Title: Chiralization of cluster structures

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Abstract:

The chiralization in the title denotes a certain procedure which turns cluster X-varieties into q-W algebras. Many important notions from cluster and q-W worlds, such as mutations, global functions, screening operators, R-matrices, etc emerge naturally in this context. In particular, we discover new bosonizations of q-W algebras and establish connections between previously known bosonizations. If time permits, I will discuss potential applications of our approach to the study of 3d topological theories and local systems with affine gauge groups.

Based on a joint project with J. Shiraishi, J.E. Bourgine, B. Feigin, A. Shapiro, and G. Schrader.

Chiralization of cluster structures

j/ w/ Shiraishi, Bourgine, Feigin, Shapiro, Schrader

$\mathbb{C}\langle e^{X_i}\rangle$ $\qquad e^{X_i}e^{X_j} = q^{2\varepsilon_{ij}}\, e^{X_j} e^{X_i}$

$$\varepsilon_{ij} \in \frac{1}{2}\mathbb{Z}$$

$\varepsilon_{ij} = 1$ $\qquad \underset{e^a}{\circ} \longrightarrow \underset{e^b}{\circ}$ $\qquad e^a e^b = q^2 e^b e^a$

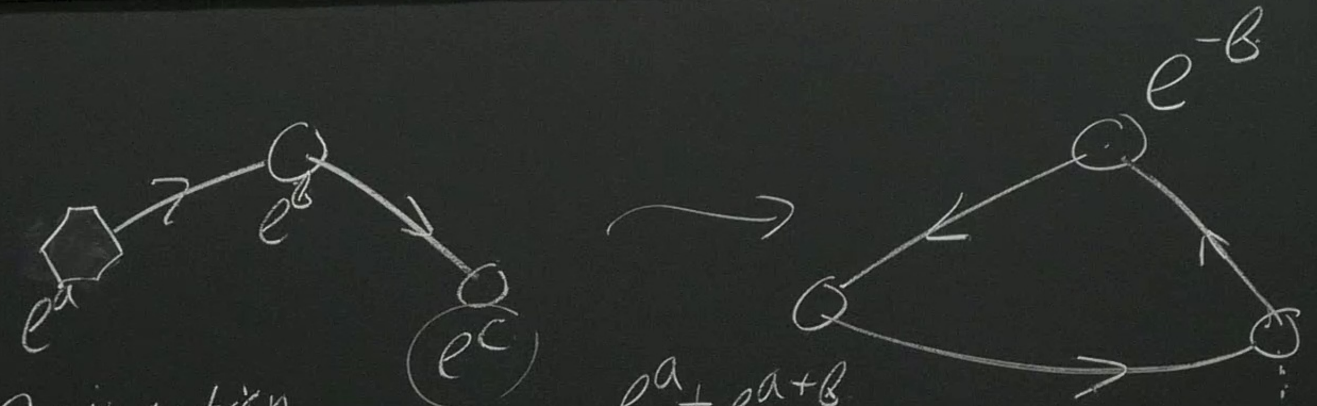
$\varepsilon_{ij} = 1/2$ $\qquad \circ \dashrightarrow \circ$

uctures

, Shapiro, Schrader

$e^{x_j} e^{x_i}$

$q^2 e^b e^a = q e^{a+b}$



Conjugation

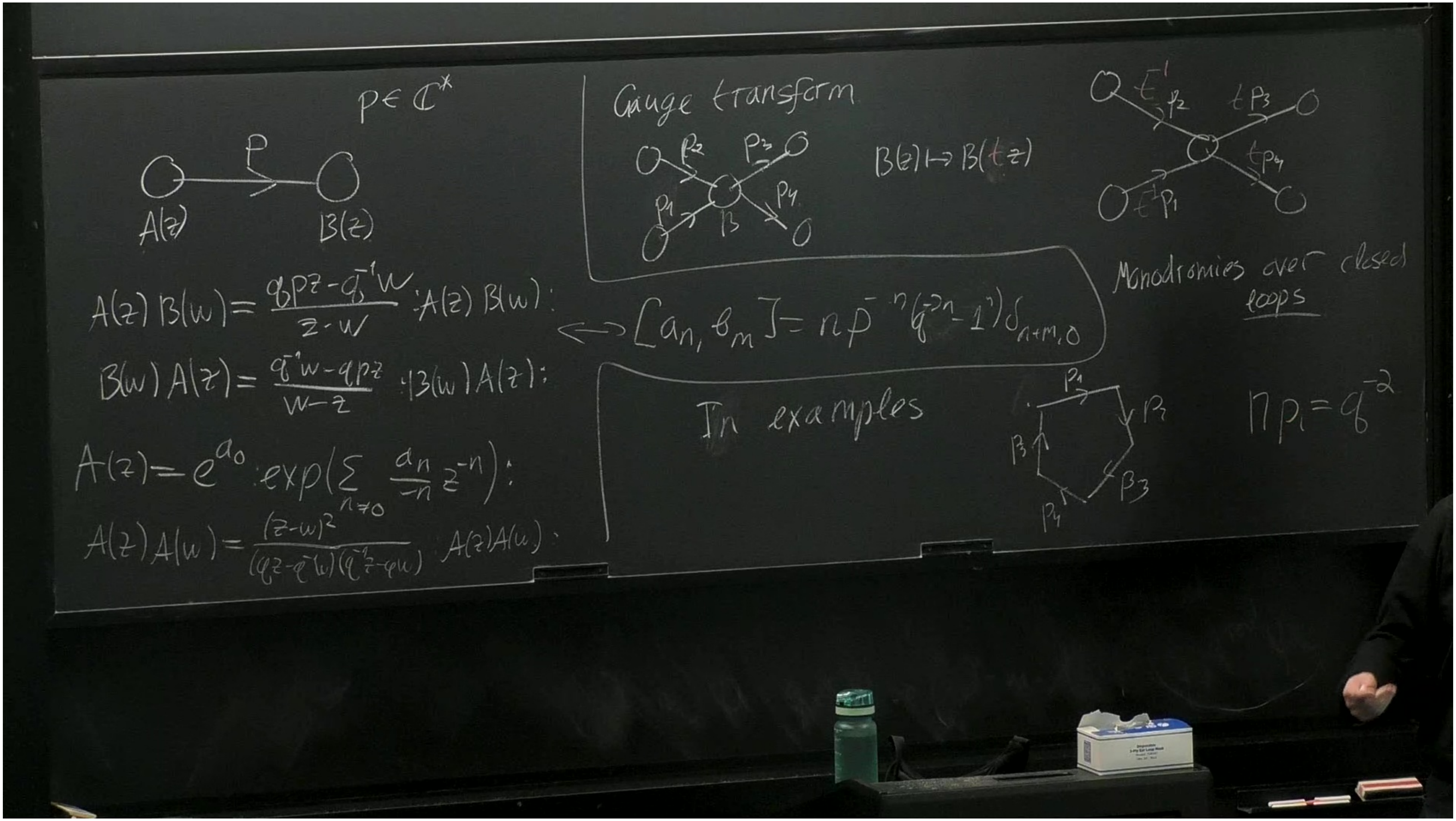
$\varphi(-b) = \prod_{k=1}^{\infty} (1+q^{2k-1} e^{-b})$

$e^a + e^{a+b} = e^a(1+q^{-1}e^b)$

$e^c(1+q^{-1}e^{-b})^{-1}$

Global functions $\mathbb{C}[X]_q$ — elements which are Laurent after $\forall$ seq of mut

$\dot{U}_q(sl_2)$ $\quad \dot{\mathbb{C}}[G]_q$ $\quad \dot{M}_{g,\Sigma,P}$ $\quad$ Coulomb branches 4d ... k-th


$p \in \mathbb{C}^*$
$A(z)$
$B(z)$
$A(z)B(w) = \frac{qpz - q^{-1}w}{z-w} :A(z)B(w):$
$B(w)A(z) = \frac{q^{-1}w - qpz}{w-z} :B(w)A(z):$
$A(z) = e^{a_0} :\exp(\sum_{n\neq 0} \frac{a_n}{-n} z^{-n}):$
$A(z)A(w) = \frac{(z-w)^2}{(qz-q^{-1}w)(q^{-1}z-qw)} :A(z)A(w):$
Gauge transform
$B(z) \mapsto B(tz)$
$[a_n, b_m] = np^{-n}(q^{-2n}-1)\delta_{n+m,0}$
Monodromies over closed loops
In examples
$\prod p_i = q^{-2}$

$$A(z)A(w) = \frac{\ \cdots\ }{(qz-q^{-1}w)(q^{-1}z-qw)} :A(z)A(w):$$

Dashed arrows

$$\underset{A}{\square} \overset{\text{"+"}\ p}{\dashrightarrow} \underset{B}{\square}$$

$A\,B \sim -1$

$B\,A \sim 1$

connects $\square^+, \square^+$

$$\underset{A}{\square} \overset{\text{"-"}\ p}{\dashrightarrow} \underset{B}{\square}$$

$A \cdot B \sim 1$

$B \cdot A \sim -1$

e^{a} e^{b} $\ldots = q\, e^{a} e^{b} = q\, e^{a+b}$

$\varepsilon_{ij} = 1/2$ □ - - → □

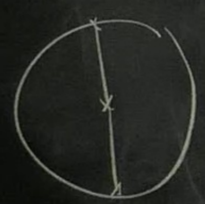
which are Laurent after ∀ seq of mut

$\dot{\mathcal{U}}_q(sy)$ $\dot{\mathcal{O}}[u]_q$ $\dot{\mathcal{M}}_{G,\varepsilon,P}$ Coulomb branches 4d ... K-th

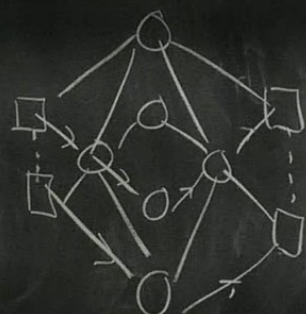
$q_1 q_2 q_3 = 1$

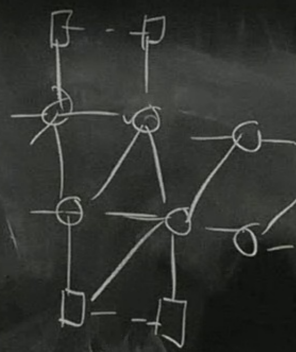
$q_2 = q_1^2$

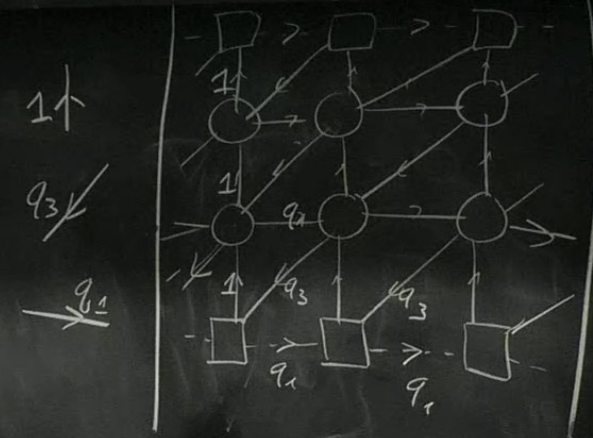
$\mathcal{U}_q(\widehat{sl}_3)$



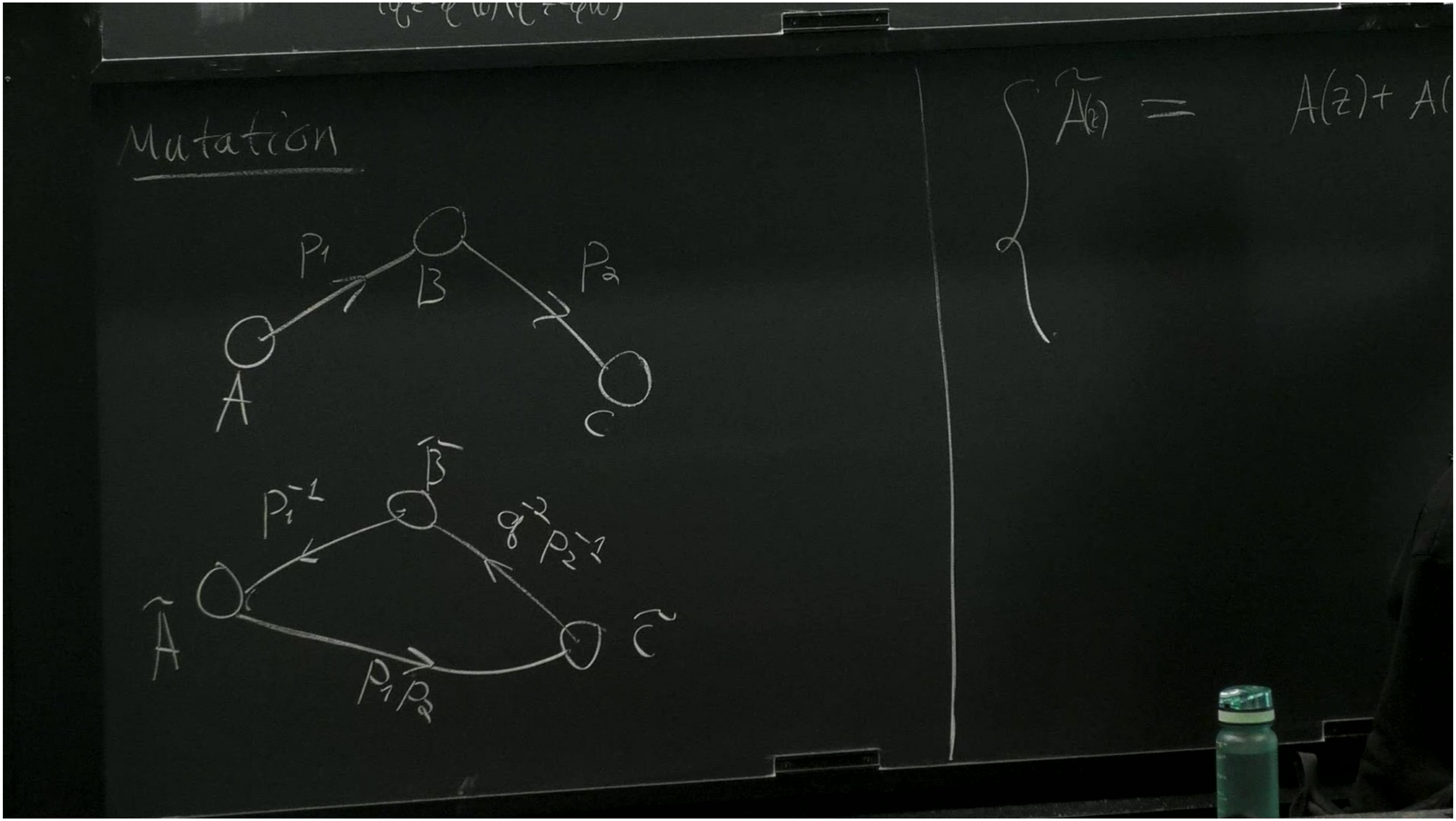
<u>Th</u> ∃ Rep of $\mathcal{U}_{\bar{q}}(\ddot{sl}_N)$ in Fock(Q)

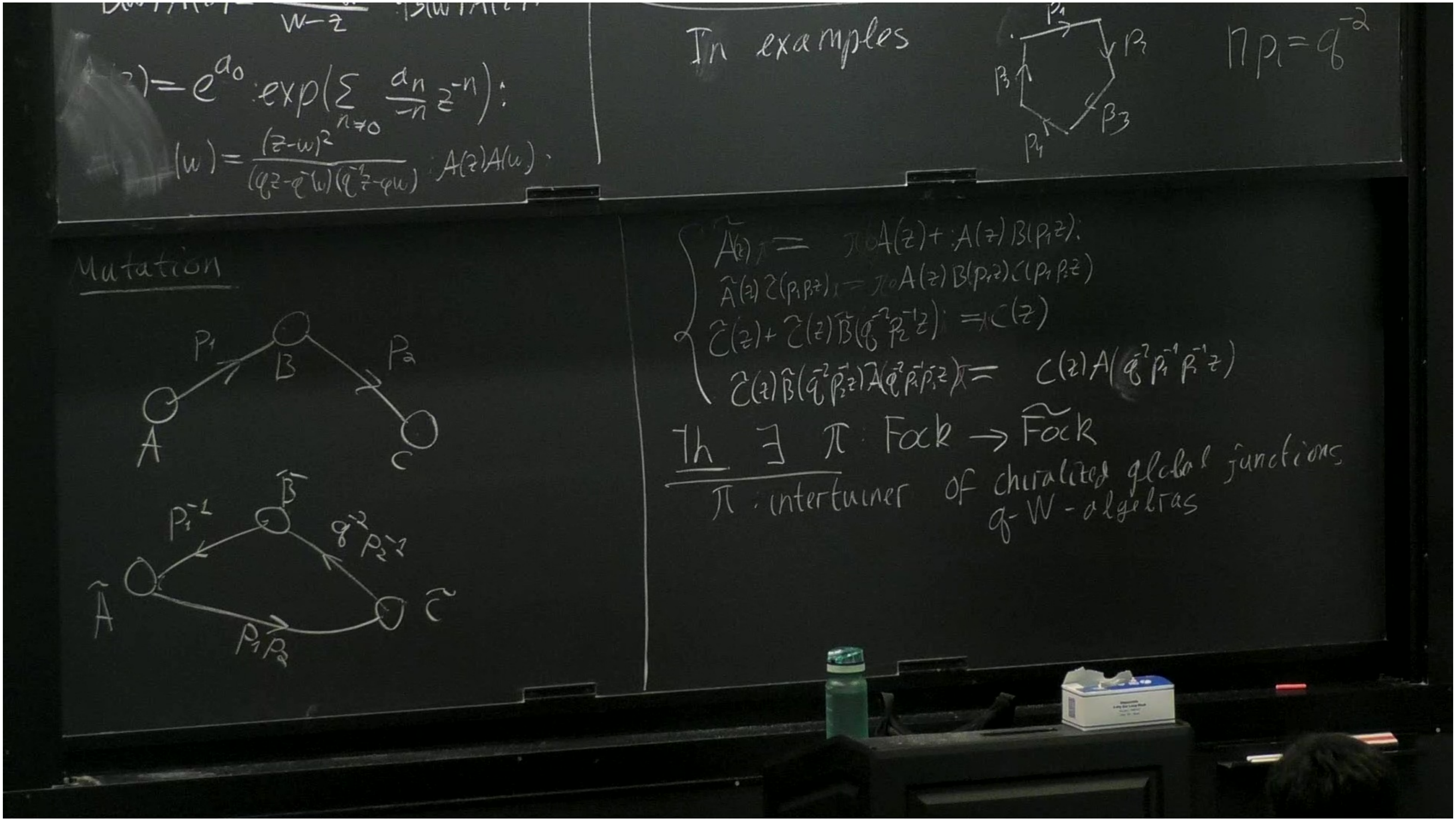


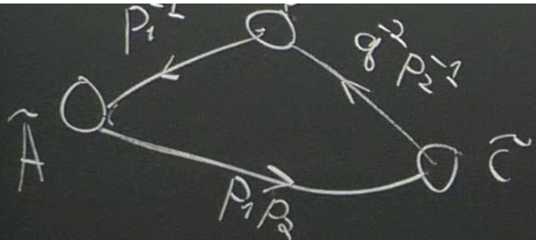




Quiver $N=3$


Mutation
P1
B
P2
A
C
B̃
P1^-1
q^-2 P2^-1
Ã
C̃
P1 P2
Ã(z) = A(z) + A(


In examples
β_1 β_2 β_3 β_4 β_5
$\prod p_i = q^{-2}$
$A(z) = e^{a_0} :\exp(\sum_{n\neq 0} \frac{a_n}{-n} z^{-n}):$
$(w) = \frac{(z-w)^2}{(qz-q^{-1}w)(q^{-1}z-qw)} :A(z)A(w):$
Mutation
p_1 p_2
A B C
p_1^{-1} $q^{-2}p_2^{-1}$ p_1p_2
$\tilde{A}$ $\tilde{B}$ $\tilde{C}$
$\tilde{A}(z)\pi = \pi\circ A(z) + :A(z)B(p_1z):$
$\tilde{A}(z)\tilde{C}(p_1p_2z)\pi = \pi\circ A(z)B(p_1z)C(p_1p_2z)$
$\tilde{C}(z) + \tilde{C}(z)\tilde{B}(q^{-2}p_2^{-1}z) = \pi C(z)$
$\tilde{C}(z)\tilde{B}(q^{-2}p_2^{-1}z)\tilde{A}(q^{-2}p_1^{-1}p_2^{-1}z)\pi = C(z)A(q^{-2}p_1^{-1}p_2^{-1}z)$
Th $\exists\ \pi$: Fock $\to \widetilde{\text{Fock}}$
π: intertwiner of chiralized global functions q-W-algebras



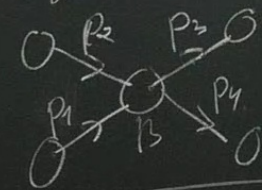
$\mathcal{H}$: intertwiner of ...
q-W-algebras

(seed)
Chart ~ Free field rep of qW-alg,

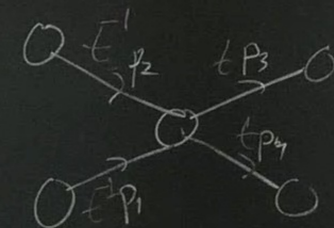
3d Coulomb branch ~ VOA

4d Coulomb branch ~ q-W.

Gauge transform



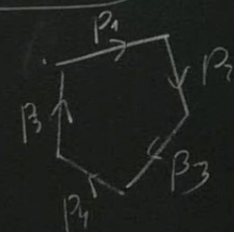
$B(z) \mapsto B(tz)$



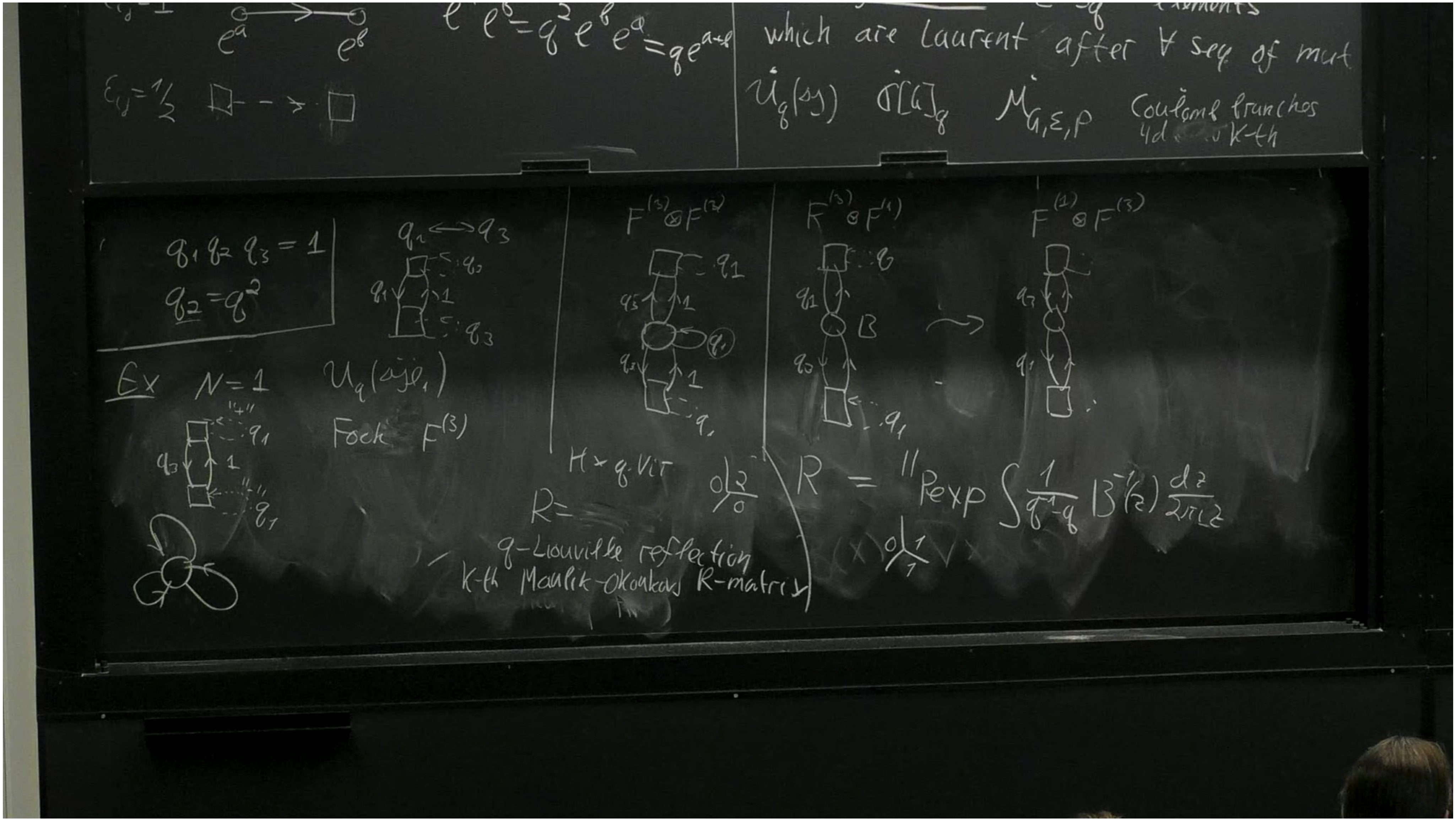
$$\longleftrightarrow [a_n, b_m] = n p^{-n}(q^{-2n}-1)\delta_{n+m,0}$$

Monodromies over closed loops

In examples



$$\prod p_i = q^{-2}$$


$e^a e^b = q^2 e^b e^a = q e^{a+b}$
which are Laurent after ∀ seq of mut
$U_q(sl_2)$
$\mathfrak{sl}[n]_q$
$\mathcal{M}_{G,\varepsilon,\rho}$
Coulomb branches
4d
K-th
$q_1 q_2 q_3 = 1$
$q_2 = q^2$
$q_1 \longleftrightarrow q_3$
$F^{(3)} \otimes F^{(2)}$
$F^{(3)} \otimes F^{(1)}$
$F^{(1)} \otimes F^{(3)}$
Ex $N=1$
$U_q(\widehat{gl}_1)$
Fock $F^{(3)}$
$H \times q\text{-Vir}$
$R =$
q-Liouville reflection
K-th Maulik-Okounkov R-matrix
$R = \text{"}P\exp \int \frac{1}{q^{-1}-q} B^{-1}(z) \frac{dz}{2\pi i z}$

Free-field realizations

screening operators $\xrightarrow{q-1}$ KL corresp

node $\rightsquigarrow$ fermionic screening operator

Casimirs $\rightsquigarrow$ bosonic screening operator

$\bigcap \mathrm{Ker} \approx W$