Title: Free-to-Interacting Maps and the Bott Spiral

Speakers: Cameron Krulewski

Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: October 15, 2024 - 11:00 AM

URL: https://pirsa.org/24100114

Abstract:

I will discuss free (i.e., noninteracting) and interacting classifications for certain fermionic symmetry-protected topological phases (SPTs) and show how to define free-to-interacting maps in terms of homotopy theory. I will apply these ideas to study the phenomenon of the "Bott spiral": as shown in work of Queiroz-Khalaf-Stern using a dimensional reduction approach, the tenfold way classification of free theories (with one additional reflection symmetry) breaks down to a large 2-torsion classification in the presence of interactions. Using K-theory and (Anderson-dual) twisted spin bordism, we can compute the same interacting classification, and with the language of fermionic groups, we can interpret the "spiral" as a failure of Morita invariance on the interacting side. Time permitting, I will also discuss how to model dimensional reduction and symmetry breaking for the Bott spiral in terms of homotopy theory.

This talk is based on upcoming work joint with Arun Debray, Natalia Pacheco-Tallaj, and Luuk Stehouwer.

Free-to-Interacting Maps & the Bott Spiral

idea:

general	SPTs
$\mathcal{M}$ moduli space of models	Hamiltonians
Δ critical locus	gapless Hams.
phase $\in (\mathcal{M} \setminus \Delta)/\sim$	$\sim$ preserves symm. + the gap

os
l

Ts

tonians

pless Hams

preserves symm
+ the gap

today: comparing $\sim_{free}$ and $\sim_{int}$

free-to-interacting map:

$$F2I: [x]_{free} \longmapsto [x]_{int}$$

$$KO^{d+l-k-2} \longrightarrow \mho^{d+2}_{E_{l,k}}$$

① K-theory. tenfold way

$$\left\{\begin{matrix}\text{free ferm SPt} \\ \text{in } d \text{ dim w/ } s \text{ symm}\end{matrix}\right\} \cong KO^{d+s-2}$$

- ex. TRS Maj. chain $(d,s)=(1,1)$
  $p+ip$ SC $(d,s)=(2,0)$

- defn. fermionic group.
  - top. group G_b
  - extension by ferm. parity $(-1)^F$

$$1 \to \mathbb{Z}/2^F \to G_f \to G_b \to 1$$

$$\rightsquigarrow \omega \in H^2(BG_b; \mathbb{Z}/2)$$

  - grading $G_f \longrightarrow \mathbb{Z}/2^T \rightsquigarrow \theta \in H^1(BG_b; \mathbb{Z}/2)$

$$(-1)^F \longmapsto 0$$

$=(1,1)$

$, 0)$

$\text{ity } (-1)^F$

$G_b \longrightarrow 1$

$\mathbb{Z}/2^T \rightsquigarrow \theta \in H^1(BG_b; \mathbb{Z}/2)$

0

$\rightsquigarrow$ cat FermGrp

defn. fermionic gp alg.

$$\mathbb{R}^f[(G_b, \theta, \omega)] := \mathbb{R}[G_f] \big/ ((-1)^F + 1)$$

- our focus: $\mathbb{Z}/2$ symms. $H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$
- ex. class BDI. $G_b = \mathbb{Z}/2$, $T^2 = 1$
  - ext. $1 \to \mathbb{Z}/2^F \to \underbrace{\mathbb{Z}/2^F \times \mathbb{Z}/2^T}_{Pin^+(1)} \to \mathbb{Z}/2 \longrightarrow 1$
  - nontrivial $\theta = id : \mathbb{Z}/2 \to \mathbb{Z}/2$

- ex $(BD\mathrm{I}$ cont'd$)$

note: $(\Theta, \omega) = (x, 1)$ corresponds to

$(w_1\sigma, w_2\sigma)$, $\sigma \to B\mathbb{Z}/2$ taut

$$\mathbb{R}^f(Pin^+(1)) \cong \mathbb{R}[a]/_{a^2=+1} \cong Cl_{+1}$$

- ex. $D\mathrm{III}$, TRS, $T^2 = (-1)^F$

$G_b = \mathbb{Z}/2$, $\Theta = id$,

$$1 \to \mathbb{Z}/2^F \to \underbrace{\mathbb{Z}/4^{T,F}}_{Pin^-(1)} \to \mathbb{Z}/2 \to 1$$

- ex. $(D\mathrm{III})$: (

...nds to

taut

$\cong Cl_{+1}$

$\longrightarrow 1$

ex. (DIII) $(\theta, \omega) = (\omega_1 \cup \sigma, \omega_2 \cup \sigma)$

or $-\sigma$

$$\mathbb{R}^f(Pin^-(1)) \cong \mathbb{R}[a]/_{a^2=-1} \cong Cl_{-1}$$

defn. ferm tensorproduct: $\hat{\times}$

$$G_f \hat{\times} H_f = (G_b \times H_b, \theta_G + \theta_H, \omega_G + \omega_H + \theta_G \theta_H)$$

defn. $E_{\ell,k} = (Pin^-(1))^{\hat{\times}\ell} \hat{\times} (Pin^+(1))^{\hat{\times}k}$

Thm. $\mathbb{R}^f[E_{\ell,k}] \cong Cl_{\ell,k}$

defn. For A a fin dim superalg.

$$KO_k(A) \cong \mathrm{Mod}^{gr}_{A\hat{\otimes}Cl_{+k}} \Big/ \mathrm{Mod}^{gr}_{A\hat{\otimes}Cl_{-(k+1)}}$$

classification:

$$\left\{ \begin{array}{l} \text{free ferm SPTs in } d \text{ dim} \\ \text{w/ } G_f \text{ symm} \\ E_{\ell,k} \end{array} \right\} \cong KO_2(C^*(\mathbb{Z}^d) \hat{\otimes} \mathbb{R}^f[G_f])$$

$$\cong KO_{2-\ell+k}(C^*(\mathbb{Z}^d))$$

$$\supset KO^{d+\ell-k-2}$$

$Cl_{\ell,k}$

· partial

peralg.

$$\mathrm{Mod}^{gr}_{A\hat{\otimes} Cl_k} \Big/ \mathrm{Mod}^{gr}_{A\hat{\otimes} Cl_{(k+1)}}$$

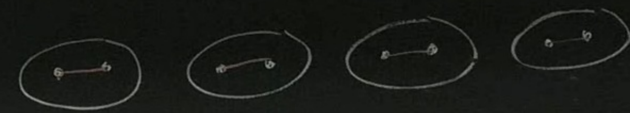
$$\Big\} \cong KO_2(C^*(\mathbb{Z}^d) \otimes \mathbb{R}^f(G_f)) \quad Cl_{\ell,k}$$

$$\cong KO_{2-\ell+k}(C^*(\mathbb{Z}^d))$$

$$\cong KO^{d+\ell-k-2}$$

ex TRS Maj. 1+1 d

$\mathbb{C}^2 \quad \mathbb{C}^2 \quad \mathbb{C}^2 \quad \mathbb{C}^2 \quad \cdots$



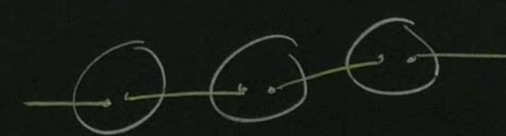
$j \qquad j+1 \qquad \cdots$

$$c_{2j-1} = a_j + a_j^+$$

$$c_{2j} = \frac{a_j - a_j^+}{i}$$

$$H_{triv} = \frac{i}{2} \sum c_{2j-1} c_{2j}$$

$$H_{Maj} = \frac{i}{2} \sum c_{2j} c_{2j+1}$$



Free-to-Interacting Maps
& the Bott Spiral

Fact: $\sum_{\alpha=1}^{n} H^{\alpha}_{Maj} \quad \chi_{\substack{free\\TRS}} \quad H_{triv}$

H_{Maj} generates $\mathbb{Z} \cong KO^0$

but $\exists$ quartic interaction trivializ[illegible]

$$\sum_{\alpha=1}^{8} H^{\alpha}_{Maj}$$

today: com[illegible]

free-to-inte[illegible]

F2I

os

l

H_{triv}

$\mathbb{Z} \cong KO^0$

trivializing

$$[H_{Maj}]_{free} \longmapsto [H_{Maj}]_{int}$$

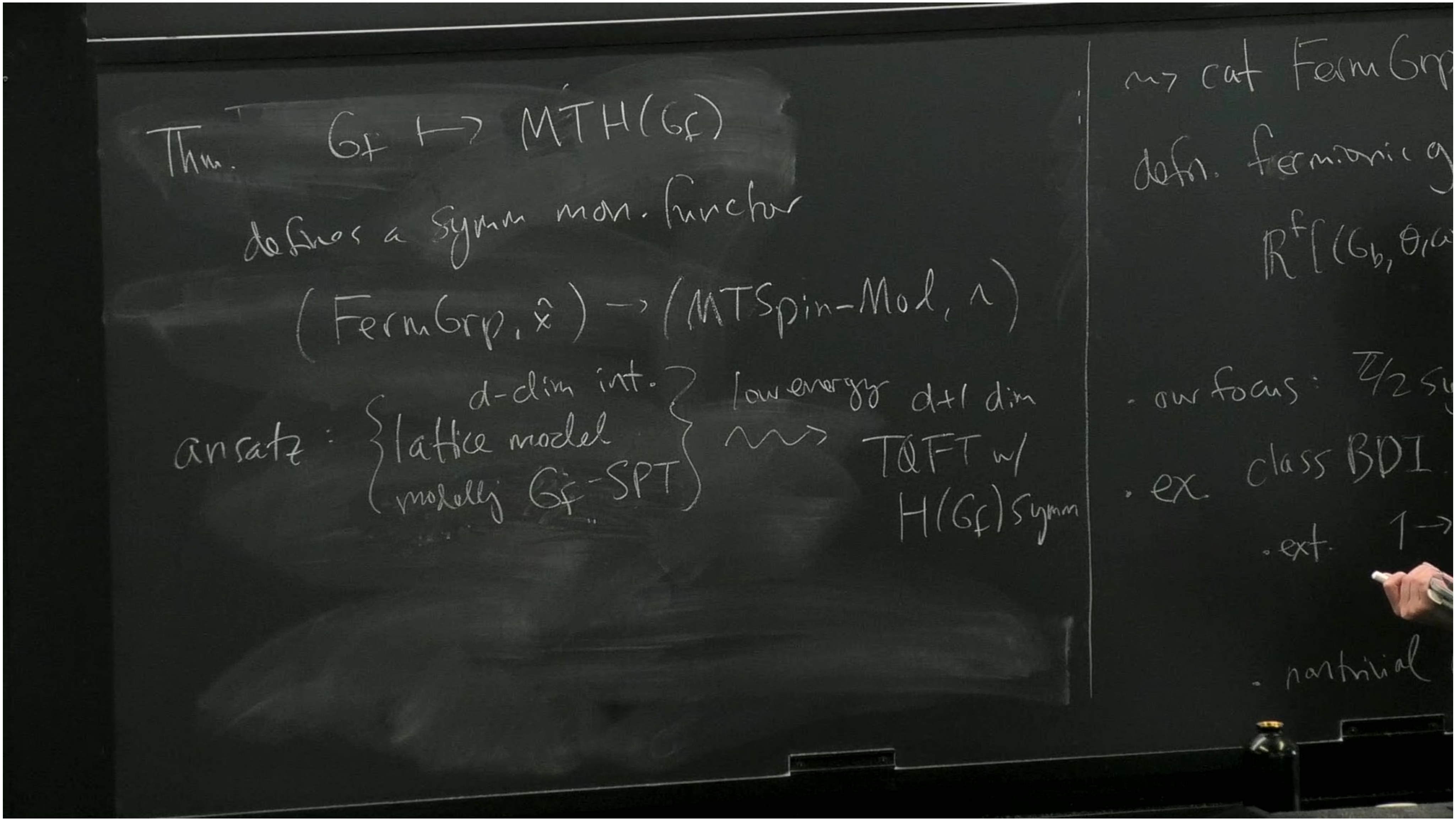
$$\mathbb{Z} \longrightarrow \mathbb{Z}/8$$

$$1 \longmapsto 1$$

② bordism + TQFTs

$(G_b, \theta, \omega) \rightsquigarrow$ tangential str $H(G_f)$ on mflds

$$\begin{tikzcd}
BH(G_f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BO \arrow[d, "{(w_1,\, w_1^2 + w_2)}"] \\
BG_b \arrow[r, "{(\theta, \omega)}"] & B\mathbb{Z}/2 \times B^2\mathbb{Z}/2
\end{tikzcd}$$


Thm. $G_f \mapsto MTH(G_f)$
defines a symm mon. functor
$(\mathrm{FermGrp}, \hat{\times}) \to (\mathrm{MTSpin\text{-}Mod}, \wedge)$
ansatz: { d-dim int. lattice model modelling G_f-SPT } $\overset{\text{low energy}}{\rightsquigarrow}$ d+1 dim TQFT w/ $H(G_f)$ symm
⇝ cat FermGrp
defn. fermionic g
$\mathbb{R}^f[(G_b, \theta, \omega$
. our focus: $\mathbb{Z}/2$ s
. ex. class BDI
. ext. $1 \to$
. nontrivial

defn. For A a fin dim superalg.

$$KO_k(A) \cong \mathrm{Mod}^{gr}_{A\hat{\otimes} C\ell_{-k}} \Big/ \mathrm{Mod}^{gr}_{A\hat{\otimes} C\ell_{-(k+1)}}$$

classification:

$$\left\{ \begin{array}{l} \text{free ferm SPTs in } d \text{ dim} \\ \text{w/ } G_F \text{ symm} \\ E_{\ell,k} \end{array} \right\} \cong KO_2(C^*(\mathbb{Z}^d) \otimes \mathbb{R}^f[G_f])$$

$$\cong KO_{2-\ell+k}(C^*(\mathbb{Z}^d))^{C\ell_{\ell,k}}$$

$$\supset KO^{d+\ell-k-2}$$

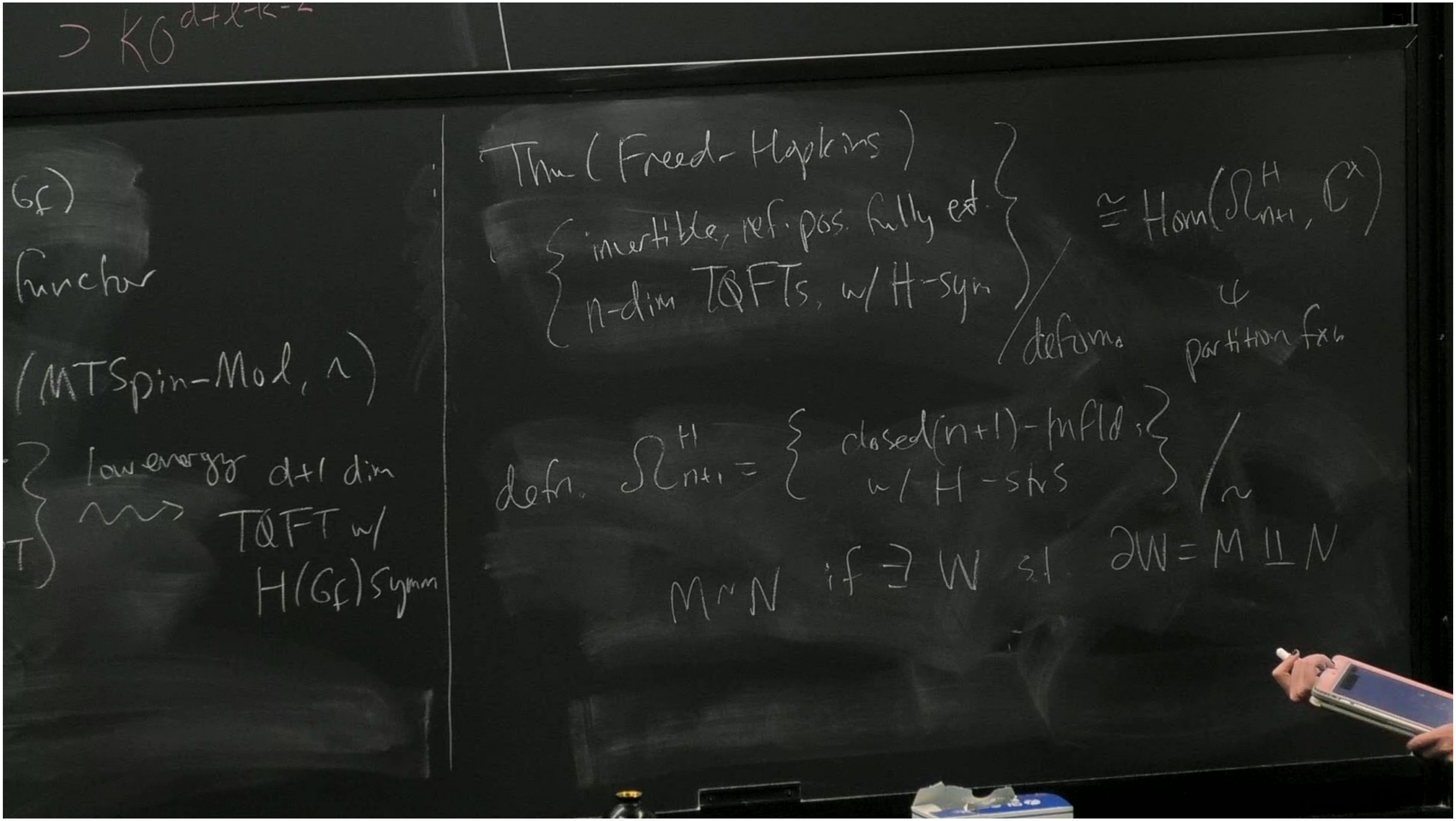
ex TRS Maj.

$\mathbb{C}^2 \quad \mathbb{C}^2$

$j \quad j+1$

$$c_{2j-1} = a_j + a_j^+$$

$$c_{2j} = \frac{a_j - a_j^+}{i}$$


⊃ KO^{d+l-k-2}
(G_f)
functor
(MTSpin-Mod, ∧)
low energy d+1 dim
⇝ TQFT w/
H(G_f) symm
Thm (Freed-Hopkins)
{invertible, ref. pos. fully ext.
n-dim TQFTs, w/ H-sym} / deform.
≅ Hom(Ω^H_{n+1}, ℂ^×)
partition fxn
defn. Ω^H_{n+1} = {closed (n+1)-mfld's w/ H-strs} / ~
M ~ N if ∃ W s.t. ∂W = M ⊔ N

③ F2I maps

classical ABS $\Omega_k^{Spin} \longrightarrow KO^{-k}$

$M \longmapsto [\ker \cancel{D}_M]$

Cl_k-linear Dirac op on $C^\infty(\$)$, $\$$ spinor bdl

$\longrightarrow KO^{-k}$

$\longmapsto [\ker \not{D}_M]$

Cl_k-linear Dirac op on $C^\infty(\$)$, $\$$ spinor bdl

defn. Let X be a space, let $V \to X$ be a vector bdl. An (X, V)-tw. Spin str on a mfld M is

- a map $M \to X$
- a spin str on $TM \oplus f^*V$

ex. $\Omega_n^{Pin^-} \cong \Omega_n^{Spin}(B\mathbb{Z}/2, \sigma)$

$\qquad\qquad\qquad\qquad\; \overset{\shortparallel}{X}$

- ex $\left\{ \begin{matrix} \text{int. SPTs on} \\ \text{pin}^- \text{ mflds} \end{matrix} \right\} \cong \mathrm{Hom}(\Omega_2^{Pin^-}, \mathbb{C}^\times)$

$$\Omega_2^{Pin^-} \cong \mathbb{Z}/8$$

defn. $ME_{\ell,k} := \left((B\mathbb{Z}/2)^{\sigma-1} \right)^{\wedge \ell} \wedge \left((B\mathbb{Z}/2)^{1-\sigma} \right)^{\wedge k}$

M has $E_{\ell,k}$ str if $TM + L_1 + \cdots + L_\ell - L_1 - \cdots - L_k$ is spin

note: (ℓ, k) matters!

- ex. (DIII) ... (

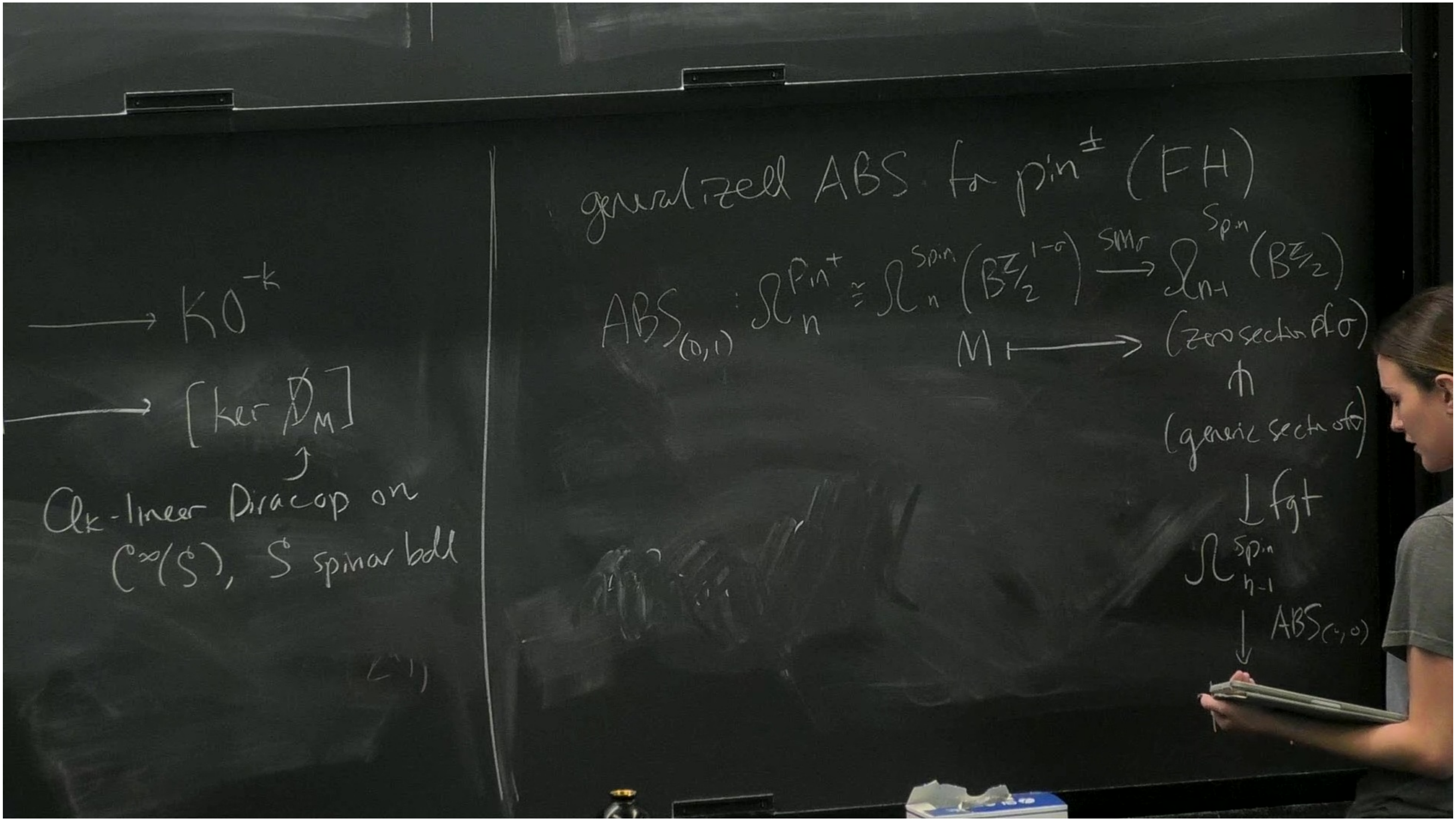
$\mathbb{R}^f(Pin^-(1)$

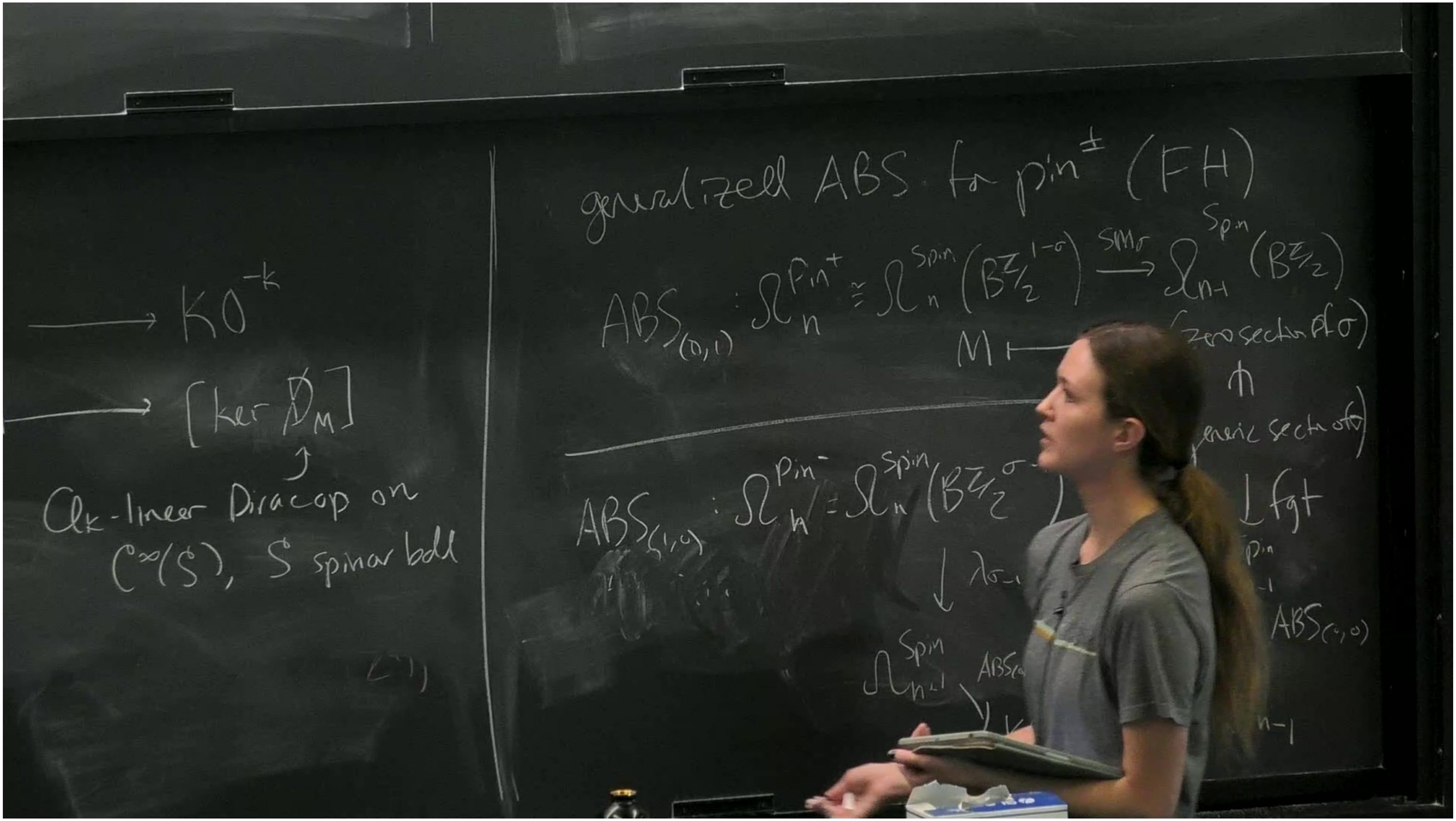
defn. ferm tws

$G_f \hat{\times} H_f$

defn. $E_{\ell,k} =$

Thm. $\mathbb{R}^f(E$


generalized ABS for pin± (FH)
ABS(0,1) : Ω_n^{Pin+} ≅ Ω_n^{Spin}(BZ/2^{1-σ}) → Ω_{n-1}^{Spin}(BZ/2)
M ↦ (zero section of σ)
(generic section of σ)
fgt
Ω_{n-1}^{Spin}
ABS(-,0)
→ KO^{-k}
→ [ker D_M]
Cl_k-linear Dirac op on
C^∞(S), S spinor bdl


generalized ABS for pin± (FH)
KO^-k
[ker D_M]
Cl_k-linear Dirac op on
C^∞(S), S spinor bdl
ABS(0,1) : Ω_n^Pin+ ≅ Ω_n^Spin(BZ/2^(1-σ)) → Ω_n-1^Spin(BZ/2)
(zero section of σ)
generic section of
ABS(1,0) : Ω_n^Pin- ≅ Ω_n^Spin(BZ/2
ABS(0,0)

KO_{n+1} | KO_{n-1}

(G_f)

functor

$(\mathrm{MTSpin\text{-}Mod}, \wedge)$

low energy $d+1$ dim $\rightsquigarrow$ TQFT w/ $H(G_f)$ symm

$$ABS_{(\ell,k)} : MTSpin \wedge ((B\mathbb{Z}/2)^{\sigma-1})^{\wedge \ell} \wedge ((B\mathbb{Z}/2)^{1-\sigma})^{\wedge k}$$

$$\Big\downarrow ABS_{(0,0)} \wedge (\lambda_{\sigma-1})^{\wedge \ell} \wedge (\Sigma\, sm_\sigma)^{\wedge k}$$

$$KO \wedge (\Sigma^{-1} KO)^{\wedge \ell} \wedge ((B\mathbb{Z}/2)_+)^{\wedge k}$$

$$\Big\downarrow id^{\wedge \ell+1} \wedge (\Sigma 1)^{\wedge k}$$

$$KO \wedge \Sigma^{-\ell} KO \wedge \Sigma^k KO$$

$$\Big\downarrow$$

$$\Sigma^{k-\ell} KO$$

$sm_\sigma : B\mathbb{Z}/2^{1-\sigma} \to \Sigma B\mathbb{Z}/2$

$0 \hookrightarrow \sigma$