Title: Large Scale White Noise and Cosmology

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Abstract:

The generation of large scales white noise is a generic property of the dynamics of physical systems described by local non-linear partial differential equations. Non-linearities prevent the small scale dynamics to be erased by smoothing. Unresolved small scale dynamics act as an uncorrelated (white or Poissonian) noise (seemingly stochastic but actually deterministic) contribution to large scale dynamics. Such is the case for cosmic inhomogeneities. In the standard model of cosmology the primordial density power spectrum is taken to be sub-Poissonian and subsequent non-linear evolutions will inevitably produce white noise which will dominate on the largest scales. Non-observation of white noise on the Hubble scale precludes a power law extrapolation of the power spectrum below one comoving parsec and places severe constraints on a wide variety of phenomena in the early universe, including phase transitions, vorticity and gravitational radiation.

Large Scale White Noise and Cosmology

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LSWN in Four Acts

ACT 1 An understanding of metric gravity

does space-time move?

ACT 2 Large scale white noise

an inconvenient truth

ACT 3 Large scale white noise in cosmology

sh— happens!

ACT 4 Parametric cosmology and the early universe

white noise: it's not just a good idea, it's the law!

ENCORE Other aspect of cosmic large scale white noise

gravity waves and vorticity

Does Space-Time Move?

Common phraseology

"*expansion of the universe*"

"*space is expanding*"

What does that mean?

Isn't this a kind of esotericism?

Doesn't this preclude the more accessible "local Newtonian" description of cosmology!

Wouldn't it be better to say

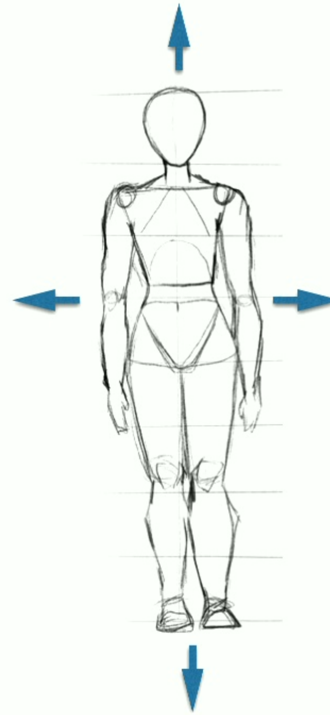
"the matter in the universe is expanding"

"*things separated by large distances are moving apart*"

For me the latter is the preferred description!

BUT

In general relativity (GR) there is a trivial way to formalize the motion of space-time!



pessimist:
am I getting fatter?

optimist:
am I getting taller?

Matter = Curvature

It is useful to interpret the equality in Einstein's equation for the Einstein curvature tensor

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$$

using

> **equal** (adjective) identical in … logical denotation: EQUIVALENT.

i.e. Ricci curvature is a property of the matter

analogy:

matter has charge which interacts with electromagnetic fields

matter has Ricci curvature which interacts with Weyl curvature

Reminder: In pseudo-Riemannian the (Riemann) curvature tensor $R_{\alpha\beta\gamma\delta}$ is the sum of contribution of the Ricci $R_{\alpha\beta}$ and Weyl $C_{\alpha\beta\gamma\delta}$ curvature:

$$R_{\alpha\beta\gamma\delta} = -\frac{R\,(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})}{(d-1)(d-2)} - \frac{R_{\alpha\delta}g_{\beta\gamma} - R_{\alpha\gamma}g_{\beta\delta} + R_{\beta\gamma}g_{\alpha\delta} - R_{\beta\delta}g_{\alpha\gamma}}{d-2} + C_{\alpha\beta\gamma\delta}$$

where

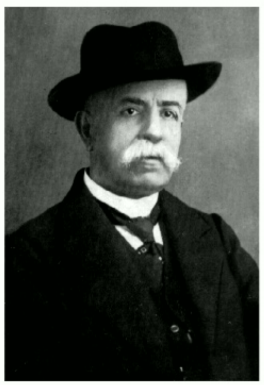
- d is the number of dimensions (space+1 time)
- $R \equiv R^{\alpha}{}_{\alpha}$ is the Ricci scalar

Aphorisms

matter tells space-time how to curve and curved space-time tells matter how to move

- Wheeler

but rather



matter has Ricci

and

Ricci tells Weyl how to curve

and

Ricci and Weyl tell Ricci how to move

Curvature is Measurable

Don't get obsessed with "gauge / coordinate invariance"

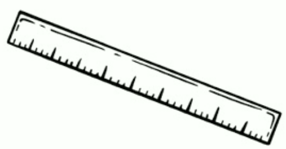
Restrict to measurable (local covariant) quantities and comparisons between them

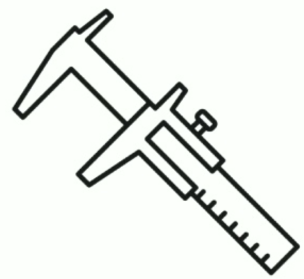
e.g. not $\delta\rho = \rho - \rho_{\text{bkg}}$ but rather ρ vs θ (expansion)

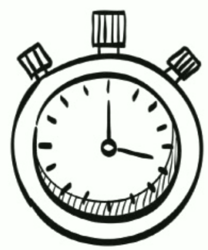
The curvature, both Ricci and Weyl, are measurable* local covariant quantities

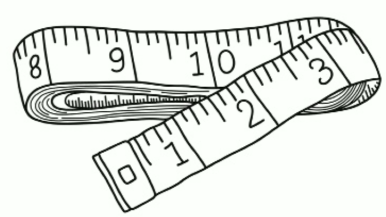
In what follows formulae involve local covariant quantities.

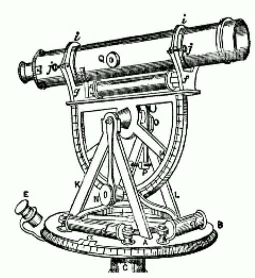
*measurable in principle though with difficulty











New Lexicon

With this equivalence one can ascribe to space-time some of the properties of (fluid) matter:

Space-time can **have**

- velocity
- density
- pressure
- anisotropic strain/pressure

Space-time can **be**

- barotropic
- irrotational
- perfect
- …….

It can be useful to not refer to the matter when discussing space-time dynamics (geometrodynamics).

Not referring to matter generalizes this formalism to any metric theory of gravity.

Space-Time Moves with Center of Momentum†

The center-of-momentum (CM) 4-velocity of matter, $\bar{u}^\alpha$, can be interpreted locally as the space-time velocity or globally as the space time flow.

It is defined from the space-time geometry as the* time-like eigenvector of the Einstein tensor

$$G^\alpha{}_\beta \, \bar{u}^\beta = -\, 8\pi G \, \bar{\rho} \bar{u}^\alpha$$

Here $\bar{\rho}$ is the matter density in the CM frame which one can ascribe to be the space-time density.

More generally the Einstein curvature tensor can be decomposed

$$G^{\alpha\beta} = 8\pi G \left((\bar{\rho} + \bar{p})\, \bar{u}^\alpha \, \bar{u}^\beta + \bar{p}\, g^{\alpha\beta} + \bar{q}^{\alpha\beta} \right)$$

$$u^\alpha \, u_\alpha = -\, 1 \quad q^{\alpha\beta} = q^{\beta\alpha} \quad q^\alpha{}_\alpha = 0 \quad u^\alpha q_{\alpha\beta} = 0$$

where $\bar{p}$ and $\bar{q}^{\alpha\beta}$ are the space-time pressure and anisotropic stress.

* uniqueness of $\bar{u}^\alpha$ requires $G^\alpha{}_\beta$ to not have a null eigenvector. This requirement is not satisfied in empty space, deSitter space and some other idealized space-times.
* one must also choose $\bar{u}^\alpha$ to be future directed and properly normalized.

† valid wherever matter is found™

CLAIMER: exact

Space-Time Flow†

Space-time streamlines are integral curves of $\bar{u}^\alpha$.

Since $\bar{u}^\alpha$ is a single-valued vector field, streamlines do not intersect and cross at caustics as generically occur for geodesic flow.

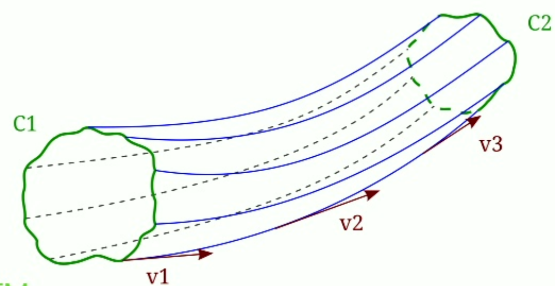
Without imposing any time-slicing one can define a universal temporal derivative of any quantity by the the convective derivative along the stream-line

$$\dot{s} \equiv u^\alpha s_{;\alpha} \qquad \dot{T}^{\alpha\cdots} \equiv u^\beta T^{\alpha\cdots}{}_{;\beta}$$

This is the derivative wrt proper time along the the streamline.

One can alway choose coordinates where this proper time is the coordinate time.

In the moving space-time picture spatial volume elements move according to $\bar{u}^\alpha$ and is distorted (compressed, stretched, rotated) according the velocity gradient, $\bar{u}_{\alpha;\beta}$.



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CLAIMER: exact

Space-Time Velocity Gradients†

Using the canonical decomposition of 4-velocity gradients,

$$\bar{u}_{\alpha;\beta} = \frac{1}{3}\bar{\mathscr{P}}_{\alpha\beta}\bar{\theta} + \bar{\sigma}_{\alpha\beta} + \bar{\omega}_{\alpha\beta} - \dot{\bar{u}}_{\alpha}\bar{u}_{\beta}\,,$$

one defines of 5 purely *spatial* quantities: 1 scalar, 1 vectors and 3 tensors:

$\bar{\mathscr{P}}_{\alpha\beta} \equiv \bar{u}_{\alpha}\bar{u}_{\beta} + g_{\alpha\beta}$ — spatial projection tensor

$\bar{\theta} \equiv \bar{u}^{\alpha}{}_{;\alpha}$ — rate of expansion scalar

$\bar{\sigma}_{\alpha\beta} \equiv \bar{\mathscr{P}}_{\alpha}{}^{\gamma}\left(\bar{u}_{\gamma;\delta} + \bar{u}_{\delta;\gamma} - \frac{1}{3}g_{\gamma\delta}\bar{u}^{\epsilon}{}_{\epsilon}\right)\bar{\mathscr{P}}^{\delta}{}_{\beta}$ — rate of shear tensor

$\bar{\omega}_{\alpha\beta} \equiv \bar{\mathscr{P}}_{\alpha}{}^{\gamma}\left(\bar{u}_{\gamma;\delta} - \bar{u}_{\delta;\gamma}\right)\bar{\mathscr{P}}^{\delta}{}_{\beta}$ — rate of rotation tensor

$\dot{\bar{u}}_{\alpha} \equiv \bar{u}^{\beta}\bar{u}^{\alpha}{}_{;\beta}$ — proper acceleration vector

Define the squared magnitudes of shear, rotation, acceleration: $\bar{\sigma}^2 \equiv \dfrac{\bar{\sigma}^{\alpha\beta}\bar{\sigma}_{\alpha\beta}}{2}$, $\bar{\omega}^2 \equiv \dfrac{\bar{\omega}^{\alpha\beta}\bar{\omega}_{\alpha\beta}}{2}$, $\bar{\alpha}^2 \equiv \dot{\bar{u}}^{\alpha}\dot{\bar{u}}_{\alpha}$.

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Using the canonical decomposition of 4-velocity gradients,

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one defines of 5 purely *spatial* quantities: 1 scalar, 1 vectors and 3 tensors:

$\bar{\mathscr{P}}_{\alpha\beta} \equiv \bar{u}_{\alpha}\bar{u}_{\beta} + g_{\alpha\beta}$	spatial projection tensor	$\bar{\mathscr{P}}_{ab} \equiv \gamma_{\alpha\beta}$
$\bar{\theta} \equiv \bar{u}^{\alpha}{}_{;\alpha}$	rate of expansion scalar	$\bar{\theta} \equiv \bar{v}^{a}{}_{:a}$
$\bar{\sigma}_{\alpha\beta} \equiv \bar{\mathscr{P}}_{\alpha}{}^{\gamma}\left(\bar{u}_{\gamma;\delta} + \bar{u}_{\delta;\gamma} - \frac{1}{3}g_{\gamma\delta}\bar{u}^{\epsilon}{}_{\epsilon}\right)\bar{\mathscr{P}}^{\delta}{}_{\beta}$	rate of shear tensor	$\bar{\sigma}_{ab} \equiv \bar{v}_{a:b} + \bar{v}_{b:a} - \frac{1}{3}\gamma_{ab}\bar{v}^{c}{}_{:c}$
$\bar{\omega}_{\alpha\beta} \equiv \bar{\mathscr{P}}_{\alpha}{}^{\gamma}\left(\bar{u}_{\gamma;\delta} - \bar{u}_{\delta;\gamma}\right)\bar{\mathscr{P}}^{\delta}{}_{\beta}$	rate of rotation tensor	$\bar{\omega}_{ab} \equiv \bar{v}_{a:b} - \bar{v}_{b;a}$
$\dot{\bar{u}}_{\alpha} \equiv \bar{u}^{\beta}\bar{u}^{\alpha}{}_{;\beta}$	proper acceleration vector	$\dot{\bar{v}}^{a} \equiv \frac{\partial}{\partial t}v^{a} + \bar{v}^{b}\bar{v}^{a}{}_{;b}$

Define the squared magnitudes of shear, rotation, acceleration: $\bar{\sigma}^2 \equiv \frac{\bar{\sigma}^{\alpha\beta}\bar{\sigma}_{\alpha\beta}}{2}$, $\bar{\omega}^2 \equiv \frac{\bar{\omega}^{\alpha\beta}\bar{\omega}_{\alpha\beta}}{2}$, $\bar{\alpha}^2 \equiv \dot{\bar{u}}^{\alpha}\dot{\bar{u}}_{\alpha}$.

The velocity gradient decomposition is nearly the same in the Galilean context of Newtonian physics.

† valid wherever matter is found™

CLAIMER: exact

Space-Time Fluid Dynamics†

Evolution of the space-time fluid is given by local energy-momentum conservation, $T^{\alpha\beta}{}_{;\beta} = 0$, or explicitly

$$\dot{\bar{\rho}} = -(\bar{\rho} + \bar{p}/c^2)\,\bar{\theta}$$ continuity equation $$\dot{\bar{\rho}} = -\bar{\rho}\,\bar{\theta}$$

$$\dot{\bar{u}}^\alpha = -\frac{\mathscr{P}^{\alpha\beta}\,\bar{p}_{;\beta} + \bar{q}^{\alpha\beta}{}_{;\beta}}{\bar{\rho} + \bar{p}/c^2}$$ Navier-Stokes equation $$\dot{\bar{u}}^a = -\frac{\gamma^{ab}\,\bar{p}_{:b} + \bar{q}^{ab}{}_{:b}}{\bar{\rho}}$$

These equations are nearly the same in a GR and Newtonian context.

Unlike in Newtonian physics, in GR the evolution of shear ($\bar{\sigma}_{\alpha\beta}$) and rotation ($\bar{\omega}_{\alpha\beta}$) is not given solely by the evolution of the velocity ($\bar{u}^\alpha$) - so these are not a closed system of equations.

† valid wherever matter is found™

CLAIMER: exact

Generalized Friedman Equations†

In an isotropically expanding universe $\bar{\theta}$ is three times the Hubble expansion rate. Thus suggests defining a local scale factor, $\bar{a}$, satisfying

$$\frac{\dot{\bar{a}}}{\bar{a}} = \frac{1}{3}\bar{\theta}$$

for each streamline.

The continuity and Raychaudhuri equation ($\dot{\bar{\theta}} = -\bar{\theta}^2/3 - 2\,\bar{\sigma}^2 + 2\,\bar{\omega}^2 + \dot{\bar{u}}^{\alpha}{}_{;\alpha} - \bar{u}^{\alpha}\bar{u}^{\beta}R_{\alpha\beta}$) or $\bar{u}^{\alpha}$ become

$$\dot{\bar{\rho}} + 3\,\frac{\dot{\bar{a}}}{\bar{a}}\,(\bar{\rho} + \bar{p}) = \bar{q}^{\alpha\beta}\,\bar{\sigma}_{\alpha\beta}$$

$$\frac{\ddot{\bar{a}}}{\bar{a}} + \frac{4\pi G}{3}\,(\bar{\rho} + 3\,\bar{p}) = -\,\frac{1}{3}\,(\bar{\sigma}^2 - \bar{\omega}^2 - \dot{\bar{u}}^{\alpha}{}_{;\alpha})$$

These are the generalized Friedman equations.

Isotropy of an FLRW space-time guarantees $\bar{\sigma}_{\alpha\beta} = \bar{\omega}_{\alpha\beta} = \dot{\bar{u}}_{\alpha} = 0$ so one recovers the usual cosmological equations.

† valid wherever matter is found™

CLAIMER: exact

Kurvature†

Just as in FLRW spacetimes one can combine the generalized Friedman equations to eliminate $\bar{p}$ by defining the "kurvature"

$$\bar{K} \equiv \frac{8\pi G}{3}\,\bar{\rho} - \frac{1}{9}\,\bar{\theta}^2 = \frac{8\pi G}{3}\,\bar{\rho} - \frac{\dot{\bar{a}}^2}{\bar{a}^2}$$

which evolves according to

$$\dot{\bar{K}} + \frac{2}{3}\,\bar{\theta}\,\bar{K} = \frac{2}{9}\,\bar{\theta}\,\left(\bar{\sigma}^2 - \bar{\omega}^2 - \dot{\bar{u}}^{\alpha}{}_{;\alpha}\right) - \frac{8\pi\,G}{3\,c^2}\,\bar{\sigma}_{\alpha\beta}\,\bar{q}^{\alpha\beta}$$

which is an additional form of the generalized Friedman equations.

In FLRW space-times $\dot{\bar{K}} = -\,2\,\dot{\bar{a}}/\bar{a}$ so $\bar{K} = k/\bar{a}^2$ where k is the curvature constant.

The corresponding equation in Newtonian fluid mechanics is

$$\dot{\bar{K}} + \frac{2}{3}\,\bar{\theta}\,\bar{K} = \frac{2}{9}\,\bar{\theta}\,\left(\bar{\sigma}^2 - \bar{\omega}^2 - \dot{\bar{v}}^{a}{}_{:a}\right)$$

which is valid even ignoring gravity: $G = 0$.

† valid wherever matter is found™

CLAIMER: exact

Kurvature and Curvature

In perturbation theory it is common to represent the "scalar" metric as

$$d\tau^2 = -\,ds^2 = a[\eta]^2\Big(\big(1 + 2\,\Psi[\eta,\vec{x}]\big)\;d\eta^2 - \big(1 - 2\,\Phi[\eta,\vec{x}]\big)\Big)\;d\vec{x}\cdot d\vec{x}$$

In terms of the Bardeen potentials Ψ and Φ.

Primordial inhomogeneities in our own universe are described by the quantity often called curvature which is

$$\mathscr{R} \equiv \Psi + \frac{2}{3}\,\frac{\Phi + \frac{\frac{\partial\Psi}{\partial\eta}}{\frac{\partial \ln a}{\partial\eta}}}{1 + \frac{p}{\rho}}$$

useful when Ψ and Φ are small. A simpler expression for $\mathscr{R}$ in linear theory is given in terms of $\bar{K}$

$$\bar{K} \approx \frac{2}{3}\frac{1}{a^2}\frac{\partial}{\partial\vec{x}}\cdot\frac{\partial}{\partial\vec{x}}\mathscr{R}$$

Since in linear theory $\sigma^2 \approx \omega^2 \approx \bar{\sigma}_{\alpha\beta}\,\bar{q}^{\alpha\beta} \approx 0$, so on *large scales* where pressure gradients are unimportant ($\dot{\bar{u}}^{\alpha}{}_{;\alpha} \approx 0$) $\bar{K} \sim 1/a^2$ so $\mathscr{R} \sim \text{constant}$. This makes $\mathscr{R}$ and hence $\bar{K}$ a useful relic from the early universe.

Act 2

large scale white noise

an inconvenient truth

when linear theory fails in the linear regime

- **CLAIM:** Large physical systems whose dynamics obey **homogeneous** **non-conservative** **non-linear** partial differential equations cannot maintain a spatial power spectrum which is both **broadband** and **sub-Poissonian** (goes to zero at small wavenumber)

  no matter how small the non-linearities!

- **HOW:** power from, very small "non-linearities" will dominate over linear evolution of even smaller initial conditions.

- **CONS:** for some initial power spectra shapes linear theory is never accurate at large scales no matter how "linear" the system is.

- **PROS:** the power spectra at large scales can provide a very sensitive probe of non-linearities.

- **click bait:** linear theory doesn't work. some truthiness

- **description:** sub-Poissonian spectra are "unstable". some truthiness

Homogeneous PDEs

- Consider the order p partial differential equation for $q[t,\vec{x}]$ and it's Taylor series in the amplitude of the dependent variable q :

$$\frac{d^p q}{dt^p} = S[t, q, \dot{q}, \nabla^2 q, \vec{\nabla} q \cdot \vec{\nabla} q, \cdots] = \sum_{n=0}^{\infty} {}_{(n)}S[t, q, \dot{q}, \nabla^2 q, \vec{\nabla} q \cdot \vec{\nabla} q, \ldots]$$

$${}_{(n)}S[t, q, \dot{q}, \nabla^2 q, \vec{\nabla} q \cdot \vec{\nabla} q] \equiv \left(\frac{1}{n!} \frac{d^n}{d\epsilon^n} S[t, \epsilon\, q, \epsilon\, \nabla^2 q, \epsilon^2\, \vec{\nabla} q \cdot \vec{\nabla} q, \epsilon\, \dot{q}, \cdots] \right)_{\epsilon=0}$$

The source function S is a smooth function of $q[t,\vec{x}]$.

- For simplicity assume that ${}_{(0)}S = 0$.

- This PDE is homogeneous (has spatial translation invariance) because S does not explicitly depend on $\vec{x}$ (also rotationally invariant though which is not important).

-

Power Transfer from Small to Large Scales

- One can express the PDE in diagramatically terms of Fourier amplitudes

$$\tilde{q}[t,\vec{k}] \equiv \int \frac{d^d\vec{x}}{(2\pi)^{d/2}}\, e^{-i\vec{k}\cdot\vec{x}}\, q[t,\vec{x}] \qquad\qquad q[t,\vec{x}] = \int \frac{d^d\vec{k}}{(2\pi)^{d/2}}\, e^{+i\vec{k}\cdot\vec{x}}\, \tilde{q}[t,\vec{k}]$$

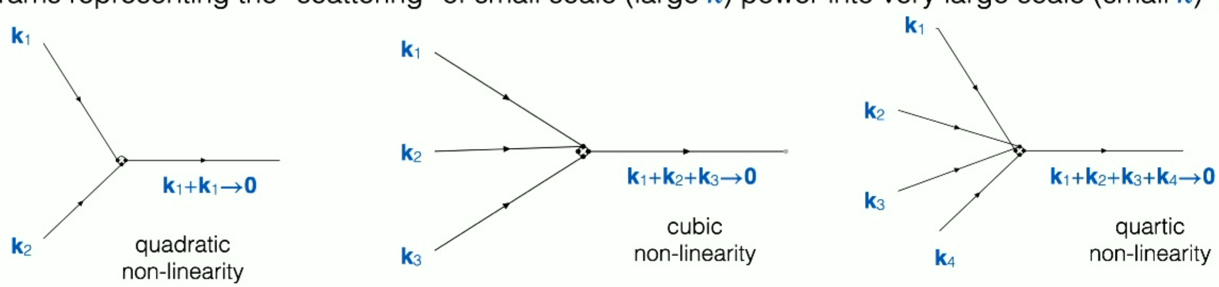
where ${}_{(n)}S$ corresponds to an interaction vertex with $n+1$ different $\vec{k}$ (Fourier mode).

- For a statistically homogeneous distribution the inhomogeneity power spectrum is defined by

$$\langle \tilde{q}[t,\vec{k}]\, \tilde{q}[t,\vec{k}'] \rangle - \langle \tilde{q}[t,\vec{k}] \rangle \langle \tilde{q}[t,\vec{k}'] \rangle = (2\pi)^d\, \delta^{(d)}[\vec{k}-\vec{k}']\, P[t,\vec{k}]$$

- One can express the PDE in diagramatically in terms of these Fourier modes.

- Diagrams representing the “scattering” of small scale (large $\vec{k}$) power into very large scale (small $\vec{k}$) power are



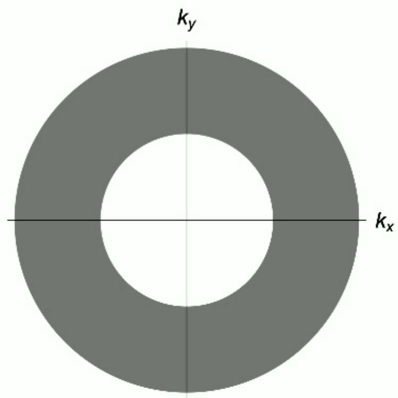
etc.

for $n = 2, 3, 4$ etc. For small non-linearities it is usually the *leading order* (smallest $n \geq 2$) which dominates.

Broadband Power

- In order to transfer power to $\vec{k} \to 0$ not only must there be a diagram with $\sum_i \vec{k}_i = 0$ but there must be power at large $\vec{k}$ to transfer to small $\vec{k}$.

- There will be power for a statistically isotropically distribution with finite bandwidth



assume NOT: Linear

- A linear systems has none of these diagrammatic interactions and cannot redistribute power to transfer power in $\vec{k}$ space.

assume NOT: Conservative Non-linearity

- In order to transfer power to $\vec{k} \to 0$ not only must there be a diagram with $\sum_i \vec{k}_i = 0$ but the amplitude of this interaction must be non-zero

- Conservative non-linearities which are spatial divergences

$$S - {}_{(1)}S = \vec{\nabla} \cdot \vec{S}$$

will have zero amplitude when $\sum_i \vec{k}_i = 0$ and not transfer power to $\vec{k} \to 0$.

Integral Formulation

- One can always formulate these PDEs as and integral equation as follows.
- To illustrate this consider 2nd order PDEs ($p = 2$) segregating the linear and non-linear terms

$$\ddot{q} - {}_{(1)}S = S_{\rm nl} \equiv {}_{(0)}S + \sum_{n=2}^{\infty} {}_{(n)}S.$$

- General linear theory solution is $\tilde{q} = a_+ \tilde{\psi}_+ + a_- \tilde{\psi}_-$ in terms of two constants $a_\pm$ and two functions $\tilde{\psi}_\pm[t, |\vec{k}|]$
- Fully non-linear $\ddot{\tilde{q}}$ evolution rewritten in terms of time varying coefficients $a_\pm[t, |\vec{k}|]$:

$$\tilde{q} = a_+ \tilde{\psi}_+ + a_- \tilde{\psi}_- \qquad \begin{pmatrix} \dot{a}_+ \\ \dot{a}_- \end{pmatrix} = \frac{\tilde{S}_{\rm nl}}{\tilde{\psi}_+ \dot{\tilde{\psi}}_- - \tilde{\psi}_- \dot{\tilde{\psi}}_+} \begin{pmatrix} -\tilde{\psi}_- \\ +\tilde{\psi}_+ \end{pmatrix}$$

where

$$\begin{pmatrix} a_+[t, \vec{k}] \\ a_-[t, \vec{k}] \end{pmatrix} = \begin{pmatrix} a_+[t_{\rm i}, \vec{k}] \\ a_-[t_{\rm i}, \vec{k}] \end{pmatrix} + \int_{t_{\rm i}}^{t} dt' \frac{\tilde{S}_{\rm nl}[t', \vec{k}]}{\tilde{\psi}_+[t', |\vec{k}|]\, \dot{\tilde{\psi}}_-[t', |\vec{k}|] - \tilde{\psi}_-[t', |\vec{k}|]\, \dot{\tilde{\psi}}_+[t', |\vec{k}|]} \begin{pmatrix} -\tilde{\psi}_-[t', |\vec{k}|] \\ +\tilde{\psi}_+[t', |\vec{k}|] \end{pmatrix}.$$

- This is an integral formulation of the 2nd order PDE.
- One can generalize this to any order PDE.

Small Non-linearity Approximations

- The Born approximation is to substitute the linear theory solution for q into $\tilde{S}_{\rm nl}[t',\vec{k}]$ in the integral formulation: $\tilde{S}_{\rm nl}[t',\vec{k}] \to \tilde{S}^{\rm B}_{\rm nl}[t',\vec{k}]$. This can be accurate when non-linearities are small.

- Leading order dynamics truncates the Taylor series for $S_{\rm nl}$ to the lowest non-zero order non-linearity. This can also be accurate when non-linearities are small.

- The leading order Born approximation applies the Born approximation to leading order dynamics: $\tilde{S}_{\rm nl}[t',\vec{k}] \to \tilde{S}^{\rm lB}_{\rm nl}[t',\vec{k}]$. This is equivalent to leading order perturbation theory. Again can be accurate when non-linearities are small.

- Usually the leading order is $n = 2$.

Illustration in Formulae

- consider the case where the quadratic non-linearities are

$$_{(2)}S = D[t]\, q^2 + E[t] q[t]\, \nabla^2 q + F[t]\, \vec{\nabla} q \cdot \vec{\nabla} q$$

and initial conditions containing only one of the two linear modes, e.g. $a_+[t_i, \vec{k}_1] \neq 0$. and $a_-[t_i, \vec{k}_1] = 0$.

- The leading order Born approximation is

$$\tilde{S}^{1B}_{nl}[t, \vec{k}] = \int \frac{d^d \vec{k}_1}{(2\pi)^{d/2}} \left(D[t] - |\vec{k} - \vec{k}_1|^2 E[t] - (\vec{k}_1 \cdot (\vec{k} - \vec{k}_1))\, F[t] \right) a_+[t_i, \vec{k}_1]\, a_+[t_i, \vec{k} - \vec{k}_1] \psi_+[t, |\vec{k}_1|]\, \psi_+[t, |\vec{k} - \vec{k}_1|] \,.$$

- We are interested in the long wavelength limit $\vec{k} \to 0$.
- And with insub-Poissonian initial condition where $a_\pm[t_i, \vec{0}] = 0$.
- The sub-Poissonian long wavelength leading order Born approximation is

$$\begin{pmatrix} a_+[t, \vec{0}] \\ a_-[t, \vec{0}] \end{pmatrix} = \int_{t_i}^{t} dt' \, \frac{\int \frac{d^d \vec{k}_1}{(2\pi)^{d/2}} |a_+[t_i, \vec{k}_1]\, \psi_+[t', |\vec{k}_1|]|^2 \left(D[t'] + |\vec{k}_1|^2 (F[t'] - E[t'])\right)}{\tilde{\psi}_+[t',0]\, \dot{\tilde{\psi}}_-[t',0] - \tilde{\psi}_-[t',0]\, \dot{\tilde{\psi}}_+[t',0]} \begin{pmatrix} -\tilde{\psi}_-[t',0] \\ +\tilde{\psi}_+[t',0] \end{pmatrix}.$$

- **Unless $D = 0$ and $E = F$ we expect some power at $\vec{k} = 0$ which is greater than the zero power of sub-Poissonian initial conditions.**
- $F = E$ doesn't contribute because $q\, \nabla^2 q + \vec{\nabla} q \cdot \vec{\nabla} q = \vec{\nabla} \cdot (q\, \vec{\nabla} q)$ is a conservative non-linearity which does not scatter power to long wavelengths.

Large Scale and Remnant White Noise

- Where the integrals have converged to asymptotic values in the sub-Poissonian long wavelength leading order Born approximation,

$$\begin{pmatrix} a_+[t,\vec{0}] \\ a_-[t,\vec{0}] \end{pmatrix} \approx \begin{pmatrix} a_+[\infty,\vec{0}] \\ a_-[\infty,\vec{0}] \end{pmatrix}$$

then the spectrum is “white noise” since $0 < P[t,0^+] < \infty$. Hence we call this large scale white noise or LSWN.

- When the integrals have converged it is unimportant whether or not the small scale non-linearities subsequently damp away.

- To the extent that the small scales non-linearities which generated the LSWN have damped away we call the remaining large scale white noise remnant white noise.

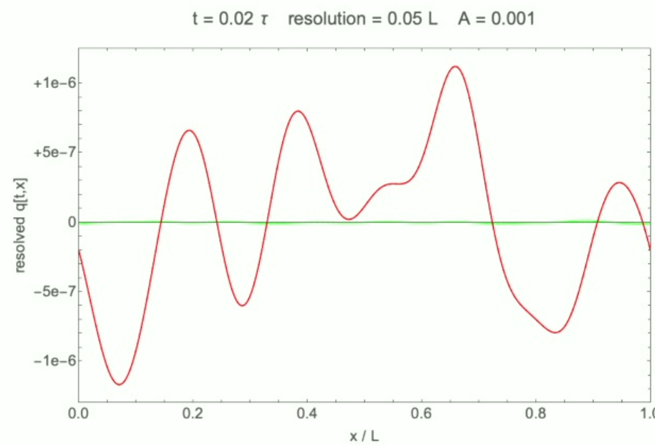
Numerical Illustration 1-D

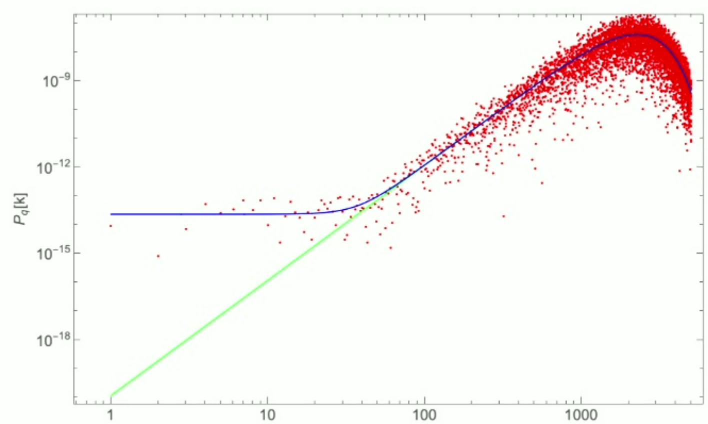
$\dot{q} = 1 - q^2 \qquad \langle q^2 \rangle_{\rm i} = 10^{-3} \qquad P[t_{\rm i}, k] \propto k^4\, e^{-(k\sigma)^2} \qquad \sigma = 10^{-2}\, L$

unresolved realization
exact

unresolved realization
linear dynamics

unresolved realization
leading order dynamics





realization
exact

expectation
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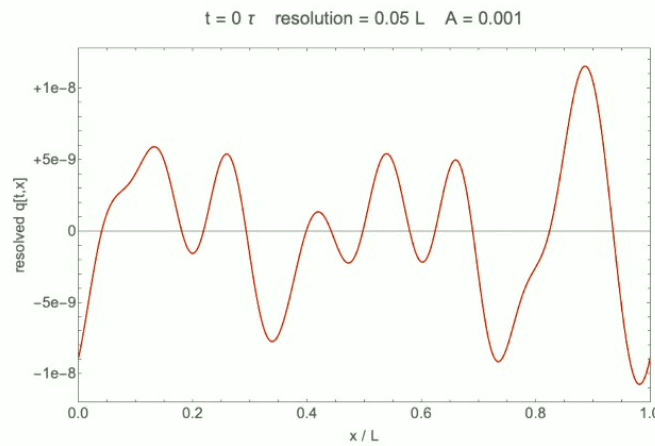

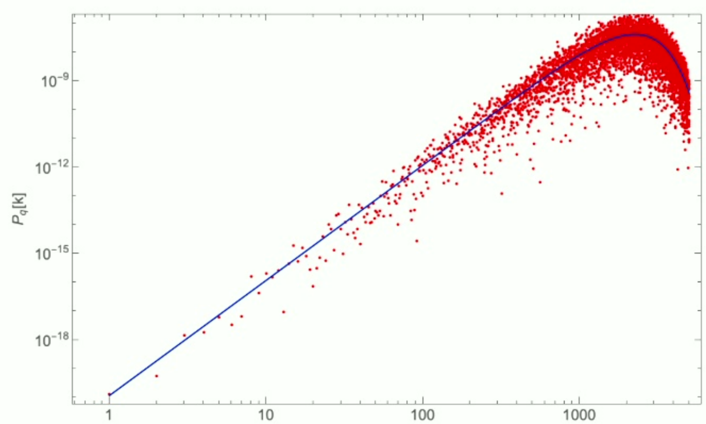

realization exact

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Analytic Illustration

$\tau \dot{q} = 1 - q^2$ $\quad P[t_i, k] \propto k^n e^{-(k\sigma)^2}$ $\quad$ arbitrary dimension d

$$P[t,k] = \underbrace{A V_\sigma c_{(d,n)} (k\sigma)^n e^{-(k\sigma)^2}}_{\text{initial power spectrum}} + \underbrace{2 A^2 V_\sigma h_{(d,n)}[\tfrac{t}{\tau}, k\sigma] e^{-\frac{(k\sigma)^2}{2}}}_{\text{leading order correction}} + \cdots \to \underbrace{2 V_\sigma A^2 \; h_{(d,n)}[\tfrac{t}{\tau}, 0]}_{\text{large scale white noise}}$$

$V_\sigma = \left(\frac{\sigma}{(2\pi)^{\frac{3}{2}}}\right)^d$ coherence volume of q

order unity factors: $c_{(d,n)} = 2^{\frac{3}{2}d} \pi^d \frac{\Gamma[\frac{d}{2}]}{\Gamma[\frac{d+n}{2}]}$ normalizes total power to A

$h_{(d,n)}[y,0] = y^2 f_{(d,n/2)}[y,0]$ integrated time dependence of transfer of power to largest scales

Large scale white noise dominates when

$$k \leq k_{\text{LSWN}} = \underbrace{\left(2\, h_{(d,p)}[\tfrac{t}{\tau}, 0] / c_{(d,p)}\right)^{1/n}}_{\text{factor of order unity}} \frac{A^{1/n}}{\sigma}$$

$f_{(d,n/2)}[0]$	$n=0$	$n=2$	$n=4$	$n=6$
$d=1$	1	$\frac{3}{4}$	$\frac{35}{48}$	$\frac{231}{320}$
$d=2$	1	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{40}{113}$
$d=3$	1	$\frac{5}{12}$	$\frac{21}{80}$	$\frac{429}{2240}$

Large scale white noise amplitude

$$P[t,0] = 2 V_\sigma A^2 \times \underbrace{h_{(d,n)}[\tfrac{t}{\tau}, 0]}_{\text{factor of order unity}}$$

ACT 3

large scale white noise in cosmology

sh— happens!

Newtonian Non-Linearities

Non-linearities which produce cosmic LSWN are not specific to GR or even to gravity:

general relativity : $\bar{K} \equiv \frac{8\pi G}{3}\,\bar{\rho} - \frac{1}{9}\,\bar{\theta}^2$ $\quad \dot{\bar{K}} + \frac{2}{3}\,\bar{\theta}\,\bar{K} = \frac{2}{9}\,\bar{\theta}\,\left(\bar{\sigma}^2 - \bar{\omega}^2 - \dot{\bar{u}}^{\alpha}{}_{;\alpha}\right) - \frac{8\pi\, G}{3\, c^2}\,\bar{\sigma}_{\alpha\beta}\,\bar{q}^{\alpha\beta}$

Newtonian gravity : $\bar{K} \equiv \frac{8\pi G}{3}\,\bar{\rho} - \frac{1}{9}\,\bar{\theta}^2$ $\quad \dot{\bar{K}} + \frac{2}{3}\,\bar{\theta}\,\bar{K} = \frac{2}{9}\,\bar{\theta}\,\left(\bar{\sigma}^2 - \bar{\omega}^2 - \dot{\bar{v}}^{i}{}_{:i}\right)$

Newtonian no gravity : $\bar{K} \equiv -\frac{1}{9}\,\bar{\theta}^2$ $\quad \dot{\bar{K}} + \frac{2}{3}\,\bar{\theta}\,\bar{K} = \frac{2}{9}\,\bar{\theta}\,\left(\bar{\sigma}^2 - \bar{\omega}^2 - \dot{\bar{v}}^{i}{}_{:i}\right)$

LSWN Issues of ΛCDM

- The simple ΛCDM narrative has a long radiation period with nearly scale invariant spectra of growing mode curvature inhomogeneities (ignoring tilt, $n_s \approx 0.973 \neq 1$, for the moment)

$$\mathscr{A} \approx \bar{\Delta}^2_{\mathscr{R}}[a_i, k] = 4\pi k^3 P_{\mathscr{R}}[a_i, k] \qquad \text{or} \qquad P_{\bar{K}}[a_i, k] \approx \frac{\mathscr{A}\, k}{9\pi\, a_i^4}$$

- This Harrison-Zel'dovich (HZ) spectrum for $\bar{K}$ is sub-Poissonian: LSWN will dominate on some scale.

- Radiation era in simplest form of ΛCDM has perfect $p = \rho/3$ fluid which undergoes undamped acoustic oscillations after horizon crossing ($\tilde{\mathscr{R}} = \tilde{\mathscr{R}}_i \sin[\varphi]/\varphi$ where $\varphi = |\vec{k}| \int_0^a da/(a\,\dot{a})/\sqrt{3}$) for all times up to the Big Bang ($a \to 0$).

- In this case LSWN in the radiation era is

$$\lim_{k\to 0} k\, \bar{\Delta}^2_{\mathscr{R}}[a, k] = \frac{1}{2}\int_{k_{\min}}^{k_{\max}} dk\, \Upsilon[\varphi, \varphi]^2 \qquad \Upsilon[\varphi, \varphi] \equiv \int_0^{\varphi} d\varphi\, \Sigma[\varphi]^2 \qquad \Sigma[\varphi] \equiv 3\,\frac{2\,\varphi\cos[\varphi] + (\varphi^2 - 2)\sin[\varphi]}{\varphi^3}$$

- The k integral diverges rapidly as $k_{\max} \to \infty$. cosmology.

- **ISSUE:** *one must impose a small scale cutoff* in the power spectrum for LSWN not to dominate!

ΛCDM LSWN

- **CON:** the simplest maximal extrapolation of ΛCDM is inconsistent with data - linear theory fails - one expects a white noise $n_s \approx 0$ not HZ $n_s \approx 1$ spectrum.
- **CON:** to solve this problem one could truncate the extrapolation and add a white noise amplitude to the power spectrum to parameterize our ignorance of the truncation.
- **PRO:** if one would detect LSWN this would tell us something about the truncation and the physics of the early universe on small scales.
- **PRO:** lack of detections puts limits physics of the early universe on small scales.
- **PRO / CON:** adding parameters always relaxes constraints on other parameters: H_0, m_ν
- **IDEAS:** possible truncation scenarios in the inflationary paradigm:
  - end of inflationary reheating
  - running of spectral index

Mandatory LSWN Cutoff scale

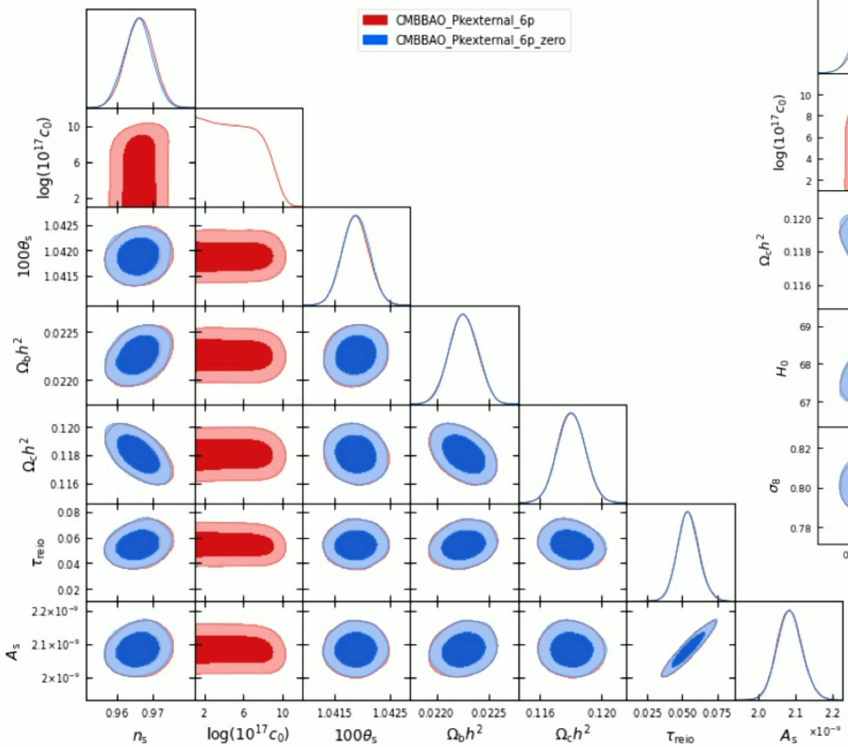
- In the cutoff model LSWN is generated by shear at horizon crossing of acoustic waves during the radiation era at some comoving cutoff scale
- The non-linearity scale is $\mathscr{R} \approx 1$ but the acoustic wave cross with amplitude $\mathscr{R} \approx \mathscr{A} \approx 10^{-9}$ so the quadratic non-linearities correspond to $\mathscr{R} \approx \mathscr{A}^2 \approx 10^{-18}$.
- From the scaling in Act 2 the wavenumber where LSWN dominates is $k_{\rm LSWN} \approx \mathscr{A}/\sigma_{\rm cut}$ since $n_s \approx 1$.
- Since this is a large scale phenomena one might these that the limit comes from the largest observable scale $k \approx H_0$
- On this scale cosmic limits accurate of the power spectra to $\delta \ln P \lesssim 1$ so one could guess that the largest allowable comoving cutoff is $\sigma_{\rm cut} \gtrsim \mathscr{A}/H_0 \approx 10\,{\rm pc}$.
- In ΛCDM this scale crosses the horizon at temperature $T \sim 100\,{\rm MeV}$

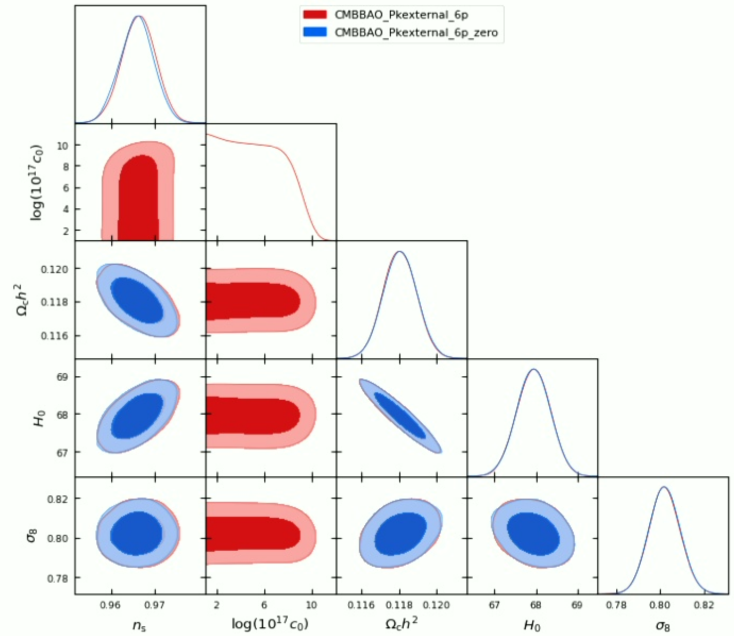
Encore

subtleties

gravity waves vorticity viscosity . . .

Parametric Cosmology





$$\lim_{k\to 0} k\,\bar{\Delta}^2_{\mathscr{R}}[a_i, k] = k_{BH}$$

$$c_0 = k_{BH}\,\mathrm{pc}$$

k_{BH} is the wavenumber at which black holes might form during a future epoch of matter dominations with this LSWN amplitude

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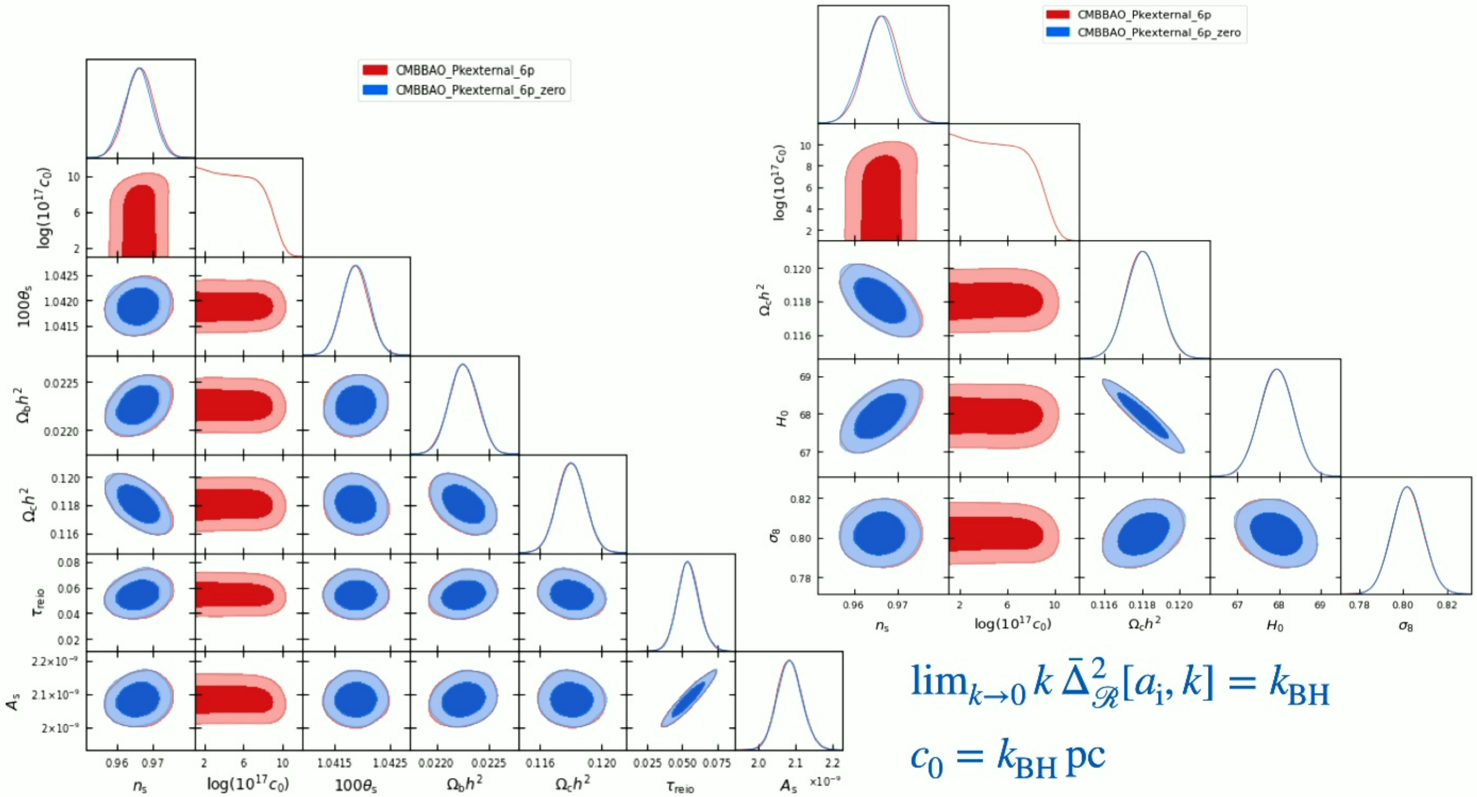

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