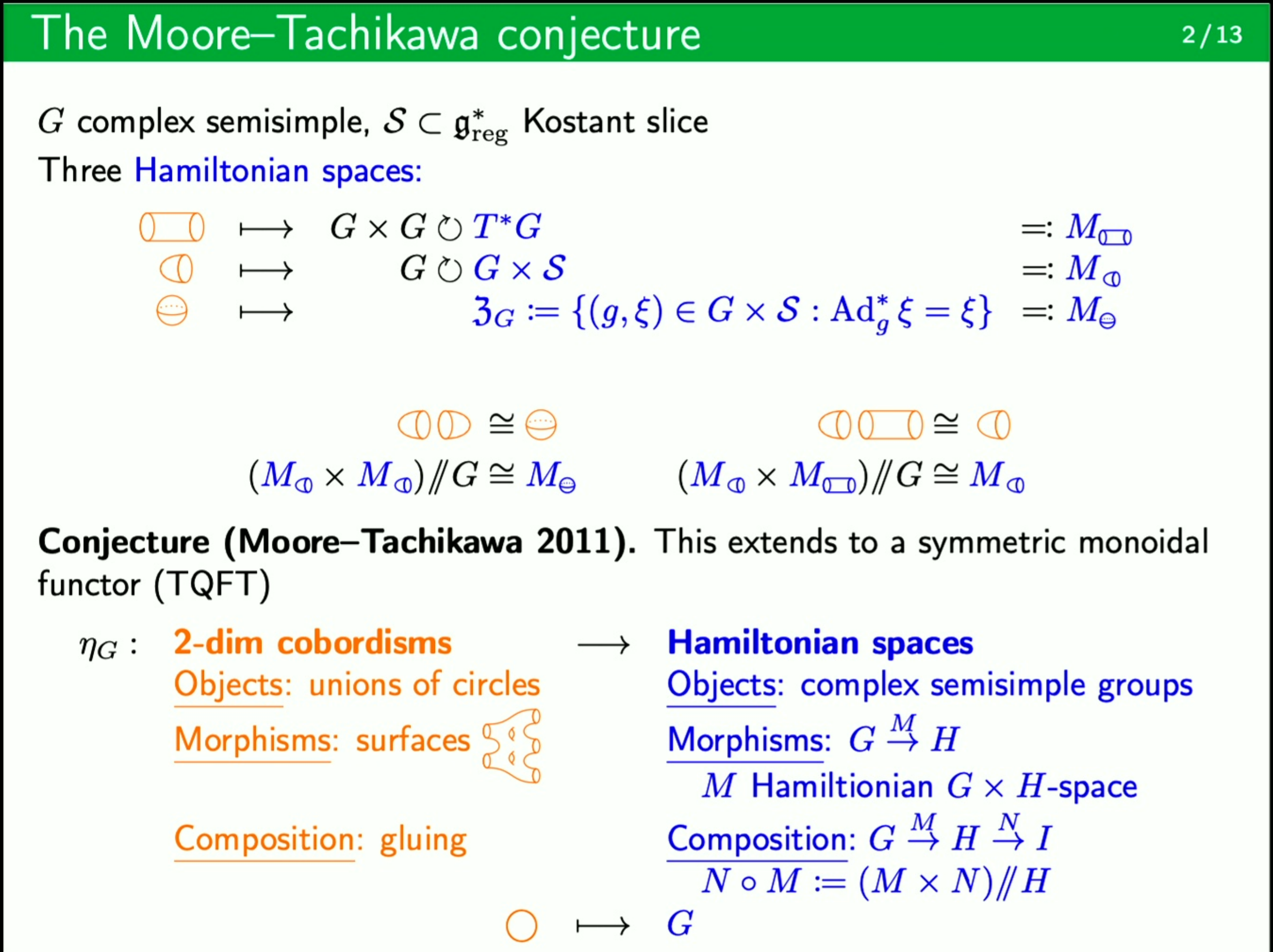
2-dimensional cobordisms

Hamiltonian spaces

Proof.

1-shifted Weinstein symplectic category

The case where $\mathfrak{g}$ is complex semisimple is the Moore–Tachikawa conjecture.



The Moore–Tachikawa conjecture

G complex semisimple, $\mathcal{S} \subset \mathfrak{g}^*_{\mathrm{reg}}$ Kostant slice

Three Hamiltonian spaces:

$$\longmapsto \quad G \times G \circlearrowright T^*G \qquad =: M$$

$$\longmapsto \quad G \circlearrowright G \times \mathcal{S} \qquad =: M$$

$$\longmapsto \quad \mathfrak{Z}_G := \{(g,\xi) \in G \times \mathcal{S} : \mathrm{Ad}^*_g \xi = \xi\} \quad =: M$$

$$\cong \qquad\qquad \cong$$

$$(M \times M)/\!\!/G \cong M \qquad (M \times M)/\!\!/G \cong M$$

Conjecture (Moore–Tachikawa 2011). This extends to a symmetric monoidal functor (TQFT)

η_G : **2-dim cobordisms** $\longrightarrow$ **Hamiltonian spaces**

Objects: unions of circles — Objects: complex semisimple groups

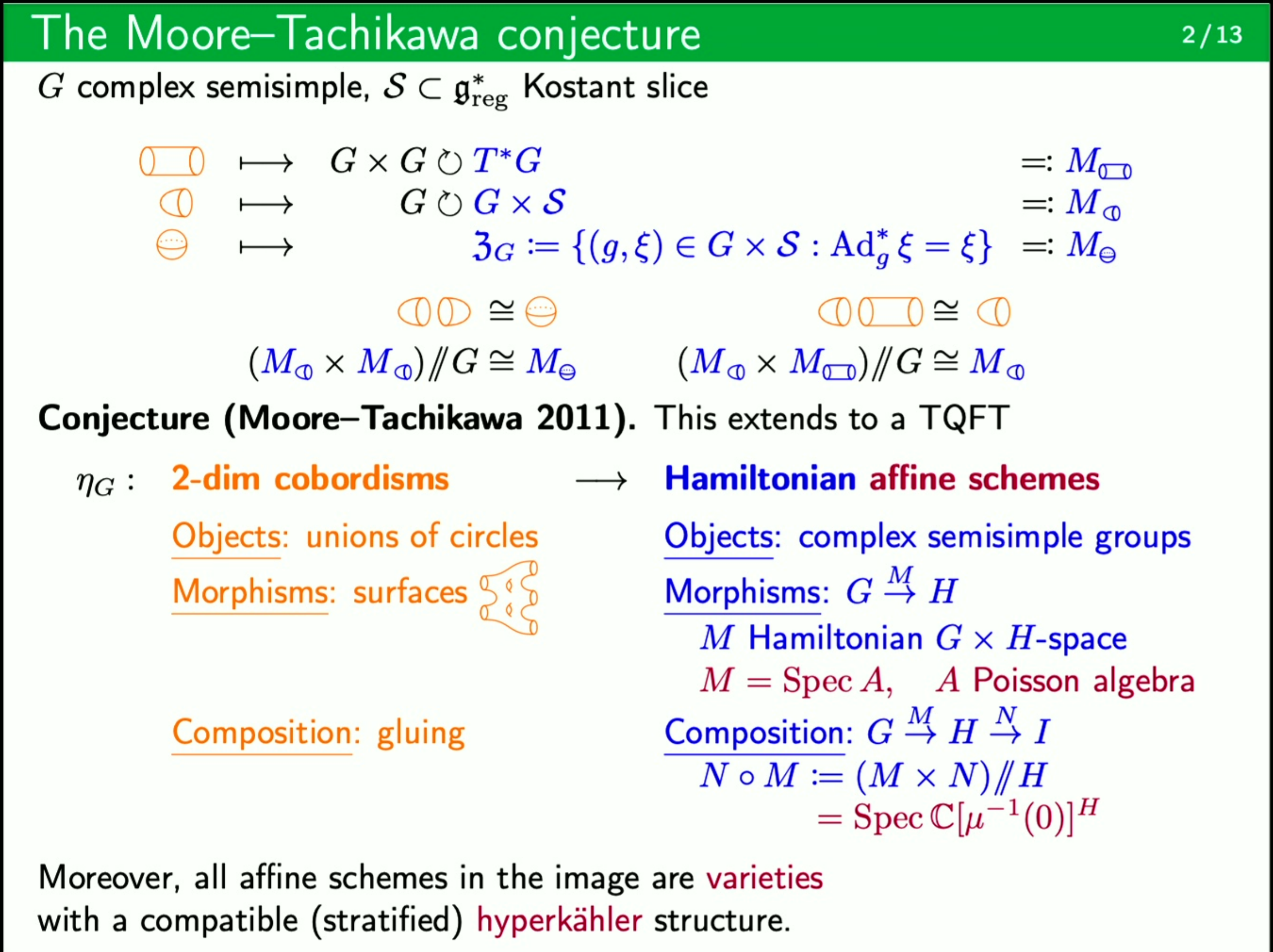
Morphisms: surfaces — Morphisms: $G \xrightarrow{M} H$

M Hamiltonian $G \times H$-space

Composition: gluing — Composition: $G \xrightarrow{M} H \xrightarrow{N} I$

$N \circ M := (M \times N)/\!\!/H$

$$\longmapsto \quad G$$



The Moore–Tachikawa conjecture

G complex semisimple, $\mathcal{S} \subset \mathfrak{g}^*_{\mathrm{reg}}$ Kostant slice

$$\text{▭} \longmapsto G \times G \circlearrowright T^*G =: M_{\text{▭}}$$

$$\text{◖} \longmapsto G \circlearrowright G \times \mathcal{S} =: M_{\text{◖}}$$

$$\ominus \longmapsto \mathfrak{z}_G := \{(g,\xi) \in G \times \mathcal{S} : \mathrm{Ad}^*_g \xi = \xi\} =: M_{\ominus}$$

$$\text{◖◗} \cong \ominus \qquad \text{◖▭} \cong \text{◖}$$

$$(M_{\text{◖}} \times M_{\text{◖}})/\!\!/G \cong M_{\ominus} \qquad (M_{\text{◖}} \times M_{\text{▭}})/\!\!/G \cong M_{\text{◖}}$$

Conjecture (Moore–Tachikawa 2011). This extends to a TQFT

η_G : **2-dim cobordisms** $\longrightarrow$ **Hamiltonian affine schemes**

Objects: unions of circles

Morphisms: surfaces

Composition: gluing

Objects: complex semisimple groups

Morphisms: $G \xrightarrow{M} H$

M Hamiltonian $G \times H$-space

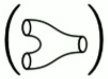
$M = \operatorname{Spec} A$, A Poisson algebra

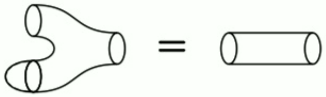
Composition: $G \xrightarrow{M} H \xrightarrow{N} I$

$$N \circ M := (M \times N)/\!\!/H$$
$$= \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^H$$

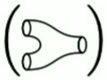
Moreover, all affine schemes in the image are varieties with a compatible (stratified) hyperkähler structure.

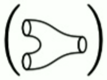
The Moore–Tachikawa conjecture

It suffices to construct η_G  and verify a finite number of relations such as



Examples.

$\eta_{\mathrm{SL}(2,\mathbb{C})}$  $= \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$\eta_{\mathrm{SL}(3,\mathbb{C})}$  $= \overline{\mathcal{O}_{\min}(E_6)}$

Partial solutions.

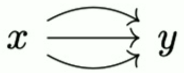
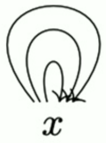
- Ginzburg–Kazhdan: scheme part of the conjecture (*ad hoc*)
- Braverman–Finkelberg–Nakajima: scheme part for general G and variety part for $\mathrm{SL}(n, \mathbb{C})$ (Coulomb branches)
- Arakawa: scheme part (vertex algebras)
- Bielawski: regular version (open dense subsets of the varieties)
- Crooks–M.: new proof of scheme part using shifted symplectic geometry $\Longrightarrow$ generalizations

Differentiable stacks

Two Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are **Morita equivalent** if

$$\exists \quad \begin{tikzcd} & \mathcal{H} \arrow[dl] \arrow[dr] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd} \qquad \text{such that } \mathcal{H} \cong f_1^*\mathcal{G}_1 \text{ and } \mathcal{H} \cong f_2^*\mathcal{G}_2.$$

Examples.

Lie groupoids	equivalence relation $x \longrightarrow y$	general Lie groupoid 	Lie group 
Morita equivalence	isomorphic quotient spaces		isomorphism of Lie groups

The ***stack*** associated to a Lie groupoid $\mathcal{G} \rightrightarrows M$ is its Morita equivalence class.

Example. $G \circlearrowright M$ (Lie group action on manifold) $x \xrightarrow{(g,x)} g \cdot x$ The quotient stack $[M/G]$ is the Morita equivalence class of the *action groupoid* $G \times M \rightrightarrows M$.

Shifted symplectic geometry

The ***"tangent bundle"*** of a stack $[\mathcal{G} \rightrightarrows M]$ is the ***Lie algebroid***

$$\mathbf{Lie}(\mathcal{G}) \longrightarrow TM \qquad \text{(vector bundles over } M\text{)}$$

up to ***quasi-isomorphisms*** of 2-term complexes.

Question. What is a ***"symplectic form"*** on a stack $[\mathcal{G} \rightrightarrows M]$?

A symplectic form on a manifold N is an

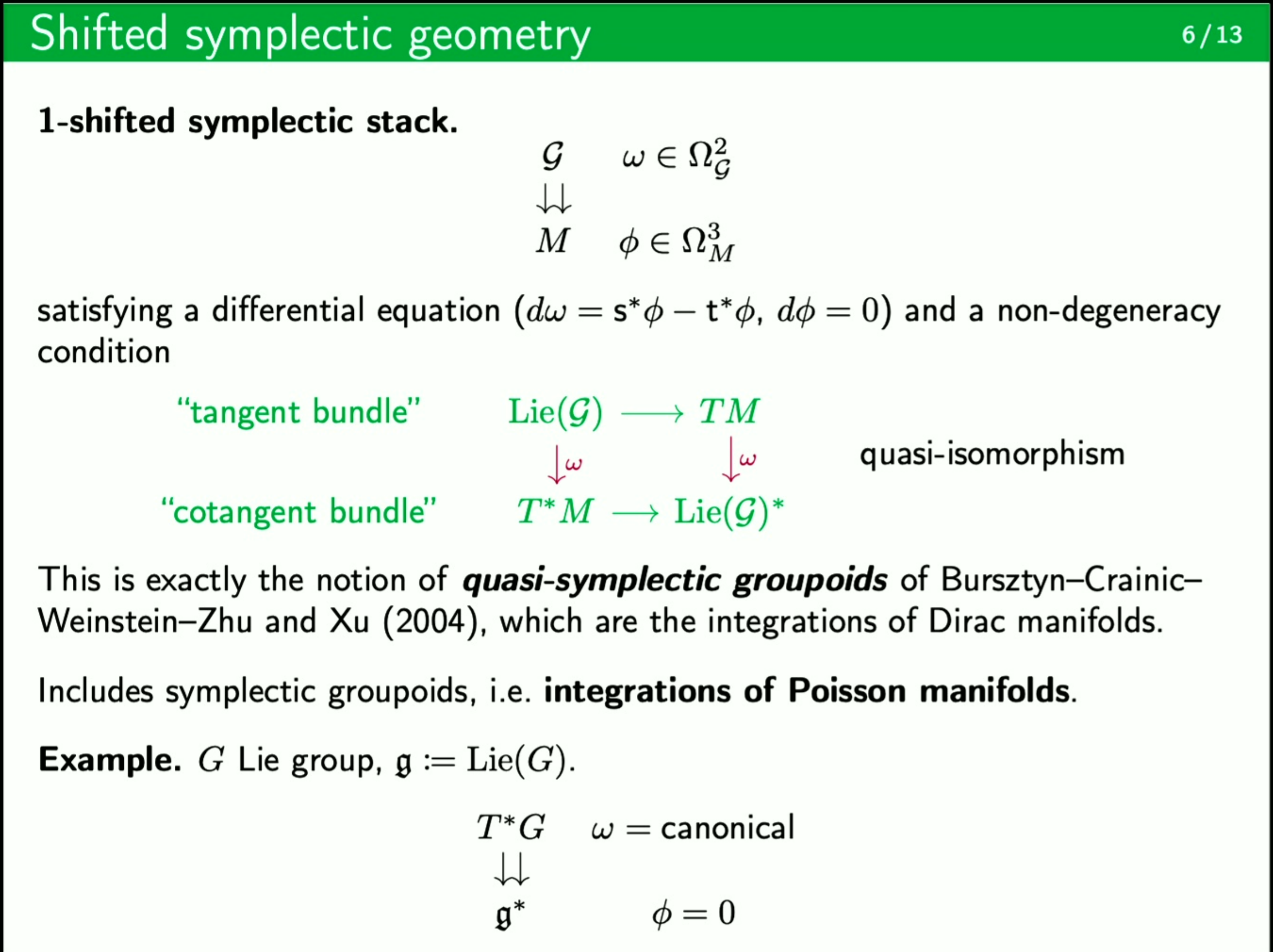
$$\textit{isomorphism} \quad TN \cong T^*N.$$

A "symplectic form" on a stack $[\mathcal{G} \rightrightarrows M]$ is a

$$\textit{quasi-isomorphism} \quad (\mathbf{Lie}(\mathcal{G}) \longrightarrow TM) \simeq (T^*M \longrightarrow \mathbf{Lie}(\mathcal{G})^*).$$

How should we align them? Three ways!

$\begin{tikzcd} \mathrm{Lie}(\mathcal{G}) \arrow[r] & TM \arrow[d] & \\ & T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* \end{tikzcd}$	$\begin{tikzcd} \mathrm{Lie}(\mathcal{G}) \arrow[r] \arrow[d] & TM \arrow[d] \\ T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* \end{tikzcd}$	$\begin{tikzcd} & \mathrm{Lie}(\mathcal{G}) \arrow[r] \arrow[d] & TM \\ T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* & \end{tikzcd}$
0-shifted symplectic symplectic geometry	**1-shifted symplectic**	**2-shifted symplectic**



Shifted symplectic geometry

1-shifted symplectic stack.

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & \omega \in \Omega^2_{\mathcal{G}} \\ M & \phi \in \Omega^3_M \end{tikzcd}$$

satisfying a differential equation ($d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$, $d\phi = 0$) and a non-degeneracy condition

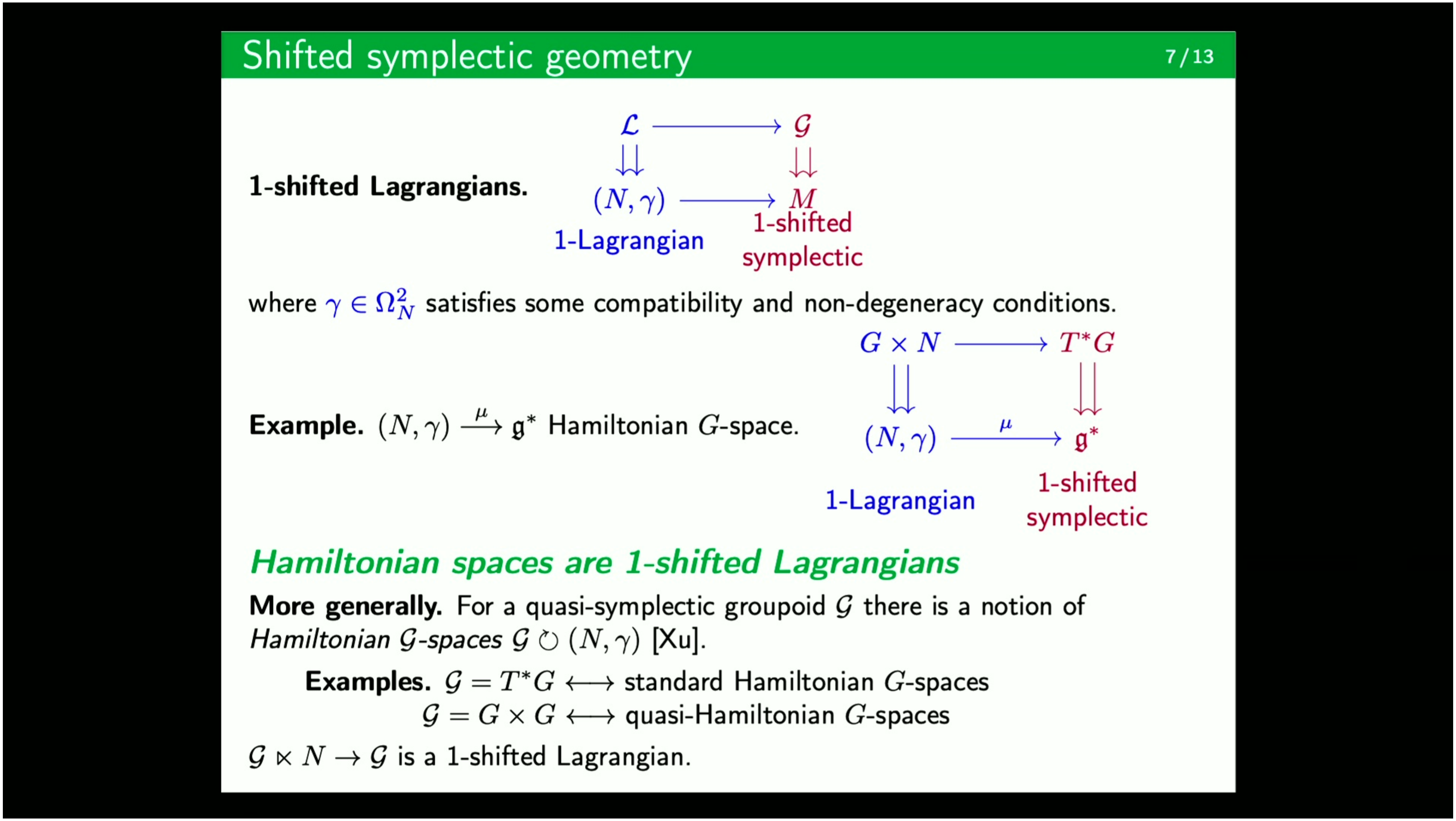
$$\begin{tikzcd} \text{“tangent bundle”} & \mathrm{Lie}(\mathcal{G}) \arrow[r] \arrow[d, "\omega"] & TM \arrow[d, "\omega"] & \text{quasi-isomorphism} \\ \text{“cotangent bundle”} & T^*M \arrow[r] & \mathrm{Lie}(\mathcal{G})^* & \end{tikzcd}$$

This is exactly the notion of ***quasi-symplectic groupoids*** of Bursztyn–Crainic–Weinstein–Zhu and Xu (2004), which are the integrations of Dirac manifolds.

Includes symplectic groupoids, i.e. **integrations of Poisson manifolds**.

Example. G Lie group, $\mathfrak{g} := \mathrm{Lie}(G)$.

$$\begin{tikzcd} T^*G \arrow[d, shift left] \arrow[d, shift right] & \omega = \text{canonical} \\ \mathfrak{g}^* & \phi = 0 \end{tikzcd}$$



Shifted symplectic geometry

1-shifted Lagrangians.

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\
(N, \gamma) \arrow[r] & M \\
\text{1-Lagrangian} & \text{1-shifted symplectic}
\end{tikzcd}$$

where $\gamma \in \Omega^2_N$ satisfies some compatibility and non-degeneracy conditions.

Example. $(N, \gamma) \xrightarrow{\mu} \mathfrak{g}^*$ Hamiltonian G-space.

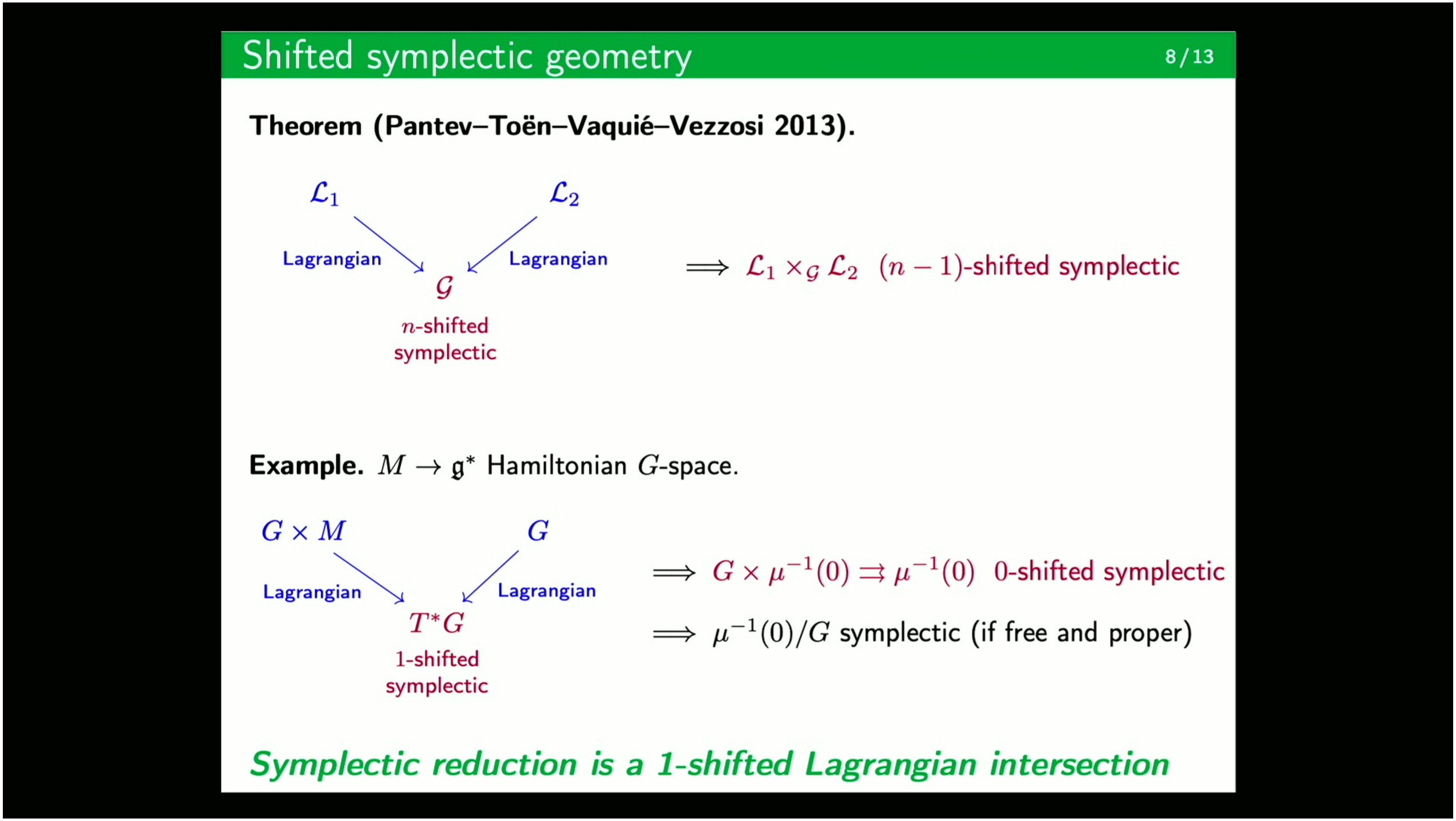
$$\begin{tikzcd}
G \times N \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & T^*G \arrow[d, shift left] \arrow[d, shift right] \\
(N, \gamma) \arrow[r, "\mu"] & \mathfrak{g}^* \\
\text{1-Lagrangian} & \text{1-shifted symplectic}
\end{tikzcd}$$

Hamiltonian spaces are 1-shifted Lagrangians

More generally. For a quasi-symplectic groupoid $\mathcal{G}$ there is a notion of *Hamiltonian $\mathcal{G}$-spaces* $\mathcal{G} \circlearrowright (N, \gamma)$ [Xu].

Examples. $\mathcal{G} = T^*G \longleftrightarrow$ standard Hamiltonian G-spaces
$\mathcal{G} = G \times G \longleftrightarrow$ quasi-Hamiltonian G-spaces

$\mathcal{G} \ltimes N \to \mathcal{G}$ is a 1-shifted Lagrangian.



Shifted symplectic geometry

Theorem (Pantev–Toën–Vaquié–Vezzosi 2013).

$$\begin{tikzcd}
\mathcal{L}_1 \arrow[dr, "\text{Lagrangian}"'] & & \mathcal{L}_2 \arrow[dl, "\text{Lagrangian}"] \\
& \mathcal{G} &
\end{tikzcd}$$

n-shifted symplectic

$\Longrightarrow$ $\mathcal{L}_1 \times_{\mathcal{G}} \mathcal{L}_2$ $(n-1)$-shifted symplectic

Example. $M \to \mathfrak{g}^*$ Hamiltonian G-space.

$$\begin{tikzcd}
G \times M \arrow[dr, "\text{Lagrangian}"'] & & G \arrow[dl, "\text{Lagrangian}"] \\
& T^*G &
\end{tikzcd}$$

1-shifted symplectic

$\Longrightarrow$ $G \times \mu^{-1}(0) \rightrightarrows \mu^{-1}(0)$ 0-shifted symplectic

$\Longrightarrow$ $\mu^{-1}(0)/G$ symplectic (if free and proper)

Symplectic reduction is a 1-shifted Lagrangian intersection

Shifted symplectic geometry

1-shifted Lagrangian correspondences. $\mathcal{L} \to \mathcal{G}_1 \times \mathcal{G}_2^-$

$$\begin{tikzcd}[column sep=small]
 & \mathcal{L}_1 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{1-Lag corr}"] & & \mathcal{L}_2 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{1-Lag corr}"] & & & & \mathcal{L}_1 \times_{\mathcal{G}_2} \mathcal{L}_2 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{1-Lag corr}"] & \\
\mathcal{G}_1 & & \mathcal{G}_2 & & \mathcal{G}_3 & \Longrightarrow & \mathcal{G}_1 & & \mathcal{G}_3
\end{tikzcd}$$

⇝ ***1-shifted Weinstein symplectic category***
(Obj: 1-shifted symplectic stacks | Mor: 1-shifted Lagrangian correspondences)

Moore–Tachikawa conjecture. Every complex semisimple group G induces a TQFT

η_G : **2-dim cobordisms** $\longrightarrow$ **Hamiltonian spaces**

Objects: complex semisimple groups

Morphisms: $G \xrightarrow{M} H : M$ Hamil. $G \times H$-space

$G \xrightarrow{M} H \xrightarrow{N} I, \quad N \circ M := (M \times N) /\!\!/ H$

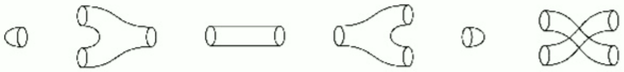
Composition in the category of Hamiltonian spaces is intersection of 1-shifted Lagrangians (Calaque 2015)

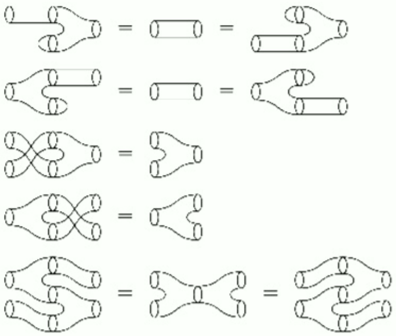
2-dim cobordisms $\longrightarrow$ ***1-shifted Weinstein symplectic category***

Commutative Frobenius objects

A 2d TQFT is a symmetric monoidal functor $\mathbf{Cob}_2 \longrightarrow \mathbf{C}$ for some symmetric monoidal category $(\mathbf{C}, \otimes, I)$.

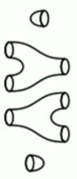
$\mathbf{Cob}_2$ is generated on **objects** by $\bigcirc$

and on **morphisms** by  with relations



$(\star)$

It suffices to specify an object $X \in \mathbf{C}$ ($\bigcirc \mapsto X$, $\bigcirc\bigcirc \mapsto X \otimes X$, $\dots$) and morphisms



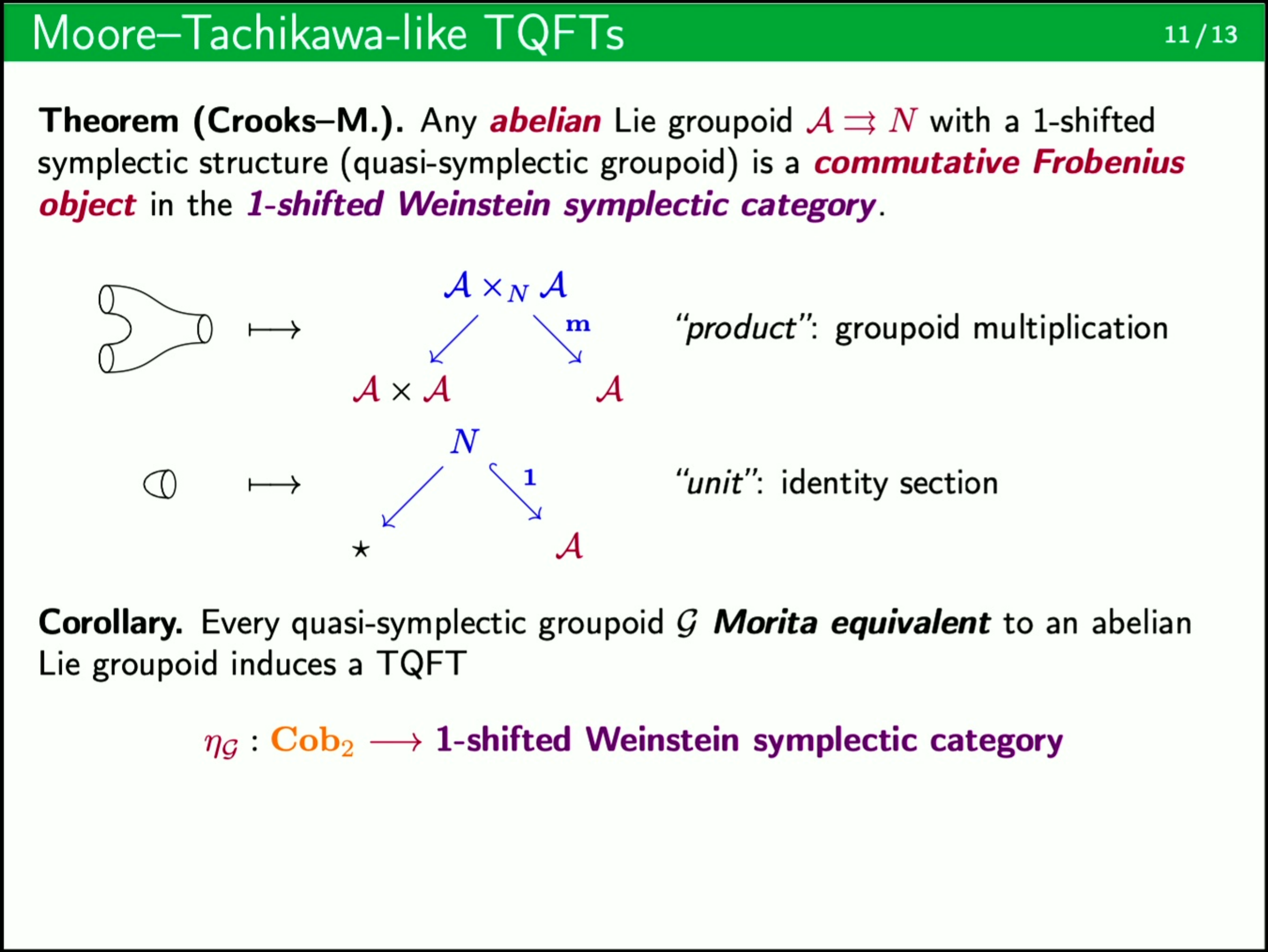
$\longmapsto \quad (I \to X)$ *"unit"*

$\longmapsto \quad (X \otimes X \to X)$ *"product"*

$\longmapsto \quad (X \to X \otimes X)$ *"co-product"*

$\longmapsto \quad (X \to I)$ *"co-unit"*

satisfying analogues of $(\star)$, i.e. X is a ***commutative Frobenius object*** in $(\mathbf{C}, \otimes, I)$.



Moore–Tachikawa-like TQFTs

Theorem (Crooks–M.). Any ***abelian*** Lie groupoid $\mathcal{A} \rightrightarrows N$ with a 1-shifted symplectic structure (quasi-symplectic groupoid) is a ***commutative Frobenius object*** in the ***1-shifted Weinstein symplectic category***.

$\longmapsto$

$$\begin{tikzcd} & \mathcal{A} \times_N \mathcal{A} \arrow[dl] \arrow[dr, "\mathbf{m}"] & \\ \mathcal{A} \times \mathcal{A} & & \mathcal{A} \end{tikzcd}$$

"product": groupoid multiplication

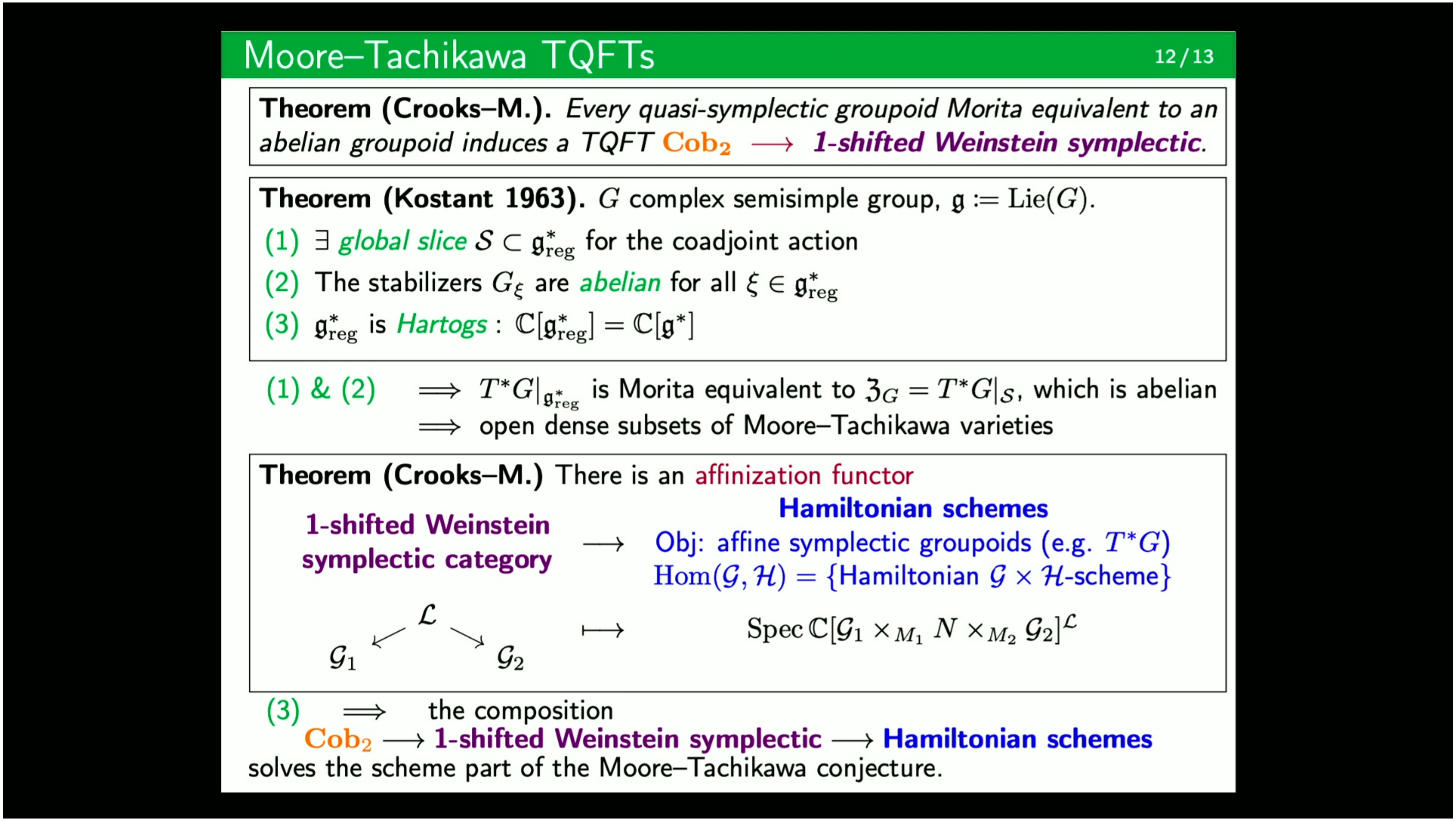
$\longmapsto$

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr, hook, "\mathbf{1}"] & \\ \star & & \mathcal{A} \end{tikzcd}$$

"unit": identity section

Corollary. Every quasi-symplectic groupoid $\mathcal{G}$ ***Morita equivalent*** to an abelian Lie groupoid induces a TQFT

$$\eta_{\mathcal{G}} : \mathbf{Cob}_2 \longrightarrow \textbf{1-shifted Weinstein symplectic category}$$



Moore–Tachikawa TQFTs

Theorem (Crooks–M.). *Every quasi-symplectic groupoid Morita equivalent to an abelian groupoid induces a TQFT* $\mathbf{Cob_2} \longrightarrow$ ***1-shifted Weinstein symplectic***.

Theorem (Kostant 1963). G complex semisimple group, $\mathfrak{g} := \mathrm{Lie}(G)$.

(1) $\exists$ *global slice* $\mathcal{S} \subset \mathfrak{g}^*_{\mathrm{reg}}$ for the coadjoint action

(2) The stabilizers G_ξ are *abelian* for all $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$

(3) $\mathfrak{g}^*_{\mathrm{reg}}$ is *Hartogs* : $\mathbb{C}[\mathfrak{g}^*_{\mathrm{reg}}] = \mathbb{C}[\mathfrak{g}^*]$

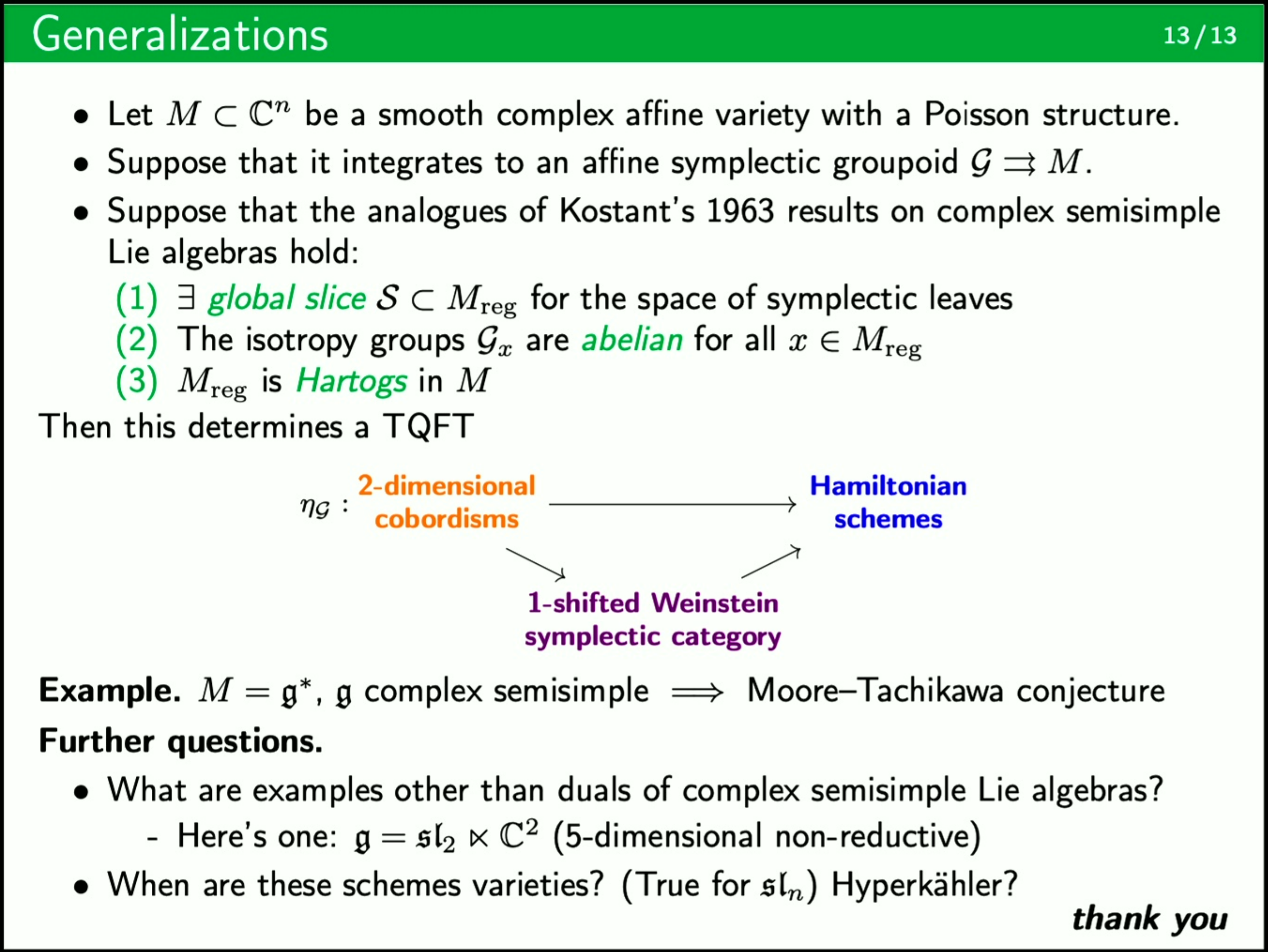
(1) & (2) $\implies$ $T^*G|_{\mathfrak{g}^*_{\mathrm{reg}}}$ is Morita equivalent to $\mathfrak{Z}_G = T^*G|_{\mathcal{S}}$, which is abelian
$\implies$ open dense subsets of Moore–Tachikawa varieties

Theorem (Crooks–M.) There is an affinization functor

1-shifted Weinstein symplectic category $\longrightarrow$ **Hamiltonian schemes**
Obj: affine symplectic groupoids (e.g. T^*G)
$\mathrm{Hom}(\mathcal{G}, \mathcal{H}) = \{$Hamiltonian $\mathcal{G} \times \mathcal{H}$-scheme$\}$

$$\begin{tikzcd} & \mathcal{L} \arrow[dl] \arrow[dr] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd} \longmapsto \operatorname{Spec} \mathbb{C}[\mathcal{G}_1 \times_{M_1} N \times_{M_2} \mathcal{G}_2]^{\mathcal{L}}$$

(3) $\implies$ the composition
$\mathbf{Cob_2} \longrightarrow$ **1-shifted Weinstein symplectic** $\longrightarrow$ **Hamiltonian schemes**
solves the scheme part of the Moore–Tachikawa conjecture.



Generalizations

- Let $M \subset \mathbb{C}^n$ be a smooth complex affine variety with a Poisson structure.
- Suppose that it integrates to an affine symplectic groupoid $\mathcal{G} \rightrightarrows M$.
- Suppose that the analogues of Kostant's 1963 results on complex semisimple Lie algebras hold:
  (1) $\exists$ *global slice* $\mathcal{S} \subset M_{\mathrm{reg}}$ for the space of symplectic leaves
  (2) The isotropy groups $\mathcal{G}_x$ are *abelian* for all $x \in M_{\mathrm{reg}}$
  (3) M_{reg} is *Hartogs* in M

Then this determines a TQFT

$$\begin{tikzcd}
\eta_{\mathcal{G}} : \textbf{2-dimensional cobordisms} \arrow[rr] \arrow[dr] & & \textbf{Hamiltonian schemes} \\
& \textbf{1-shifted Weinstein symplectic category} \arrow[ur] &
\end{tikzcd}$$

Example. $M = \mathfrak{g}^*$, $\mathfrak{g}$ complex semisimple $\Longrightarrow$ Moore–Tachikawa conjecture

Further questions.

- What are examples other than duals of complex semisimple Lie algebras?
  - Here's one: $\mathfrak{g} = \mathfrak{sl}_2 \ltimes \mathbb{C}^2$ (5-dimensional non-reductive)
- When are these schemes varieties? (True for $\mathfrak{sl}_n$) Hyperkähler?

thank you