Title: String amplitudes in AdS: an emergent worldsheet picture

Speakers: Maria Nocchi

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Abstract:

The study of string scattering in general curved spacetimes presents a stark contrast to the relatively well-understood framework in flat space, where perturbative techniques and worldsheet methods can be used. However, the AdS/CFT correspondence offers a powerful indirect method to compute string amplitudes in AdS. In this talk, I will present recent developments in the AdS Virasoro-Shapiro program. The tree-level amplitude of four gravitons in AdS^5 x S^5 is mapped to a four-point correlator in N=4 SYM at large central charge, and studied by CFT methods combined with single-valuedness, which echoes its importance in flat space. The amplitude itself is defined and constructed as an expansion around flat space. While the goal is to compute it to all orders, attention can be placed on more tractable limits. I will present our results for the High Energy limit, with dual insights from the spacetime and worldsheet perspectives, and the richer case of the Regge limit, which encodes full information on the intermediate operators in the leading Regge trajectory.

String amplitudes in *AdS*: an emergent worldsheet picture

Maria Nocchi
University of Oxford

Based on work with L. F. Alday, T. Hansen, C. Virally, and X. Zhou

[2312.02261], [2409.03695]

Perimeter Institute, Quantum Fields & String Seminar
15/10/24

Outline

- Introduction
- Mathematical structure of scattering amplitudes
- The (AdS) Virasoro-Shapiro amplitude
- The High Energy limit
- The Regge limit
- Summary & Conclusions

Outline

*Introduce a **natural language** to make contact with the string worldsheet.*

- Introduction
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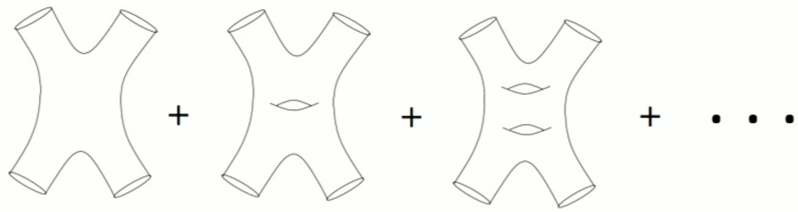
Introduction



- Scattering amplitudes encode the differential probability for a certain process to happen. This predictive power makes them an essential object in Particle Physics, Mathematics, and String Theory.

- String scattering in flat space
  → perturbative String Theory
  → worldsheet methods

$$A^{(n)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi} \frac{1}{\text{Vol}} \int DXDg \; e^{-S_{\text{Poly}}} \prod_{i=1}^{n} V_{\Lambda_i}(p_i)$$

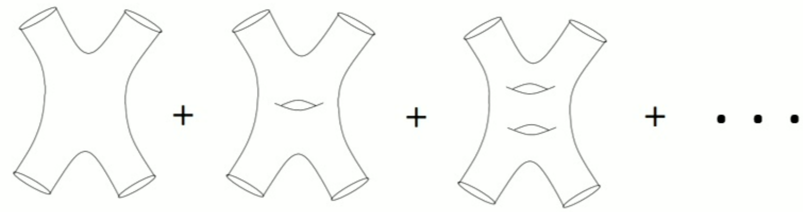
- When considering scattering processes, we sum over all possible configurations of WSs.

- Sum over Riemann surfaces of **increasing genus** with insertions of vertex operators for the initial/final states (different topologies).

- **In some regimes, the sum is dominated by a saddle point.**

Parameters:
g_s (string coupling constant)
α' (size of the string)

Introduction



- An outstanding question: what about curved spacetimes?
  → difficulties with standard formulations
  → **perturbative genus expansion** but no direct worldsheet approach, even at tree-level

We need additional tools!

- AdS/CFT: string scattering amplitudes on AdS from CFT correlators of the dual boundary theory.

- Our strategy: combine worldsheet intuition with alternative powerful tools.

This program: how to compute string amplitudes on $\boldsymbol{AdS}$

→ Analyze and include curvature corrections systematically and efficiently.
→ Exploit and emphasize the interplay between String Theory and **Number Theory.**

Main object of study of this program

- Scattering of four graviton states at tree level.

- Flat space : Virasoro Shapiro amplitude

Prefactor: polarisation vectors

$$A_4(\varepsilon_i, p_i) = K(\varepsilon_i, p_i) \int d^2 z |z|^{-2S-2} |1-z|^{-2T-2}$$

The integrand is a **single-valued** function of z.

- $AdS_5 \times S^5$: correlator of 4 stress-tensor multiplets, to leading non-trivial order in a *1/c* expansion.

- Right language: Mellin space. [Penedones]

- Then, Borel transform: AdS analog of the Virasoro Shapiro amplitude.

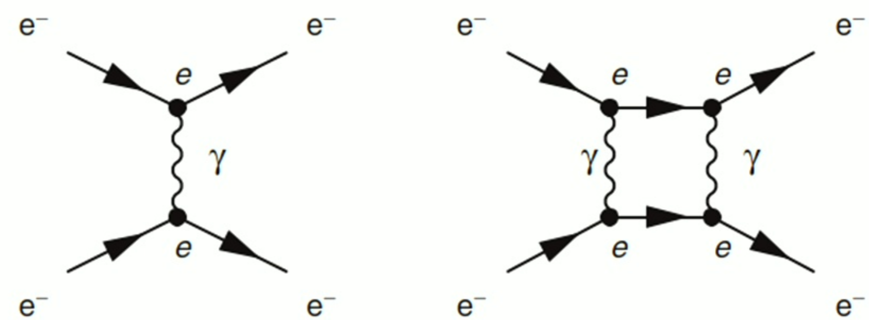
2 Mathematical structure of scattering amplitudes

Let’s start from a general problem:

Reveal and understand the hidden **mathematical structure of scattering amplitudes** in Field Theory/String Theory.

Scattering amplitudes in QFT

- Use scattering amplitudes to:
  - test predictions for the theory
  - uncover structure and reveal symmetries
- Perturbation theory:
  - sum over Feynman diagrams
  - **loop** integrals
  - **complicated functions with branch cuts** (intermediate virtual particles going on-shell)



Strategy: study loop integrals from a purely mathematical and algebraic point of view.

Special numbers and functions in loop computations

$$B(p^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2\,(k+p)^2}\,,$$

$$T(p_1^2,p_2^2,p_3^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2\,(k+p_1)^2\,(k+p_1+p_2)^2}$$

$$B(p^2) = \frac{1}{\epsilon} + 2 - \log(-p^2) + \epsilon\left[\frac{1}{2}\log^2(-p^2) - 2\log(-p^2) - \frac{1}{2}\zeta_2 + 4\right] + \mathcal{O}(\epsilon^2)$$

$$T(p_1^2,p_2^2,p_3^2) = \frac{2}{\sqrt{\lambda}}\left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(z\bar{z})\,\log\frac{1-z}{1-\bar{z}}\right] + \mathcal{O}(\epsilon)\,.$$

Rational functions are insufficient to write down the answer!

Notice the appearance of **zeta values** (Riemann ζ function at integer values):

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}\,,\ n>1$$

and the (powers) of logarithms as well as their generalisations (**polylogs**):

$$\log z = \int_1^z \frac{dt}{t}$$

Scattering amplitudes in String Theory

- Building blocks of **closed string theory** amplitudes at genus 0 (tree-level):

$$M_{N+3}(\mathbf{s},\mathbf{n},\tilde{\mathbf{n}}) = \left(\frac{i}{2\pi}\right)^N \int_{\mathbb{C}^N} \prod_{0<i<j<N+1} |z_i - z_j|^{2s_{ij}} \, (z_i - z_j)^{n_{ij}} \, (\bar{z}_i - \bar{z}_j)^{\tilde{n}_{ij}} \prod_{i=1}^{N} dz_i d\bar{z}_i$$

- Functions of the complex variables s_{ij}.

  s : collection of Mandelstam kinematic invariants: $s_{ij} = \alpha' p_i \cdot p_j$

$$z_0 = 0\,, \qquad z_{N+1} = 1, \qquad N \in \mathbb{N}, \qquad n_{ij}, \tilde{n}_{ij} \in \mathbb{Z}$$

  $N = 1,\ n_{12} = \tilde{n}_{12} = -1$: Virasoro-Shapiro amplitude

- The global (any s) and local properties are related to the theory of single-valued periods, such as **zeta values** and **polylogs**. [Vanhove, Zerbini]

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The nature of the underlying worldsheet describing the string interaction is fundamental!

- The global (any s) and local properties are related to the theory of single-valued periods, such as **zeta values** and **polylogs**. [Vanhove, Zerbini]

Polylogarithms

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad \rightarrow \quad \int_0^z dz' \, \frac{\mathrm{Li}_{n-1}(z')}{z'}$$

- Classical polylogs:
  - convergent series on the unit disk $|z| < 1$
  - can be continued to the cut plane $\mathbb{C} \setminus [1, \infty)$ by an iterated integral representation
  - reduce to zeta-values at *z=1* $\mathrm{Li}_n(1) = \zeta_n \,, n > 1$
  - reduce to the standard logarithm at *n=1*

- Multiple polylogs (MPLs):
  - *w* = word with letters in the alphabet {0,1}
  - function of a single variable labelled by the word *w*, whose length we call weight:

$$\frac{d}{dz} L_{0w}(z) = \frac{1}{z} L_w(z) \,, \qquad \frac{d}{dz} L_{1w}(z) = \frac{1}{z-1} L_w(z)$$

$$L_{0^p}(z) = \frac{\log^p z}{p!} \qquad L_{0^{n-1}1} = -\mathrm{Li}_n(z)$$

multiple zeta values (MZVs)
= MPLs at unity

$$\log|z|^2 = \log z + \log \bar{z}$$

Single-valued polylogs

- For the classical polylog, the discontinuity across the cut is

$$\mathrm{Disc}(\mathrm{Li}_n(z)) = 2\pi i \frac{\log^{n-1} z}{(n-1)!}$$

Polylogs = special analytic functions of a single complex variable, with branch points (**multi-valued** functions on the complex plane).

- Can consider combinations of polylogs such that all the branch cuts cancel, and define single-valued functions in the $(z, \bar{z})$ plane!
- Can do the same for multiple polylogs: SVMPLs.
- $L_w(1) = \zeta(w)$

SVMPLs

- At any given weight, there is a **finite-dimensional** vector space of available functions!
- **SVMZVs** = single-valued projection of MZVs = **SVMPLs at unity**

Building blocks of the AdS Virasoro-Shapiro amplitude

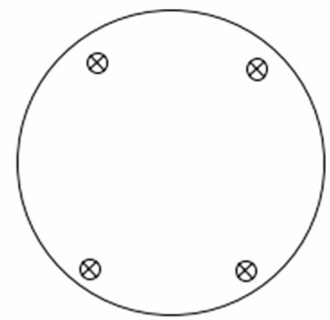
Technically: determine the coefficients of the allowed SMPLs at a given weight.

SPOILER: this is the *right function space* in our program!

Why?

The Virasoro-Shapiro amplitude

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$



- Crossing symmetry in the 3 Mandelstam variables $S+T+U=0$
- Fix *T* and vary *S*.
  Poles at mass of the tachyon + higher states of the closed string

⇒ STRING AMPLITUDE AS AN INFINITE NUMBER OF (s-channel) TREE-LEVEL QFT DIAGs

- Regge behaviour
- Low/High Energy $S = -\frac{\alpha'}{4}(p_1+p_2)^2\,, \qquad T = -\frac{\alpha'}{4}(p_1+p_3)^2\,, \qquad U = -\frac{\alpha'}{4}(p_1+p_4)^2$

Virasoro-Shapiro amplitude and single-valued periods

- Low Energy expansion of VS

$$A^{(0)}(S,T) = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \sigma_2^a \sigma_3^b \alpha_{a,b}^{(0)} \qquad \sigma_2 = \frac{1}{2}(S^2+T^2+U^2)\ , \sigma_3 = STU$$

SUGRA + TOWER OF STRINGY CORRECTIONS

$$\alpha_{a,0} = \zeta(3+2a)$$

- Only **odd ζ values** appear!
  The Wilson coefficients live in the ring of SVMZVs.

$$A^{(0)}(S,T) = \frac{\exp\left(\sum_{n=1}^{\infty} \frac{\zeta^{\rm sv}(2n+1)(S^{2n+1}+T^{2n+1}+U^{2n+1})}{2n+1}\right)}{STU}$$

- This reflects the single-valued nature of the integral representation.

$$A^{(0)}(S,T) = \frac{1}{U^2}\int d^2z |z|^{-2S-2}|1-z|^{-2T-2}$$

Key Takeaways

- The infinite number of vibration modes in string spectra introduces transcendental numbers already at tree-level!

3

The AdS Virasoro-Shapiro amplitude

The AdS Virasoro-Shapiro amplitude

Scattering of massless strings: 4 **gravitons** on $\boldsymbol{AdS_5 \times S_5}$ in Type IIB superstring theory

$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$

AdS/CFT

$g_s \sim \frac{1}{N}$

Correlator of four **stress-tensor** multiplets in Mellin space, at large central charge

Borel

$$A(S,T) = 2\lambda^{\frac{3}{2}} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{d\alpha}{2\pi i} e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

$$= A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \dots$$

VS in flat space

Curvature corrections

The AdS Virasoro-Shapiro amplitude

Scattering of massless strings: 4 **gravitons** on $\boldsymbol{AdS_5 \times S_5}$ in Type IIB superstring theory

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A natural language!

Borel

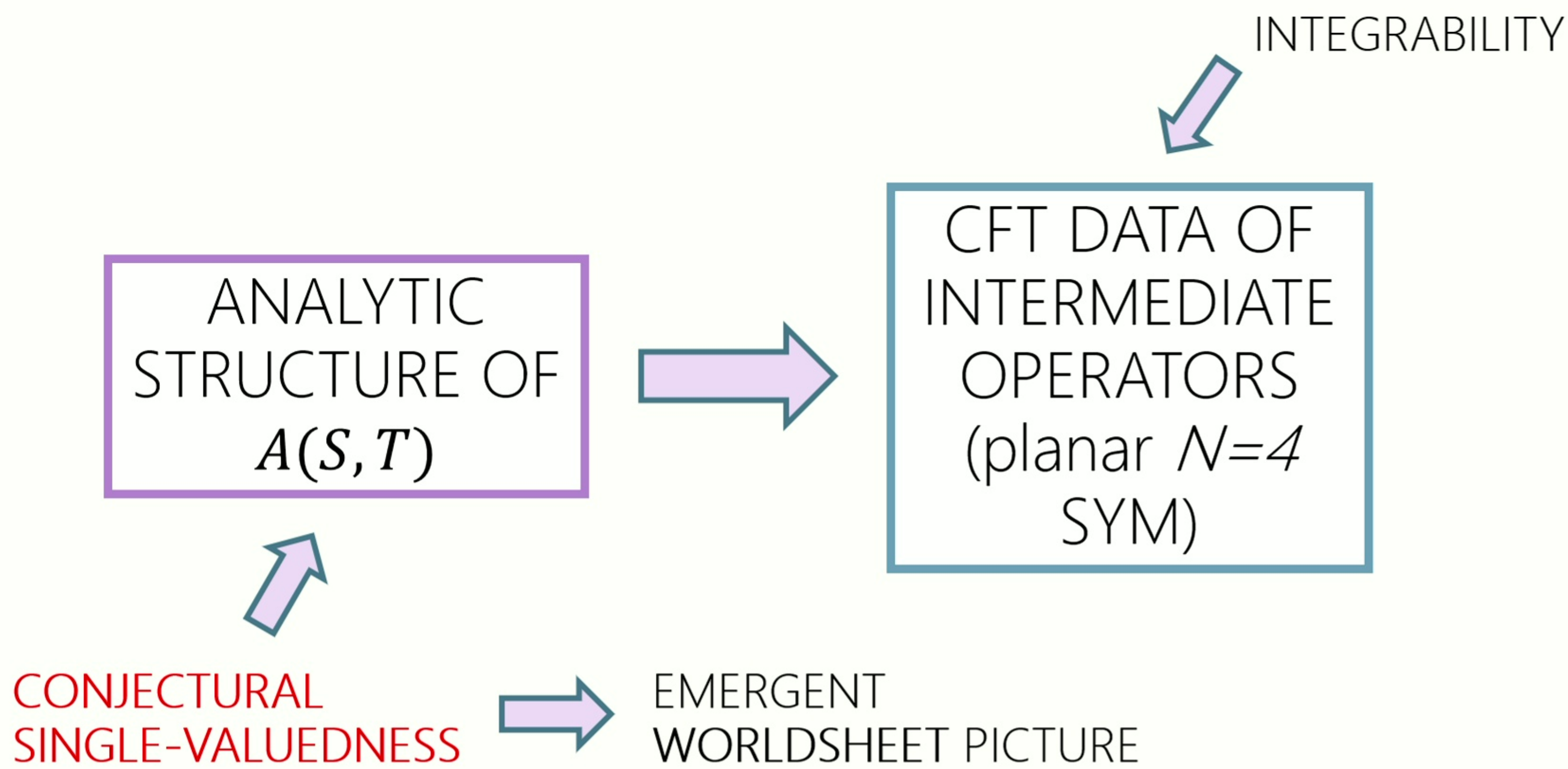
$$A(S,T) = 2\lambda^{\frac{3}{2}} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{d\alpha}{2\pi i}\, e^{\alpha}\alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

$$= A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \dots$$

VS in flat space

Curvature corrections

AdS amplitude as a curvature expansion around flat space



INTEGRABILITY
ANALYTIC STRUCTURE OF $A(S,T)$
CFT DATA OF INTERMEDIATE OPERATORS (planar *N=4* SYM)
CONJECTURAL SINGLE-VALUEDNESS
EMERGENT WORLDSHEET PICTURE

The AdS Virasoro-Shapiro amplitude

Three key points:

- **Structure of poles** (from the expansion of the AdS propagator around flat-space and dispersive sum-rules). [Alday, Hansen, Silva]

AdS CORRECTIONS: HIGHER ORDER POLES (jump by 3)

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta)}{(S-\delta)^{3k+1}} + \frac{R^{(k)}_{3k}(T,\delta)}{(S-\delta)^{3k}} + \ldots + \frac{R^{(k)}_{1}(T,\delta)}{S-\delta} + \text{regular}\,, \quad \delta = 1, 2, \ldots$$

- Each correction admits a **Low Energy** expansion: assume the unknown coefficients to be **single-valued** zetas as in flat space!

- Intuition from the **worldsheet:** $A^{(k)}(S,T)$ from worldsheet integrals similar to the one in flat space.

$$\int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G(z,\bar{z})$$

The AdS Virasoro-Shapiro amplitude

What is the relevant space of functions? Linear combination of single-valued functions such that the Low Energy expansion contains only SVMZVs.

The k-th order answer takes the form of a genus 0 worldsheet integral involving weight 3k SVMPLs, and rational in *S,T*. [Alday, Hansen]

$$A(S,T) = \int d^2z |z|^{-2S} |1-z|^{-2T} W_0(z,\bar{z}) \left(1 + \frac{S^2}{R^2} W_3(z,\bar{z}) + \frac{S^4}{R^4} W_6(z,\bar{z}) + \dots \right)$$

$$W_0(z,\bar{z}) = \frac{1}{2\pi U^2 |z|^2 |1-z|^2}$$

$$\alpha' = 1$$

Precise proposal for the structure of the tree-level amplitude on $AdS_5 \times S^5$!

NOTE: This is NOT the result of a direct worldsheet computation!

Big goal

Determine *A(S,T)* to all orders in *1/R*.

Key Takeaways

- **Single valuedness** plays a fundamental role in the construction of AdS scattering amplitudes, as in flat space!

Big goal

Determine *A(S,T)* to all orders in *1/R*.

- In the meanwhile… focus on more accessible limits.
- How to make connections to more direct worldsheet computations?

Flat space result [Gross & Mende]

- HE limit: $|S|, |T| \gg 1$ and S/T fixed

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

- Use Stirling's formula to access this regime:

$$A^{(0)}(S,T)_{HE} \sim e^{-2S\log|S| - 2T\log|T| - 2U\log|U|}$$

$\to$ soft exponential behavior!

Analogous to the universality of singularities of the OPE in field theory.

- The exponential behavior is universal: independent of the particular String Theory and the quantum numbers of the scattered particles.

- We can understand this also from the WS integral representation:

$$A(S,T) \sim \int d^2z |z|^{-2S}|1-z|^{-2T} W_0(z,\bar{z})$$

HE limit: saddle point approximation $z = \bar{z} = \frac{S}{S+T} = z_0$

What do we expect for AdS?

Given the "WS representation" for AdS, given that the transcendental functions $W_n(z, \bar{z})$ are polynomials in S, T, the location of the saddle is not modified in a $1/R$ expansion!

The AdS VS in the HE limit can be computed by evaluating the WS integral representation on the saddle point:

$$A_4^{AdS}(S,T)_{\text{HE}} \sim e^{-2S\log|S|-2T\log|T|-2U\log|U|} W_0(z_0)\left(1+\frac{S^2}{R^2}W_3(z_0)+\frac{S^4}{R^4}W_6(z_0)+\cdots\right)$$

where we keep the **leading large energy** contribution at each order.

We are looking at a regime with large R, S and S^2/R^2 finite.

Flat space result [Gross & Mende]

Alternatively, we can understand HE from the point of view of spacetime.

$$A^{(0)}(S,T) \sim \int Dg\ DX \exp\left[-\frac{1}{4\pi}\int d\zeta_1 d\zeta_2 \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right]\prod_i V_i(p_i)$$

WS coordinates: $(\zeta, \bar{\zeta})$
Punctures: $z_k \in \mathbb{R}$

$$V_i(p_i) \sim \int d^2 z_i \sqrt{g} e^{ip_i\cdot X(z_i)}$$

$$p_i^2 = 0\ ,\ p_1\cdot p_2 = -2S\ ,\ p_1\cdot p_3 = -2T\ ,\ p_1\cdot p_4 = -2U$$

At HE, the path integral is dominated by a classical solution:

$$X^\mu(\zeta) = -i\sum_k p_k^\mu \log\left|1 - \frac{\zeta}{z_k}\right|$$

Plug the classical solution into the PI: correct HE result!

Let's carry this classical analysis for AdS!

Classic scattering problem in AdS_d

Embedding coordinates labeled by $M = (0; \mu) = (0, 1, \ldots, d)$

Constraint: $X^M X_M = -R^2$

Vertex operators: $V_i(P_i) \sim \int d^2 z_i \sqrt{g} e^{i P_i^M X_M(z_i)}$

$$\mathcal{L} = \frac{1}{2\pi} \partial X^M \bar{\partial} X_M + \Lambda(X^M X_M + R^2) - i \sum_k P_k^M X_M \delta^{(2)}(\zeta - z_k)$$

Expectation: the HE behavior of the amplitude is captured by classical solutions, now in AdS!

Solve for X^0 using the constraint and take $R \to \infty$. The X^μ coordinates are constant in this limit and identified with the flat space coordinates.

$$X^0 = R + \frac{1}{R} X_1^0 + \cdots ,$$

$$X^\mu = X_0^\mu + \frac{1}{R^2} X_1^\mu + \cdots ,$$

Classic scattering problem in AdS_d

Flat space solution: single-valued as we move around each puncture on the worldsheet

$$X_0^\mu = -\frac{i}{2}\sum_k p_{k,0}^\mu \mathcal{L}_{z_k}(\zeta)$$

Higher orders?

-solve EOMs and Virasoro constraints in a ***1/R* expansion**

-write the solution in terms of **SVMPLs whose letters are the locations of the punctures**

$$\partial\bar{\partial}X^M = \frac{\partial X^N \bar{\partial} X_N}{R^2} X^M$$

Integrate $\partial\bar{\partial}X$ at each order with the rules:

$$\int d\zeta \frac{\mathcal{L}_w(\zeta)}{\zeta - z_i} \to \mathcal{L}_{z_i w}(\zeta)$$

$$\int d\bar{\zeta} \frac{\mathcal{L}_w(\zeta)}{\bar{\zeta} - z_i} \to \mathcal{L}_{w z_i}(\zeta) + \dots$$

Sum of terms of uniform weight $|w| + 1$

Explicit solution for the first correction

$$\partial\bar{\partial}X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{8}\sum_{i,j,k}\frac{p_{i,0}\cdot p_{j,0}}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_{k,0}^\mu \mathcal{L}_{z_k}(\zeta)$$

$$\int d\bar{\zeta}\frac{\mathcal{L}_{z_k}(\zeta)}{(\bar{\zeta} - z_j)} \to \mathcal{L}_{z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_k}(\zeta)$$

$$X_1^\mu = \frac{i}{8}\sum_{i,j,k=1}^{4} p_{i,0}\cdot p_{j,0}\; p_{k,0}^\mu \left(\mathcal{L}_{z_i z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_i z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_i z_k}(\zeta)\right)$$

$$X^M = -iP_k^M \log\left|1 - \frac{\zeta}{z_k}\right| + Q_k^M + \dots$$

Classic scattering problem in AdS_d

$$X^\mu = \mathcal{L}_1(\zeta) + \frac{1}{R^2}\mathcal{L}_3(\zeta) + \frac{1}{R^4}\mathcal{L}_5(\zeta) + \dots$$

FINAL SOLUTION

$$X^0 = \sqrt{R^2 + X_\mu X^\mu} \Rightarrow X^0 = R\mathcal{L}_0(\zeta) + \frac{1}{R}\mathcal{L}_2(\zeta) + \frac{1}{R^3}\mathcal{L}_4(\zeta) + \dots$$

$\mathcal{L}_n(\zeta)$ are linear combinations of pure SVMPLs of weight n, with either ζ or z_i as their arguments and letters in the alphabet $\{z_1, z_2, z_3, z_4\}$.

Once the seed solution is given, the **whole** tower in *1/R* is fixed by the EOMs and integration!

Evaluate the action

Plugging our classical solution into the action:

$$A_{4,\text{bos}}^{AdS}(S,T)_{\text{HE}} \sim A_4^{\text{flat}}(S,T)_{\text{HE}} \times e^{\frac{S^2}{R^2}V_3(z_0)+\frac{S^3}{R^4}V_5(z_0)+\cdots}$$

$V_i(z_0)$ = combinations of transcendental functions of weight i

Our result correctly reproduces the HE limit (after an appropriate redefinition of the Mandelstam variables):

$$A_4^{AdS}(S,T)_{\text{HE}} \sim e^{-2S\log|S|-2T\log|T|-2U\log|U|}W_0(z_0)\left(1+\frac{S^2}{R^2}W_3(z_0)+\frac{S^4}{R^4}W_6(z_0)+\cdots\right)$$

$$W_3(z_0) = V_3(z_0)$$

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$$W_3(z_0) = V_3(z_0)$$

$$W_6(z_0) = \frac{1}{2}V_3(z_0)^2$$

Curvature corrections exponentiate!

$$A_4^{AdS}(S,T)_{HE} \sim A_4^{flat}(S,T)_{HE} \times e^{\frac{S^2}{R^2}W_3(z_0)}$$

- The **full** High Energy limit to **all orders** in S^2/R^2 is determined by the subleading exponent.
- High Energy limit: regime where the amplitude can be computed to all orders in the curvature expansion.
- This result can be explicitly checked to order $1/R^4$ by comparison with AdS VS.
- Now, can we explore a richer limit?

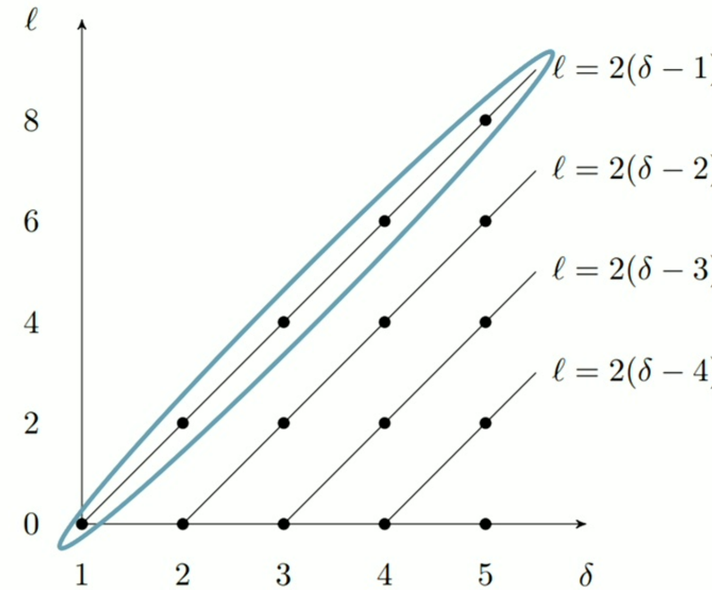
Why Regge?

- Very useful in the presence of **infinitely many resonances** in scattering amplitudes.

- Look for a generalization to AdS to go beyond SUGRA [Costa, Gonçalves, Penedones].

- Virasoro-Shapiro amplitude in flat space: infinitely many poles for the exchanged particles, which organize in Regge trajectories.

- Big achievement: amplitude in the Regge limit **without** knowing the full result! Need only spectrum of the particles in the leading trajectory and cubic couplings.
  Here: Konishi-like operators (target of integrability)



$\delta = 1,2, \ldots$
flat space string mass level

What do we expect for AdS?

- **Flat space:** large T, S finite
  The exchange of a spin J state makes the amplitude scaling as T^J.

$$A^{(0)}_{\text{Regge}}(S,T) = e^{i\pi S}\frac{\Gamma(-S)}{\Gamma(S+1)}T^{2S-2}$$

- **AdS corrections:** higher and higher order poles (quartic, seventh order,...)

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T,\delta)}{(S-\delta)^{3k+1}} + \frac{R^{(k)}_{3k}(T,\delta)}{(S-\delta)^{3k}} + \ldots + \frac{R^{(k)}_{1}(T,\delta)}{S-\delta} + \text{regular}\,, \quad \delta = 1,2,\ldots$$

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⇒ we expect logarithmic corrections of the form $(log\, T)^{\#}$, where the power of the log is fixed by the order of the poles.

The AdS Virasoro-Shapiro in the Regge limit

- We determined the AdS Virasoro-Shapiro amplitude in the Regge limit in terms of the CFT data of the exchanged operators (*N=4* SYM at strong coupling):

$$A_{Regge}(S,T) = :\hat{\mathcal{R}}(y): \frac{1+(-1)^J}{2\sin(\pi J)} \mathcal{C}(J) \frac{1}{\beta(J)} T^J \Big|_{J=J^*(S)}$$

OPE

Effect of the curvature of the background

Dimension of the operators in the leading Regge trajectory

- At **all orders** in *1/R*, we have explicit solutions as derivatives on the flat space result in the Regge limit.

- The derivatives produce powers of $\log T$.

Results to all orders

$$A^{LL}_{Regge}(S,T) = A^{(0)}_{Regge}(S,T) \times e^{-\frac{4}{3}\frac{S^2}{R^2}\log^3 T}$$

$$\times \log^2 T \left(\frac{16 S^3 \log^3 T}{5R^2} + 2\pi S^2 \cot(\pi S) + 4S^2 \psi^{(0)}(S) - \frac{4S}{3} - 2i\pi S^2 \right)$$

- Can resum all leading logs.
- Leading logs exponentiate!
- Can also resum subleading logs.

⇨ **All order** results in the limit of large *R*, *T* with $\frac{log^3\, T}{R^2}$ fixed!

Worldsheet: from the integrand to the right Regge limit

$$A(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G(S,T,z)$$

$$\hat{G}(S,T,z) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} G\left(S, \frac{T}{\epsilon}, \epsilon z\right)$$

- At each order in $1/\mathrm{R}$, $\widehat{\mathrm{G}}$ is a polynomial in $\mathbf{z}, \bar{\mathbf{z}}, \log|\mathbf{z}|$ (asymptotics of SVMPLs).
- Regge limit of the Virasoro-Shapiro to order $1/\mathrm{R}^4$ from the worldsheet perspective and it agrees with our prediction!

- Ambiguities when constructing integrands (since $S+T+U=0$).
  -We assume the integrand remains finite as we take $\epsilon \to 0$.
  -This is true for G_1, G_2 provided the ambiguities are chosen appropriately.
  -For this choice, we can take the Regge limit at the level of the integrand.

Worldsheet: from the Regge limit to the right integrand

- Now let's revers our logic: look for integrands that lead to the correct result in the Regge limit, to all orders in *1/R*.

$$\frac{1}{T^2}\int d^2z|z|^{-J-4}|1-z|^{-2T-2}F\left(S,-\log|z|\right)\simeq -F(S,\partial_J)e^{i\pi\frac{J}{2}}\frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)}T^J$$

- We get a functional equation:

$$-F(S,\partial_J)e^{i\pi\frac{J}{2}}\frac{\Gamma(-1-J/2)}{\Gamma(2+J/2)}T^J\bigg|_{J=J^*(S)} =: \hat{\mathcal{R}}\left(\partial_J\frac{1}{\beta(J)}\right) : \frac{1+(-1)^J}{2\sin(\pi J)}\mathcal{C}(J)\frac{1}{\beta(J)}T^J\bigg|_{J=J^*(S)}$$

- Families of solutions, for the world-sheet integral, that lead to the correct Regge behaviour **to all orders** in the curvature expansion.

- Up to ambiguities that integrate to zero in this limit.

Summary and conclusions

- **Compute string amplitudes on AdS** from
  - *AdS/CFT*
  - *Number Theory*
  - *Integrability*
  - *Worldsheet intuition*
- **Single valuedness** to understand/construct scattering amplitudes in AdS (as in flat space).
- AdS Virasoro-Shapiro amplitude as a «worldsheet» integral.
- Further step towards the worldsheet theory: High Energy and Regge limit.
- More direct comparison to a putative worldsheet description of strings on $AdS_5 \times S^5$.

Summary and conclusions

- High Energy limit:

  -Curvature corrections **exponentiate.**

  -The leading behavior is captured by a **bosonic model** describing scattering of classical strings on *AdS*.

- Regge limit:

  -Leading logs **exponentiate.**

  -Contact with integrability.

  -Manifest single-valuedness of the integrand around $z=0$.

→ Correct Regge limit + single valuedness at 1?

→ Ambiguities?

The *full* AdS Virasoro-Shapiro amplitude seems now within our grasp! ☺

Thanks for your attention!