Title: Free-to-Interacting Maps and the Bott Spiral

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: October 15, 2024 - 11:00 AM

URL: https://pirsa.org/24100114

Abstract:

I will discuss free (i.e., noninteracting) and interacting classifications for certain fermionic symmetry-protected topological phases (SPTs) and show how to define free-to-interacting maps in terms of homotopy theory. I will apply these ideas to study the phenomenon of the "Bott spiral": as shown in work of Queiroz-Khalaf-Stern using a dimensional reduction approach, the tenfold way classification of free theories (with one additional reflection symmetry) breaks down to a large 2-torsion classification in the presence of interactions. Using K-theory and (Anderson-dual) twisted spin bordism, we can compute the same interacting classification, and with the language of fermionic groups, we can interpret the "spiral" as a failure of Morita invariance on the interacting side. Time permitting, I will also discuss how to model dimensional reduction and symmetry breaking for the Bott spiral in terms of homotopy theory.

This talk is based on upcoming work joint with Arun Debray, Natalia Pacheco-Tallaj, and Luuk Stehouwer.

Free-to-Interacting Maps & the Bott Spiral

idea:

general	SPTs
$\mathcal{M}$ modulispace of models	Hamiltonians
Δ critical locus	gapless Hams
phase $\in (\mathcal{M} \setminus \Delta)/\sim$	$\sim$ preserves symm. + the gap

os

Ts

tonians

pless Hams

preserves symm
+ the gap

today: comparing ν_{free} and ν_{int}

free-to-interacting map:

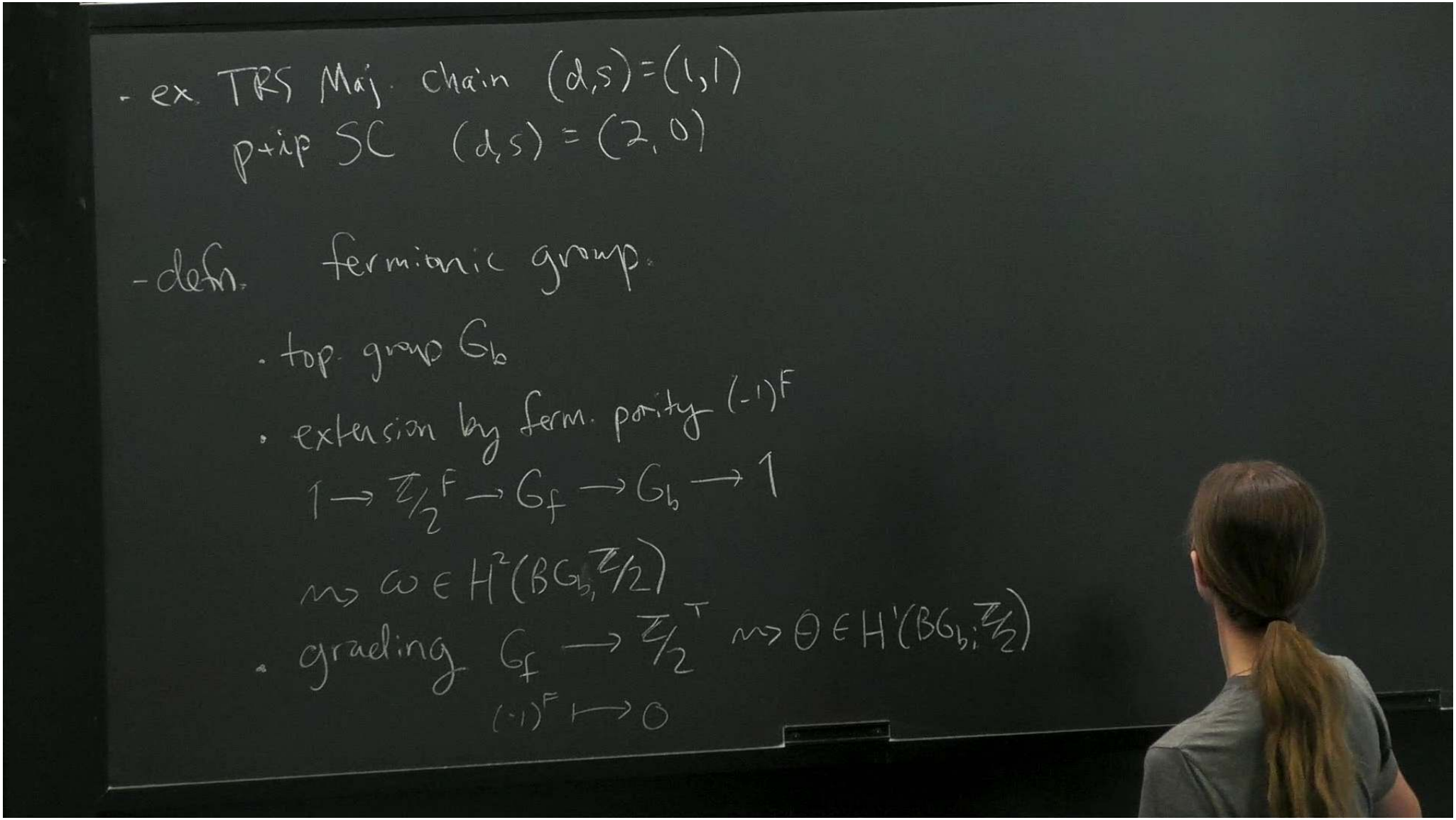
$$F2I: [x]_{free} \longmapsto [x]_{int}$$

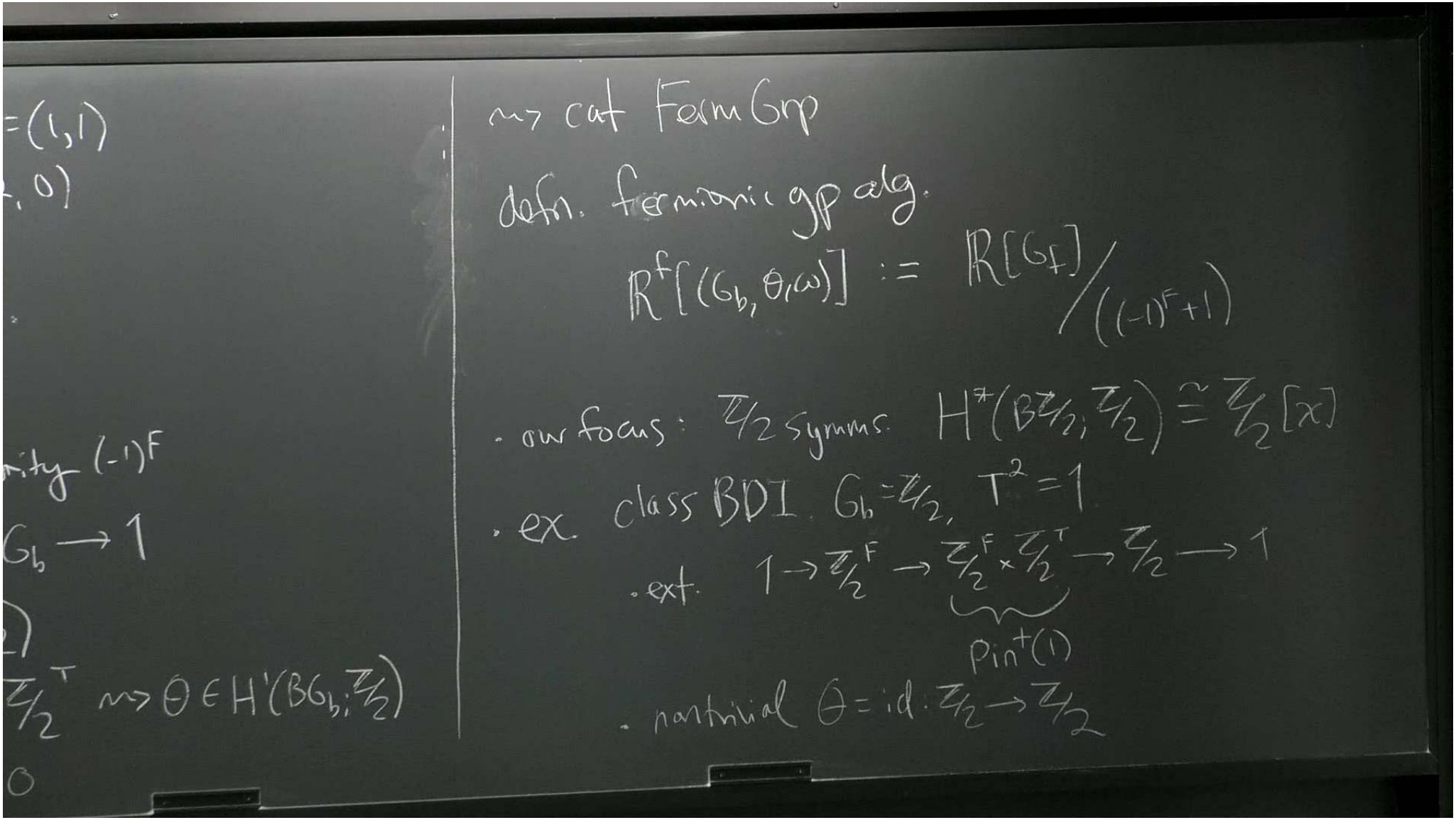
$$\uparrow \qquad \uparrow$$

$$KO^{d+\ell-k-2} \longrightarrow \mathcal{V}^{d+2}_{E_{\ell,k}}$$

① K-theory tenfold way

$$\left\{ \begin{array}{l} \text{free fern SPT} \\ \text{in } d \text{ dim w/ } s \text{ symm} \end{array} \right\} \cong KO^{d+s-2}$$


- ex. TRS Maj. chain $(d,s)=(1,1)$
p+ip SC $(d,s)=(2,0)$
- defn. fermionic group
• top. group G_b
• extension by ferm. parity $(-1)^F$
$1 \to \mathbb{Z}_2^F \to G_f \to G_b \to 1$
$\rightsquigarrow \omega \in H^2(BG_b; \mathbb{Z}_2)$
• grading $G_f \to \mathbb{Z}_2^T \rightsquigarrow \theta \in H^1(BG_b; \mathbb{Z}_2)$
$(-1)^F \mapsto 0$


=(1,1)
,0)
ity $(-1)^F$
$G_b \to 1$
$\mathbb{Z}_2^T \rightsquigarrow \theta \in H^1(BG_b; \mathbb{Z}_2)$
0
$\rightsquigarrow$ cat FermGrp
defn. fermionic gp alg.
$R^f[(G_b, \theta, \omega)] := R[G_f] / ((-1)^F + 1)$
• our focus: $\mathbb{Z}_2$ symms $H^*(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$
• ex. class BDI $G_b = \mathbb{Z}_2$, $T^2 = 1$
• ext. $1 \to \mathbb{Z}_2^F \to \mathbb{Z}_2^F \times \mathbb{Z}_2^T \to \mathbb{Z}_2 \to 1$
$\mathrm{Pin}^+(1)$
• nontrivial $\theta = \mathrm{id} : \mathbb{Z}_2 \to \mathbb{Z}_2$

- ex (BDI cont'd)

note: $(\theta, \omega) = (x, 0)$ corresponds to $(\omega_1 \sigma, \omega_2 \sigma)$, $\sigma \to B^{\mathbb{Z}/2}$ twist

$$R^f(Pin^+(1)) \cong R[a]/_{a^2=+1} \cong Cl_{+1}$$

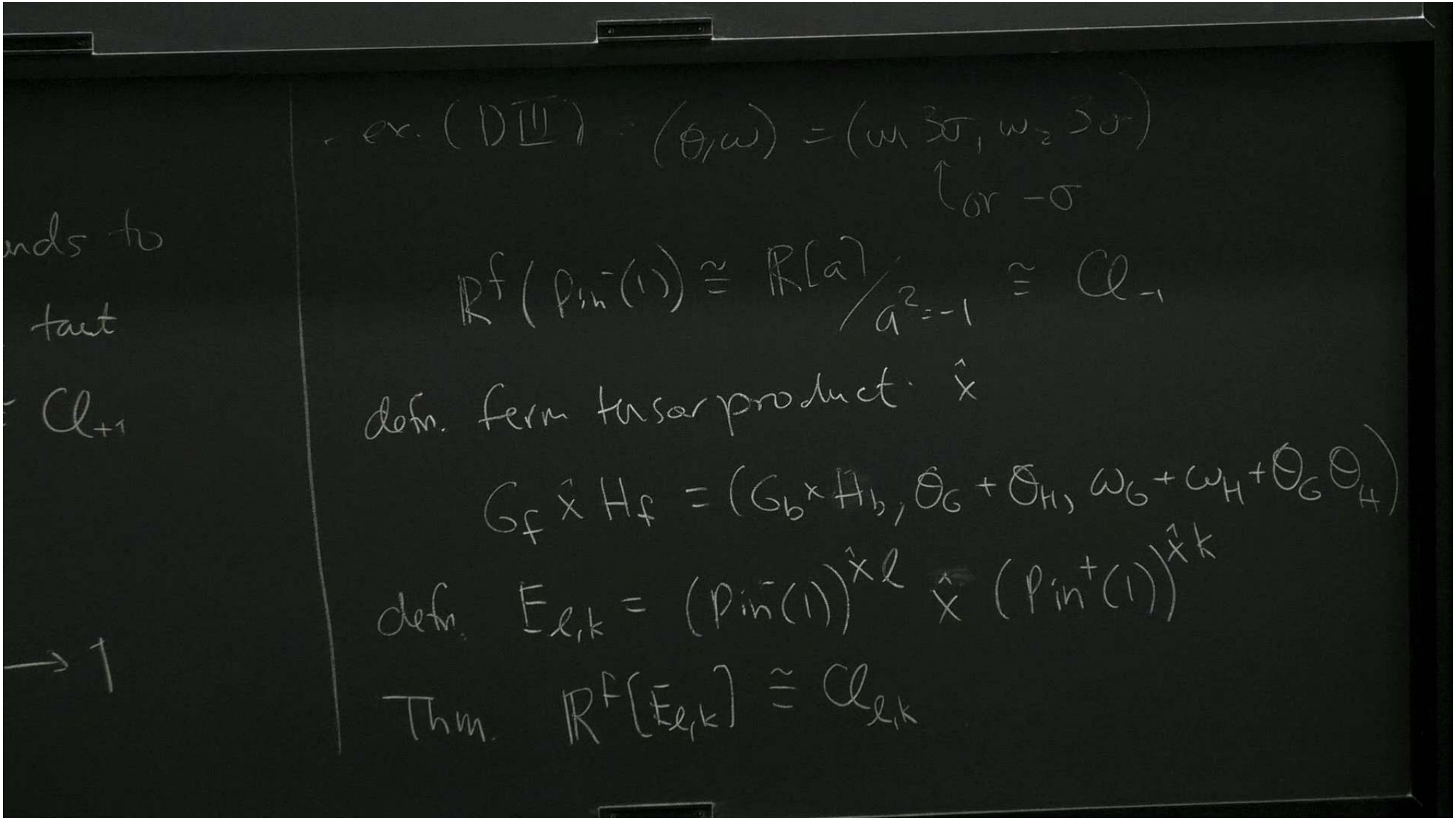
- ex. DIII, TRS, $T^2 = (-1)^F$

$G_b = \mathbb{Z}/2$, $\theta = id$,

$$1 \to \mathbb{Z}/2^F \to \mathbb{Z}/4^{T,F} \to \mathbb{Z}/2 \to 1$$

$Pin^-(1)$

- ex. (DIII)


ex. (DIII) $(\theta,\omega) = (\omega_1 \sigma_1, \omega_2 \sigma_2)$
σ or $-\sigma$
$\mathbb{R}^f(Pin^-(1)) \cong \mathbb{R}[a]/a^2=-1 \cong Cl_{-1}$
defn. fermionic tensor product $\hat{\times}$
$G_f \hat{\times} H_f = (G_b \times H_b, \theta_G + \theta_H, \omega_G + \omega_H + \theta_G \theta_H)$
defn. $E_{\ell,k} = (Pin^-(1))^{\hat{\times}\ell} \hat{\times} (Pin^+(1))^{\hat{\times}k}$
Thm. $\mathbb{R}^f[E_{\ell,k}] \cong Cl_{\ell,k}$

defn. For A a fin dim superalg.

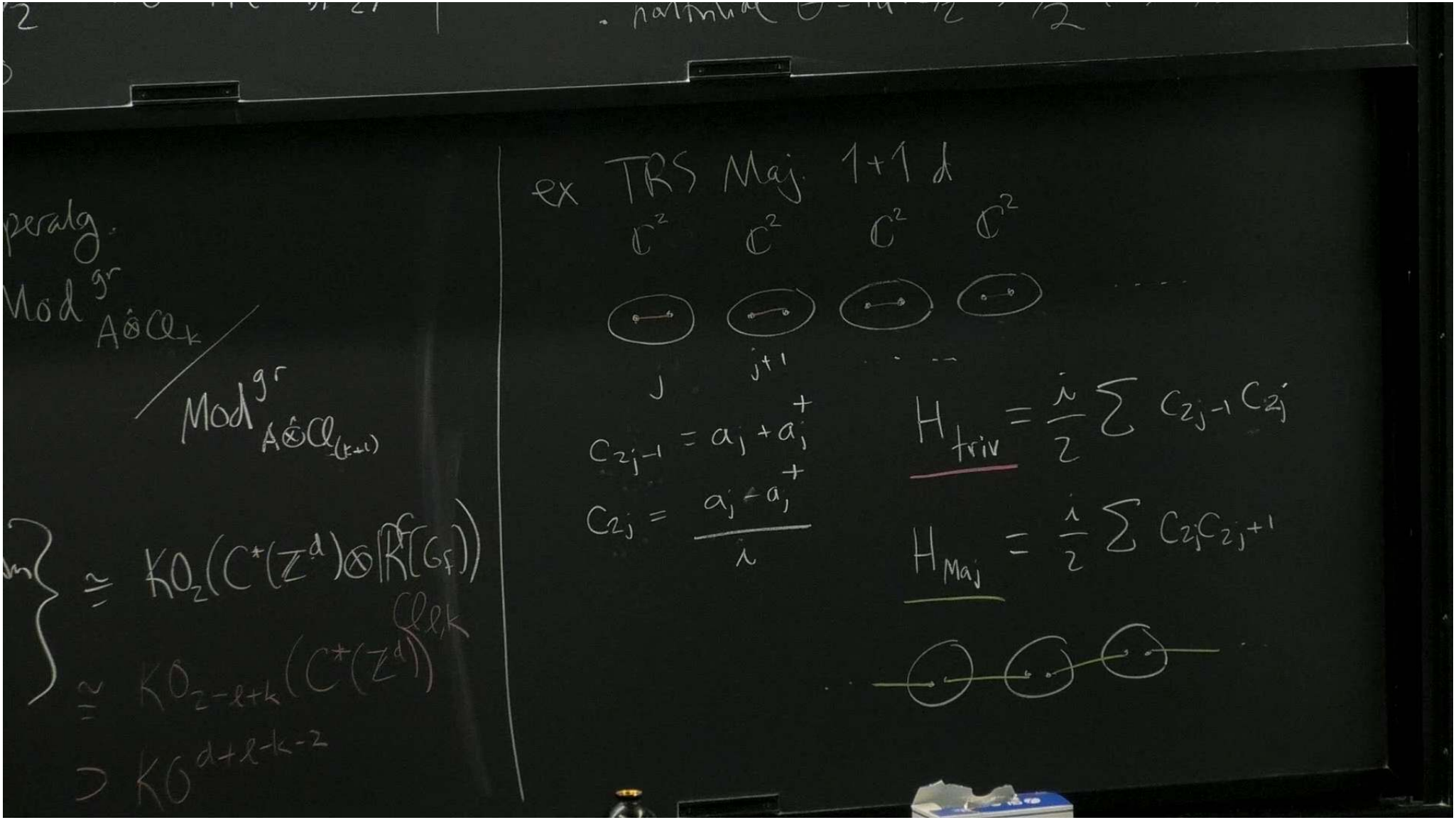
$$KO_k(A) \cong \frac{\mathrm{Mod}^{gr}_{A\hat{\otimes}Cl_k}}{\mathrm{Mod}^{gr}_{A\hat{\otimes}Cl_{(k-1)}}}$$

classification:

$$\left\{ \begin{array}{l} \text{free fermionic SPTs in } d \text{ dim} \\ \text{w/ } G_f \text{ symm} \\ E_{\ell k} \end{array} \right\} \cong KO_2(C^*(\mathbb{Z}^d) \otimes \mathbb{R}[G_f])^{Cl_{\ell k}}$$

$$\cong KO_{2-\ell+k}(C^*(\mathbb{Z}^d))$$

$$\supset KO^{d+\ell-k-2}$$


ex TRS Maj 1+1 d
$\mathbb{C}^2$ $\mathbb{C}^2$ $\mathbb{C}^2$ $\mathbb{C}^2$
j j+1
$c_{2j-1} = a_j + a_j^\dagger$
$c_{2j} = \frac{a_j - a_j^\dagger}{i}$
$H_{triv} = \frac{i}{2} \sum c_{2j-1} c_{2j}$
$H_{Maj} = \frac{i}{2} \sum c_{2j} c_{2j+1}$
$\cong KO_2(C^*(\mathbb{Z}^d) \otimes R^{Cl_{l,k}}(G_f))$
$\cong KO_{2-l+k}(C^*(\mathbb{Z}^d))$
$= KO^{d+l-k-2}$

Free-to-Interacting Maps & the Bott Spiral

today: Com

free-to-inte

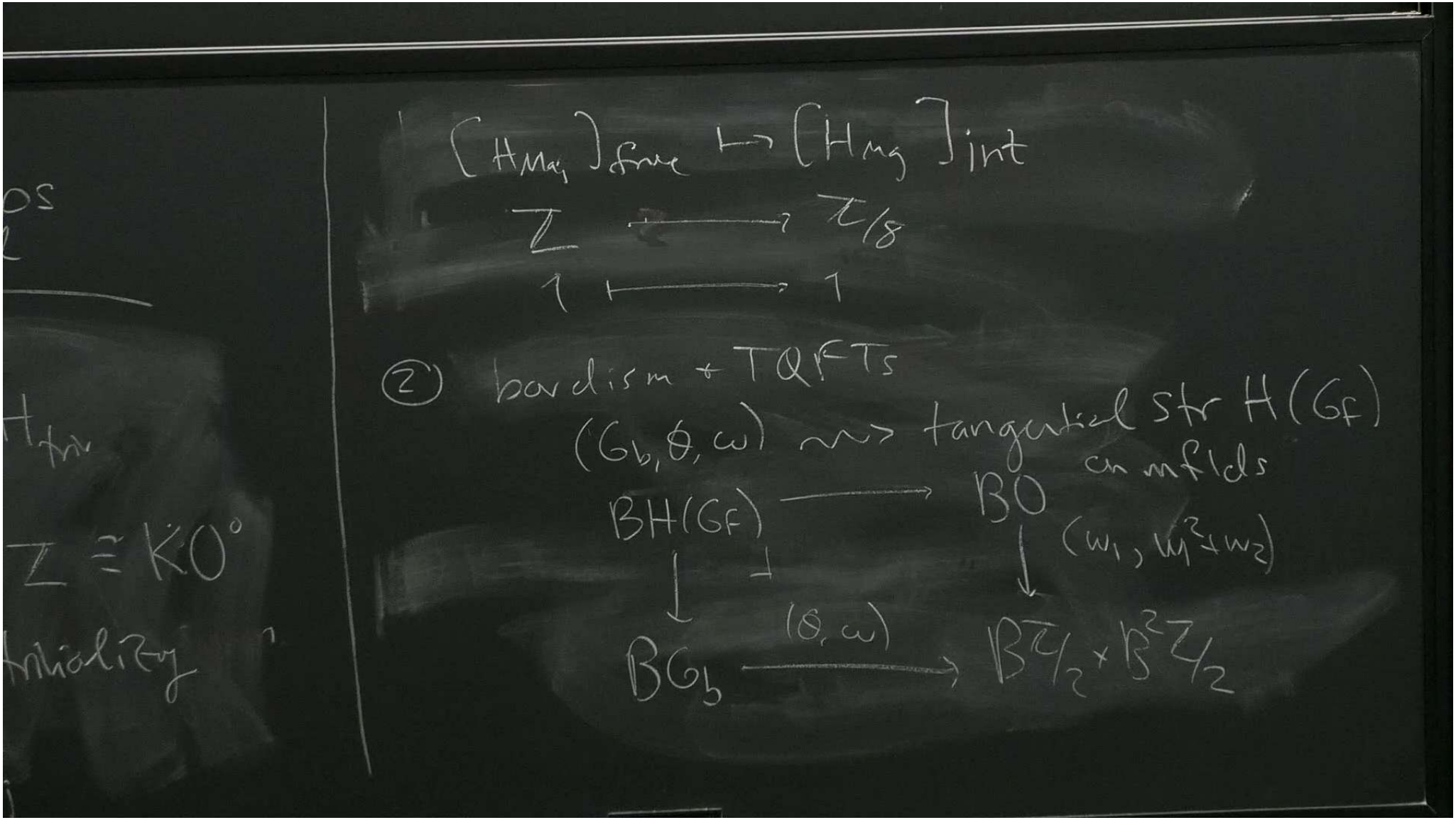
F2I

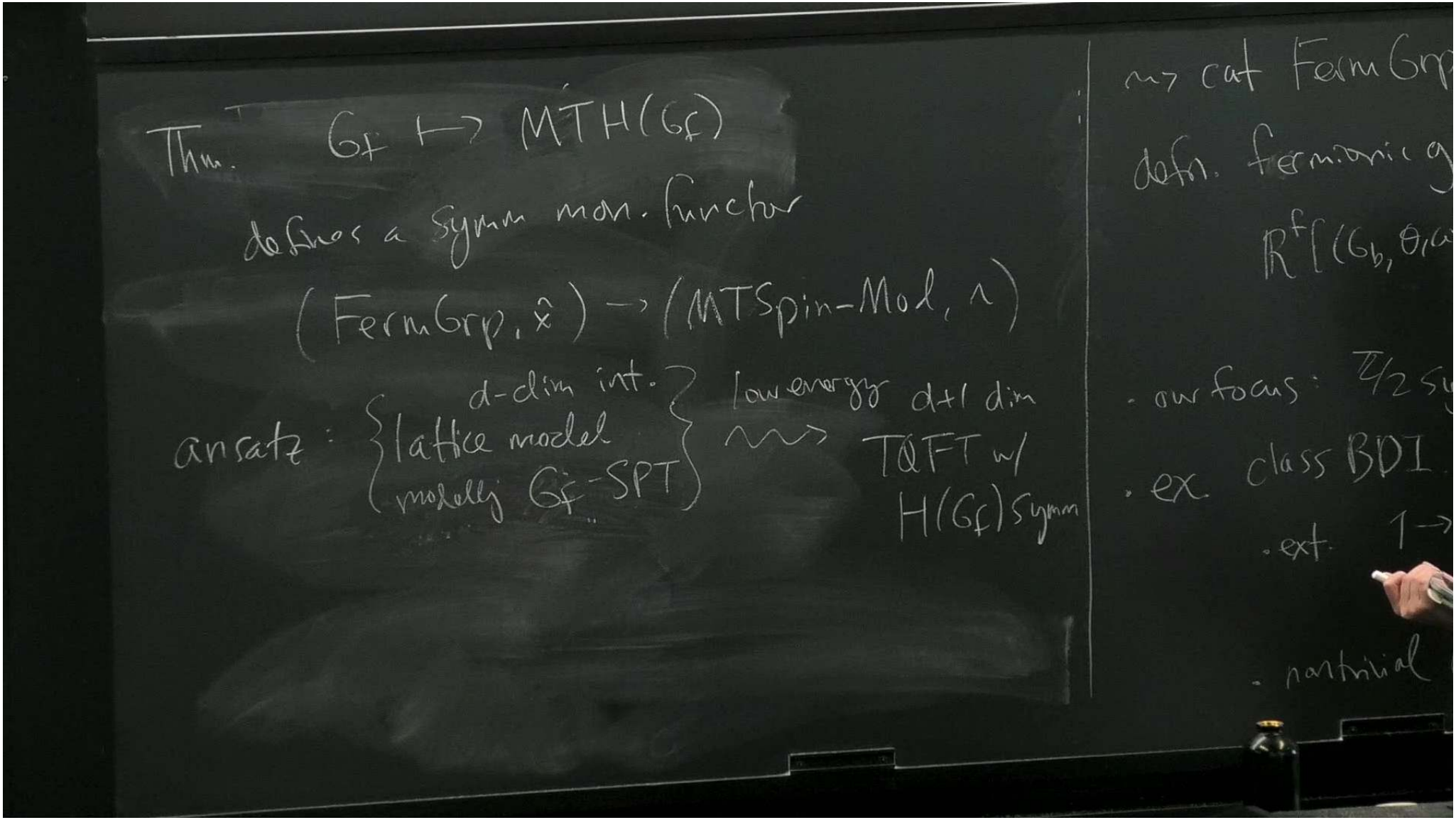
Fact: $\sum_{\alpha=1}^{n} H_{Maj}^{\alpha} \chi_{\substack{\text{free} \\ \text{TRS}}} H_{triv}$

H_{Maj} generates $Z \cong KO^0$

but $\exists$ quartic interaction trivializer

$\sum_{\alpha=1}^{8} H_{Maj}^{\alpha}$


$[H_{Mq}]_{free} \mapsto [H_{Mg}]_{int}$
$\mathbb{Z} \longrightarrow \mathbb{Z}/8$
$1 \longmapsto 1$
(2) bordism + TQFTs
$(G_b, \theta, \omega) \rightsquigarrow$ tangential str $H(G_f)$ on mflds
$BH(G_f) \longrightarrow BO$
$(w_1, w_1^2 + w_2)$
$BG_b \xrightarrow{(\theta, \omega)} B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$


Thm. $G_f \mapsto MTH(G_f)$
defines a symm mon. functor
$(FermGrp, \hat{x}) \to (MTSpin\text{-}Mod, \wedge)$
ansatz: { d-dim int. lattice model modelly G_f-SPT } low energy ⇝ d+1 dim TQFT w/ $H(G_f)$ symm
⇝ cat FermGrp
defn. fermionic
$R^f[(G_b,$
• our focus: $\mathbb{Z}/2$
• ex. class BDI
• ext. $1 \to$
• nontrivial

defn. For A a fin dim superalg.

$$KO_k(A) \cong \frac{\mathrm{Mod}^{gr}_{A\hat{\otimes}Cl_{-k}}}{\mathrm{Mod}^{gr}_{A\hat{\otimes}Cl_{-(k+1)}}}$$

classification:

$$\left\{\begin{array}{l}\text{free fermion SPTs in } d \text{ dim} \\ \text{w/ } G_f \text{ symm}\end{array}\right\} \cong KO_2(C^*(Z^d)\otimes \mathbb{R}^{Cl_{\ell,k}}(G_f))$$

$$\cong KO_{2-\ell+k}(C^*(Z^d))$$

$$E_{\ell k} \qquad \supset KO^{d+\ell-k-2}$$

ex TRS Maj.

$$c_{2j-1} = a_j + a_j^\dagger$$

$$c_{2j} = \frac{a_j - a_j^\dagger}{i}$$

$\supset KO^{d+2-k-2}$

Gf)

functor

(MTSpin-Mod, $\wedge$)

} low energy d+1 dim ⇝ TQFT w/ H(Gf) symm

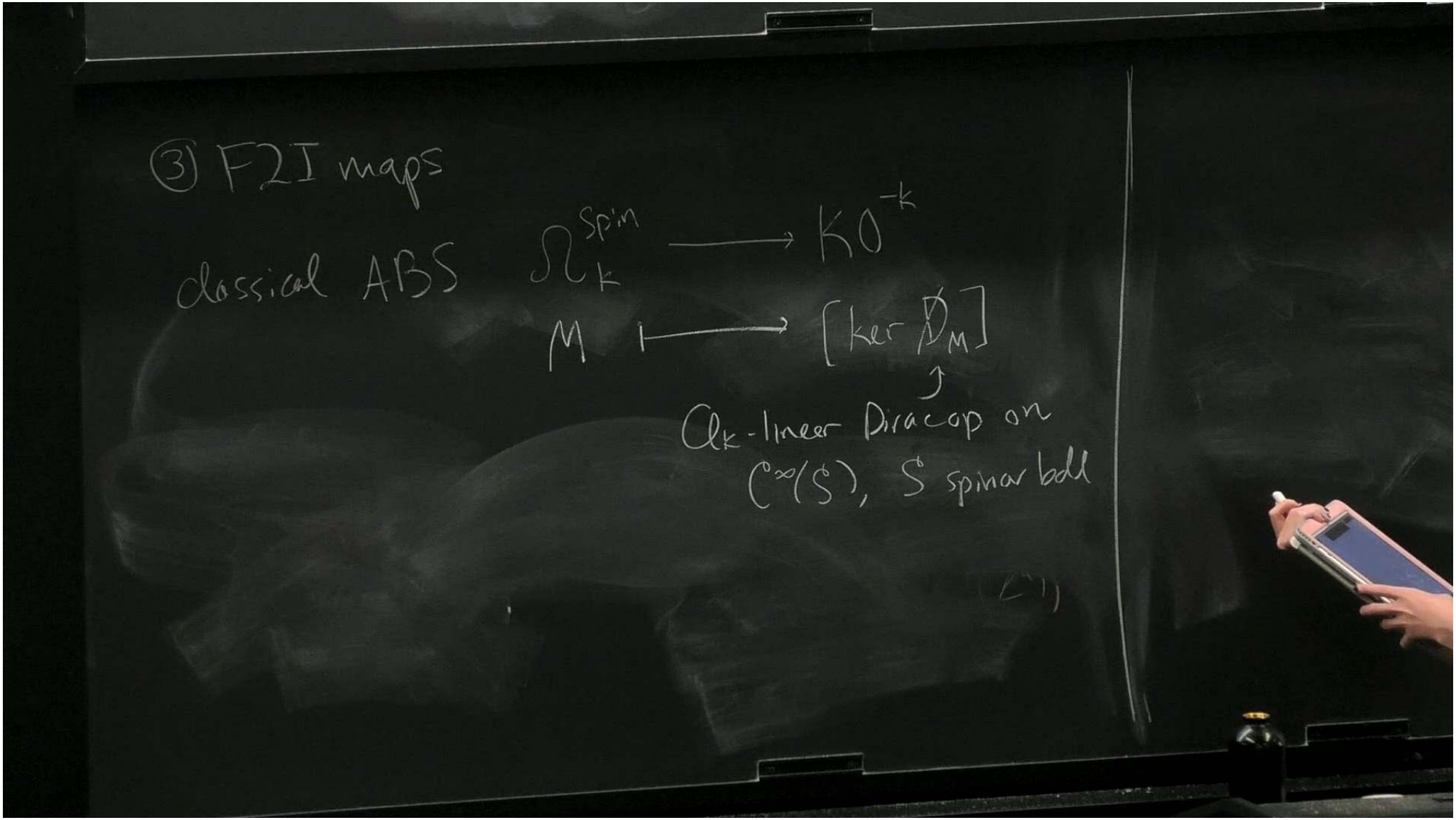
T)

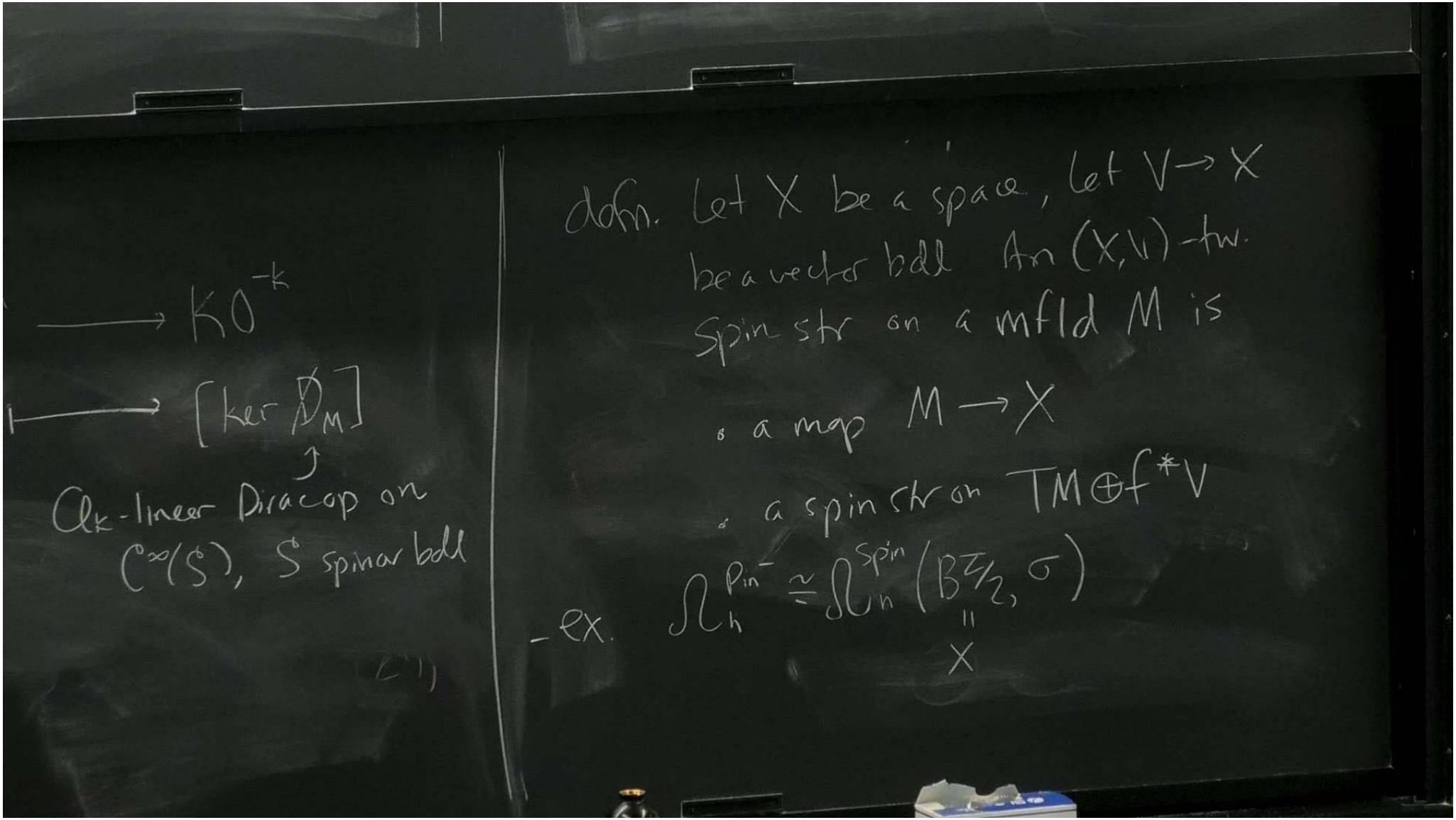
Thm (Freed-Hopkins)

{ invertible, ref. pos. fully ext. n-dim TQFTs, w/ H-sym } / defms $\cong$ Hom(Ω^H_{n+1}, $\mathbb{C}^\times$) ↔ partition fcts

defn. Ω^H_{n+1} = { closed (n+1)-mfld s w/ H-strs } / $\sim$

$M \sim N$ if $\exists$ W s.t. $\partial W = M \amalg N$


③ F2I maps
classical ABS
$\Omega_k^{Spin} \longrightarrow KO^{-k}$
$M \longmapsto [\ker \not{D}_M]$
Cl_k-linear Diracop on
$C^\infty(S)$, S spinor bdl


$\longrightarrow KO^{-k}$
$\longmapsto [\ker \not{D}_M]$
Cl_k-linear Diracop on
$(C^\infty(S), S$ spinor bdl
defn. Let X be a space, let $V \to X$ be a vector bdl $An(X,V)$-tw. Spin str on a mfld M is
• a map $M \to X$
• a spin str on $TM \oplus f^*V$
- ex. $\Omega_n^{Pin^-} \simeq \Omega_n^{Spin}(B\mathbb{Z}/2, \sigma)$
X

- ex $\left\{ \begin{array}{l} \text{int. SPTs on} \\ \text{pin}^- \text{ mflds} \end{array} \right\} \cong \mathrm{Hom}(\Omega_2^{\mathrm{Pin}^-}, \mathbb{C}^\times)$

$\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$

defn. $ME_{\ell,k} := \left((B\mathbb{Z}/2)^{\sigma-1} \right)^{\wedge \ell} \wedge \left((B\mathbb{Z}/2)^{1-\sigma} \right)^{\wedge k}$

M has $E_{\ell,k}$ str if $TM + L_1 + \cdots + L_\ell - \xi_1 - \cdots - \xi_k$ is spin

note: (ℓ, k) matters!

- ex. (DIII)

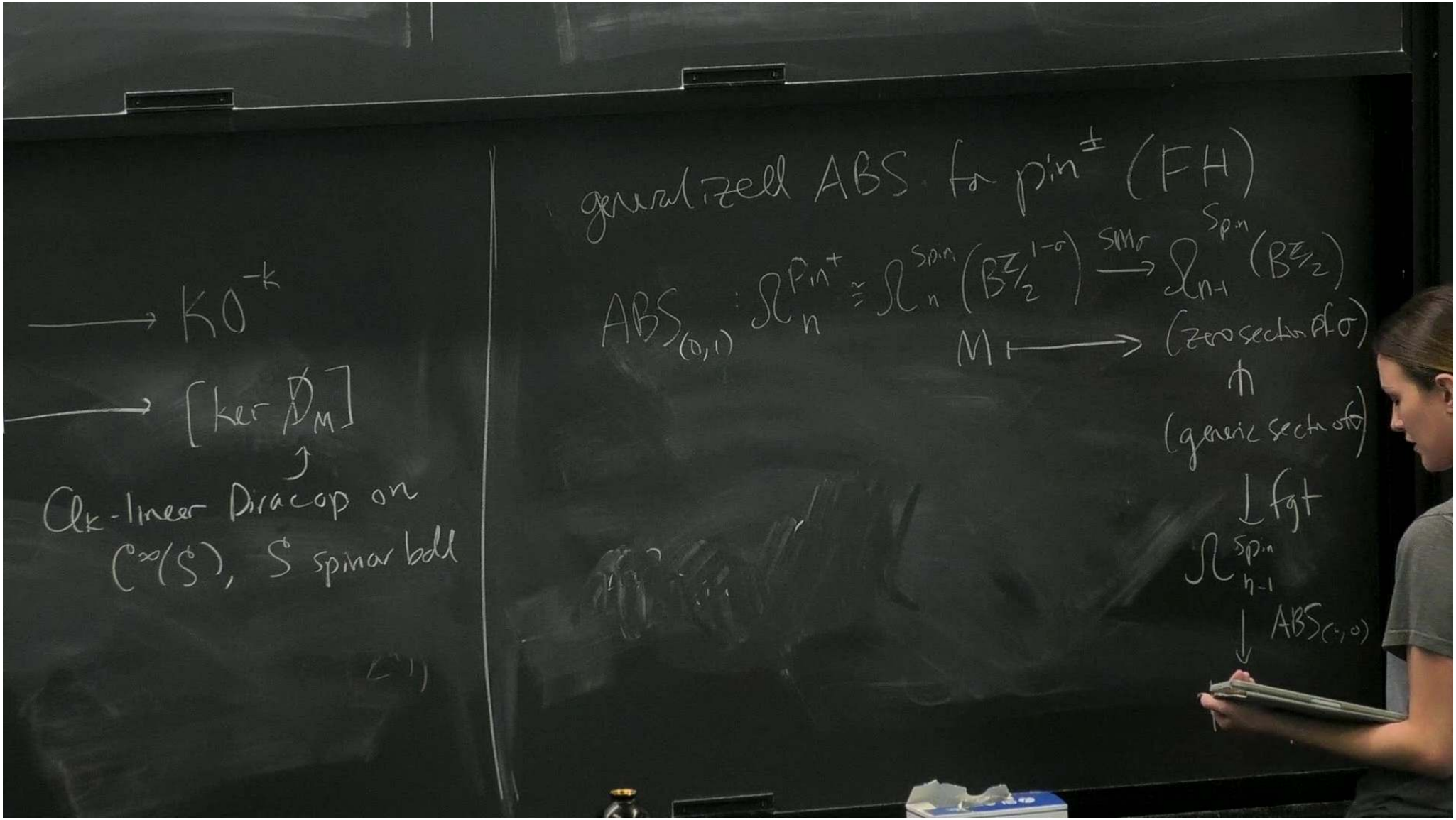
$R^f(\mathrm{Pin}^-(1)$

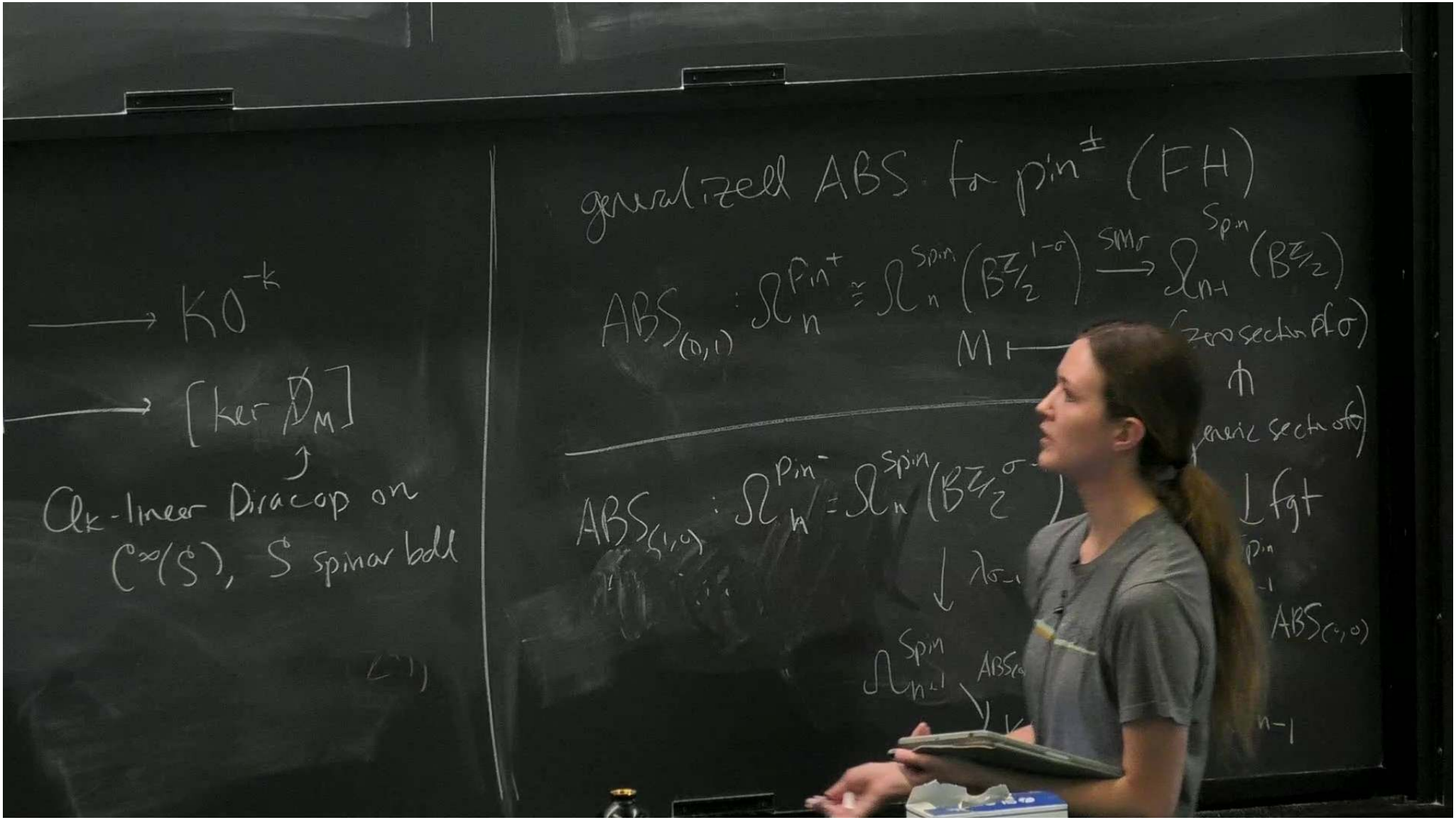
defn. ferm tns

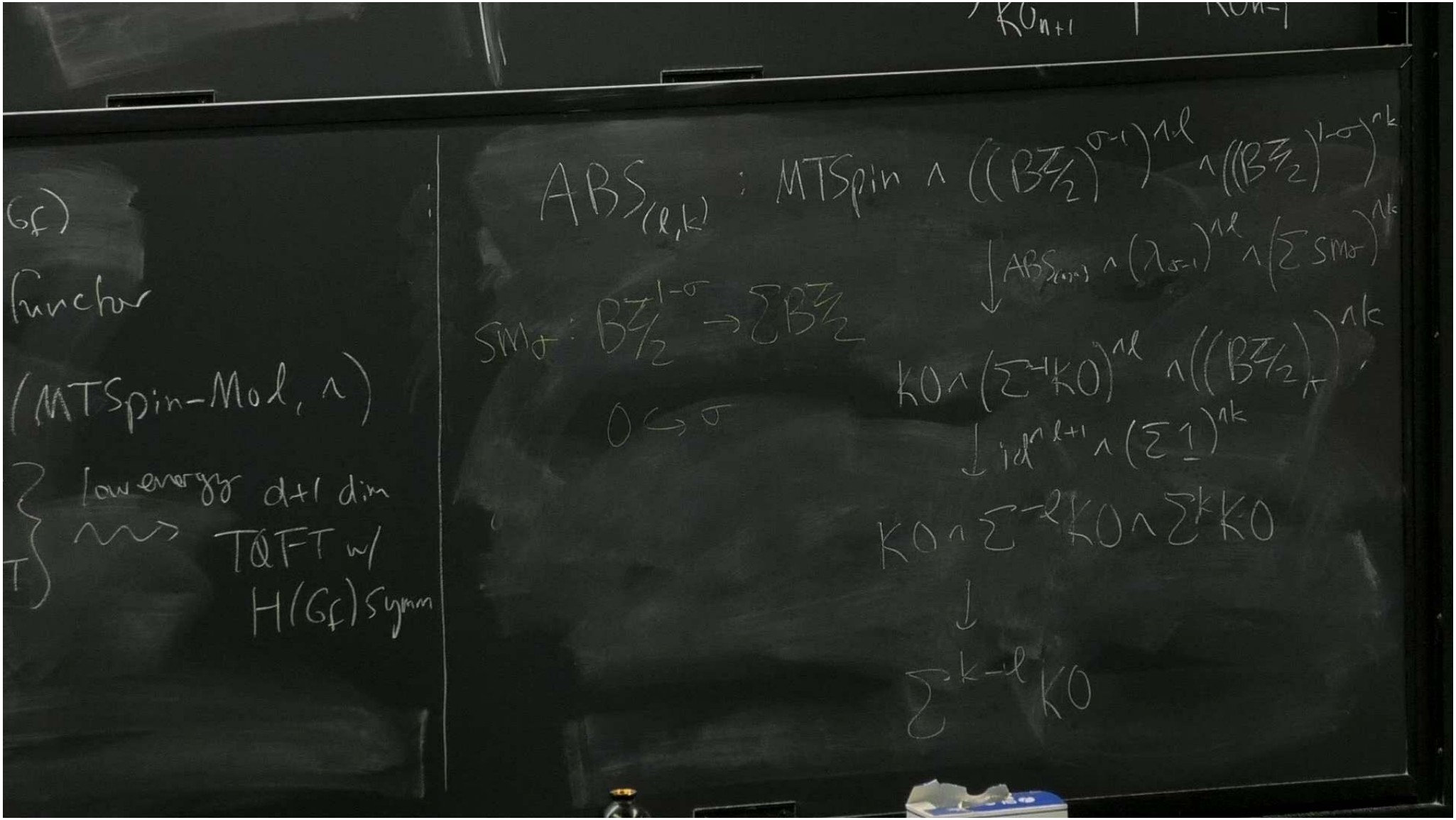
$G_f \tilde{\times} H_f$

defn. $E_{\ell,k} =$

Thm. $R^f(E$


generalized ABS for pin±
$\longrightarrow KO^{-k}$
$\longrightarrow [\ker \not{D}_M]$
Cl_k-linear Diracop on
$C^\infty(S)$, S spinor bdl
$ABS_{(0,1)}: \Omega_n^{Pin^+} \cong \Omega_n^{Spin}(B\mathbb{Z}/2^{1-\sigma}) \xrightarrow{sus\sigma} \Omega_{n+1}^{Spin}(B\mathbb{Z}/2)$
$M \longmapsto$ (zero section of σ)
(generic section of σ)
$\downarrow fgt$
Ω_{n-1}^{Spin}
$\downarrow ABS_{(-1,0)}$


generalized ABS for pin±
Clk-linear Diracop on
Clk-1


KO_{n+1}
$ABS_{(\ell,k)} : MTSpin \wedge ((B\mathbb{Z}/2)^{\sigma-1})^{\wedge \ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k}$
$SM_\sigma : B\mathbb{Z}/2^{1-\sigma} \to \Sigma B\mathbb{Z}/2$
$0 \hookrightarrow \sigma$
$\downarrow ABS_{(0,1)} \wedge (\lambda_{\sigma-1})^{\wedge \ell} \wedge (\Sigma SM_\sigma)^{\wedge k}$
$KO \wedge (\Sigma^{-1} KO)^{\wedge \ell} \wedge ((B\mathbb{Z}/2)_+)^{\wedge k}$
$\downarrow id^{\wedge \ell+1} \wedge (\Sigma 1)^{\wedge k}$
$KO \wedge \Sigma^{-2} KO \wedge \Sigma^k KO$
$\downarrow$
$\Sigma^{k-\ell} KO$
G_f)
functor
$(MTSpin\text{-}Mod, \wedge)$
T) Low energy $d+1$ dim ⇝ TQFT w/ $H(G_f)$ Symm