Title: Non-vanishing of quantum geometric Whittaker coefficients

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

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Abstract:

We will discuss the functor of geometric Whittaker coefficients in the context of quantum geometric Langlands. We will prove that tempered twisted D-modules on the stack of G-bundles on a smooth projective curve have non-vanishing Whittaker coefficients. Roughly, this means that a certain natural subcategory of twisted D-modules on the stack of G-bundles can be controlled by the category of twisted D-modules on the Beilinson-Drinfeld affine Grassmannian. The proof will combine generalizations of representation-theoretic and microlocal methods from the preceding works of Faergeman-Raskin and Nadler-Taylor respectively.

X - proj. smooth conn curve / $\mathbb{C}$

G - simple red gp $G^\vee$ - Langlands dual gp

Bun_G - stack of G-bundles on X

1. Motivation

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G - simple red gp $\quad {}^L G$ - Langlands dual gp

Bun_G - stack of G-bundles on X

1. Motivation

Conj. (quantum GLC)

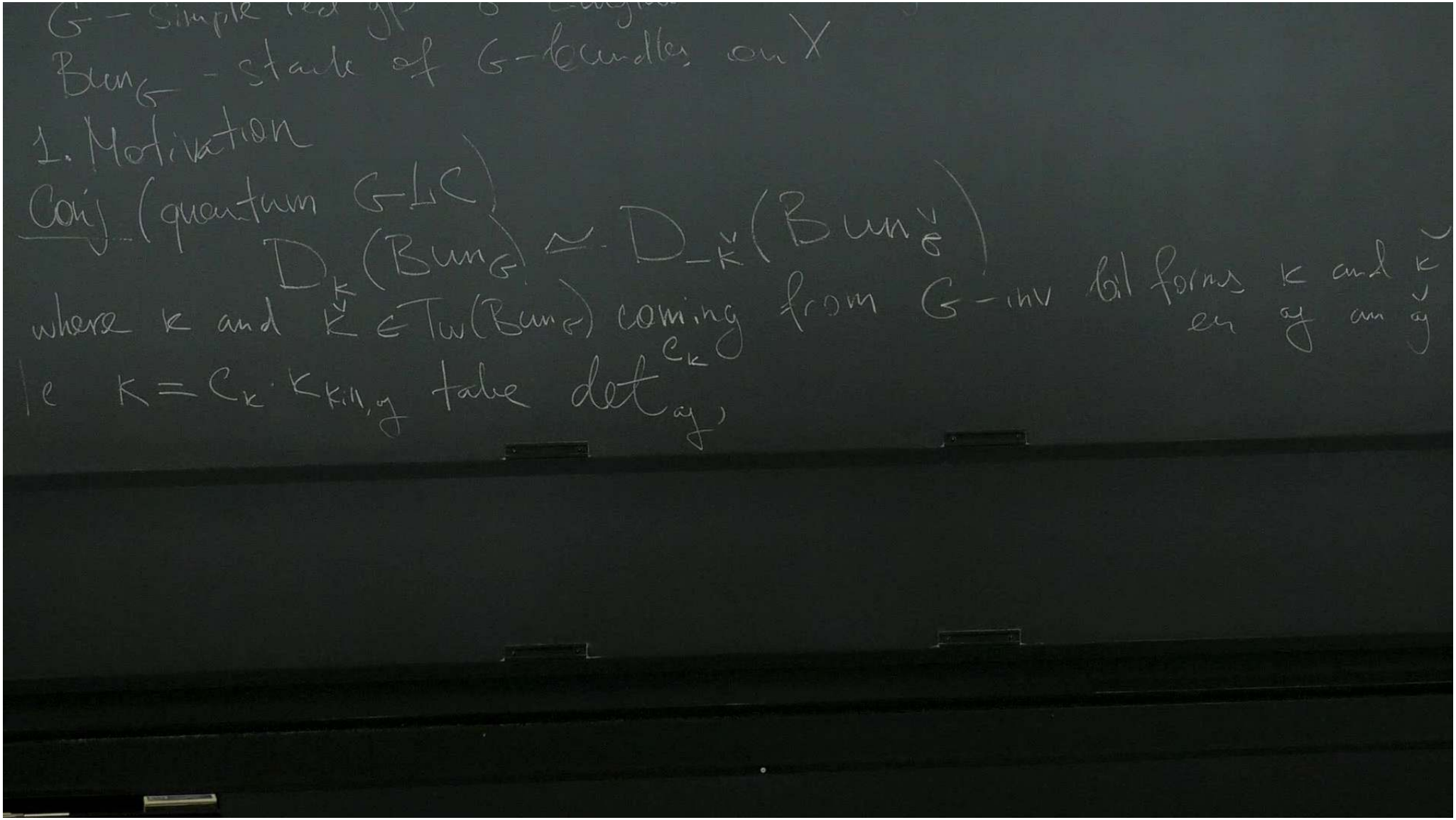
Bun_G - stack of G-bundles on X

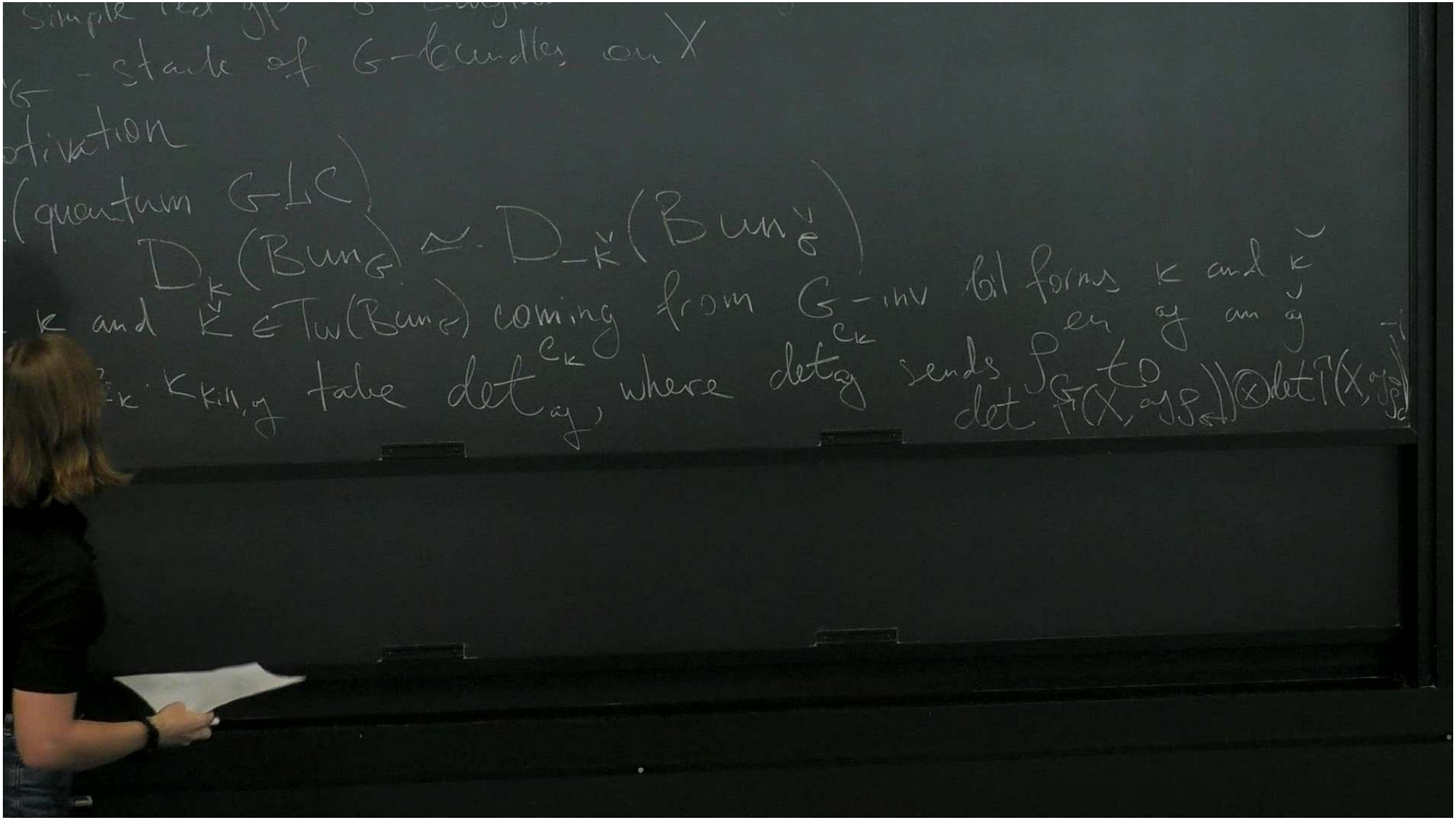
1. Motivation

Conj. (quantum Geometric Langlands)

$$D_{\kappa}(\mathrm{Bun}_G) \simeq D_{-\check{\kappa}}(\mathrm{Bun}_G^{\check{}})$$

where κ and $\check{\kappa} \in \mathrm{Tw}(\mathrm{Bun}_G)$ coming from G-inv bil


G - simple
Bun_G - stack of G-bundles on X
1. Motivation
Conj (quantum G-LC)
$D_k(\mathrm{Bun}_G) \simeq D_{-\check{k}}(\mathrm{Bun}_{\check{G}})$
where k and $\check{k} \in \mathrm{Tw}(\mathrm{Bun}_G)$ coming from G-inv bl forms k and $\check{k}$
on $\mathfrak{g}$ on $\check{\mathfrak{g}}$
$k = c_k \cdot \kappa_{\mathrm{Kill},\mathfrak{g}}$ take $\det^{c_k}_{\mathfrak{g}}$


- stack of G-bundles on X
otivation
(quantum GLC)
$D_\kappa(\mathrm{Bun}_G) \simeq D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}})$
κ and $\check{\kappa} \in \mathrm{Tw}(\mathrm{Bun}_G)$ coming from G-inv bil forms κ and $\check{\kappa}$
$\kappa_{\mathrm{Kill},\mathfrak{g}}$ take $\det_{\mathfrak{g}}^{c_\kappa}$, where $\det_{\mathfrak{g}}^{c_\kappa}$ sends ρ_G to
$\det \Gamma(X, \mathfrak{g}_{\rho_G}) \otimes \det \Gamma(X, \mathfrak{g}_{\rho_G})$

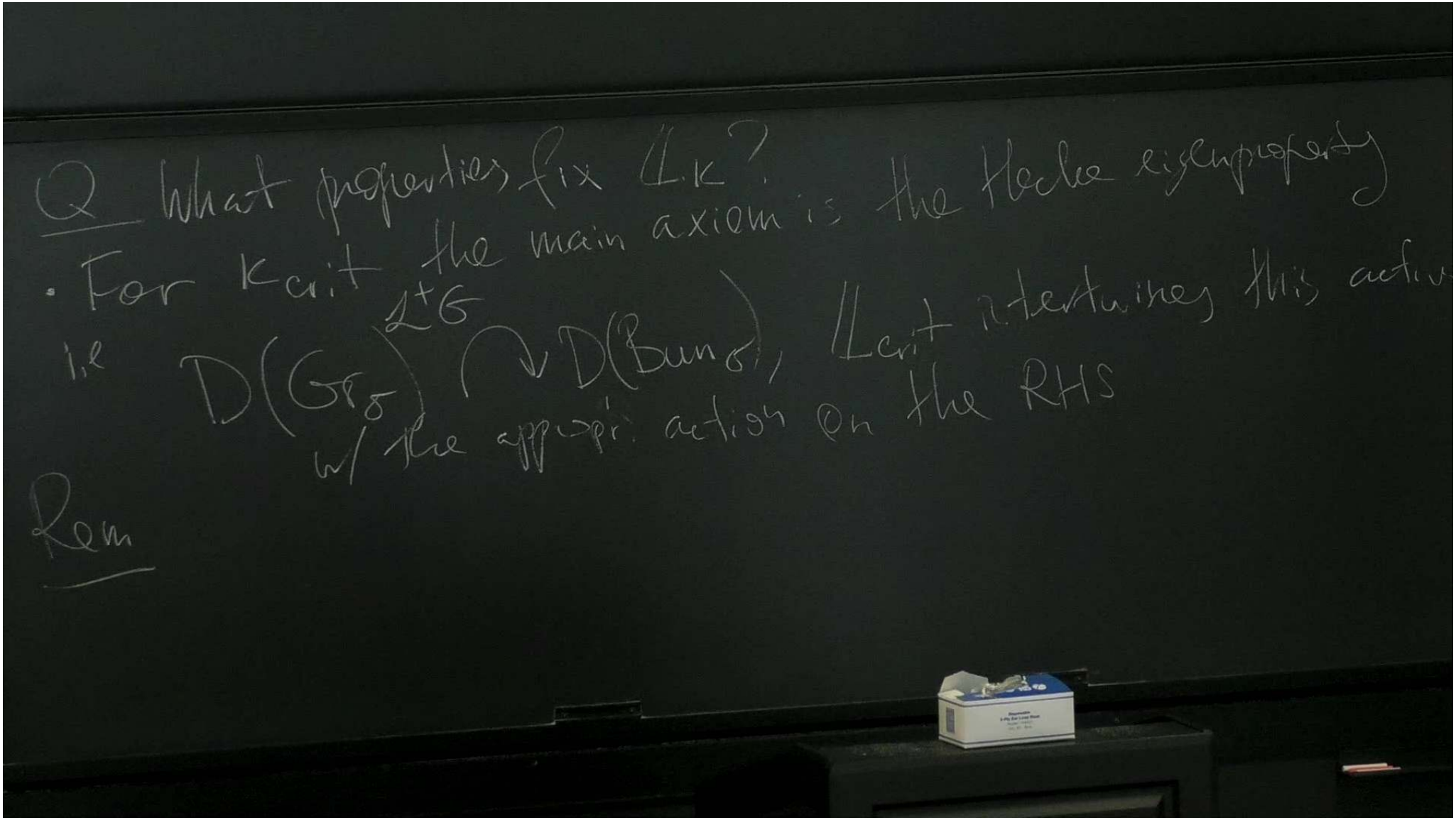
i.e. $\kappa = c_\kappa \cdot \kappa_{\mathrm{Kill},\mathfrak{g}}$ take $\det_{\mathfrak{g}}^{c_\kappa}$, where $\det_{\mathfrak{g}}^{c_\kappa}$ sends ρ_G to $\det \Gamma(X, \mathfrak{g}_{\rho_G}) \otimes \det$

For us: κ rational, i.e. $c_\kappa \in \mathbb{Q}$, $\kappa - \underbrace{\left(-\tfrac{1}{2}\right)\mathrm{Kill}_{\mathfrak{g}}}_{\kappa_{crit}}$ are non-deg.

Rem (relation between κ and $\check{\kappa}$)
roots of $\kappa - \kappa_{crit}$ to t and $\check{\kappa} - \check{\kappa}_{crit}$ to $\check{t} = t^*$ are dual bi forms.

For $\kappa = \kappa_{crit}$

Thm $D(\mathrm{Bun}_G) \overset{\mathbb{L}_{\kappa_{crit}}}{\text{"}\simeq\text{"}} \mathrm{QCoh}(LS_{\check{G}}(X))$



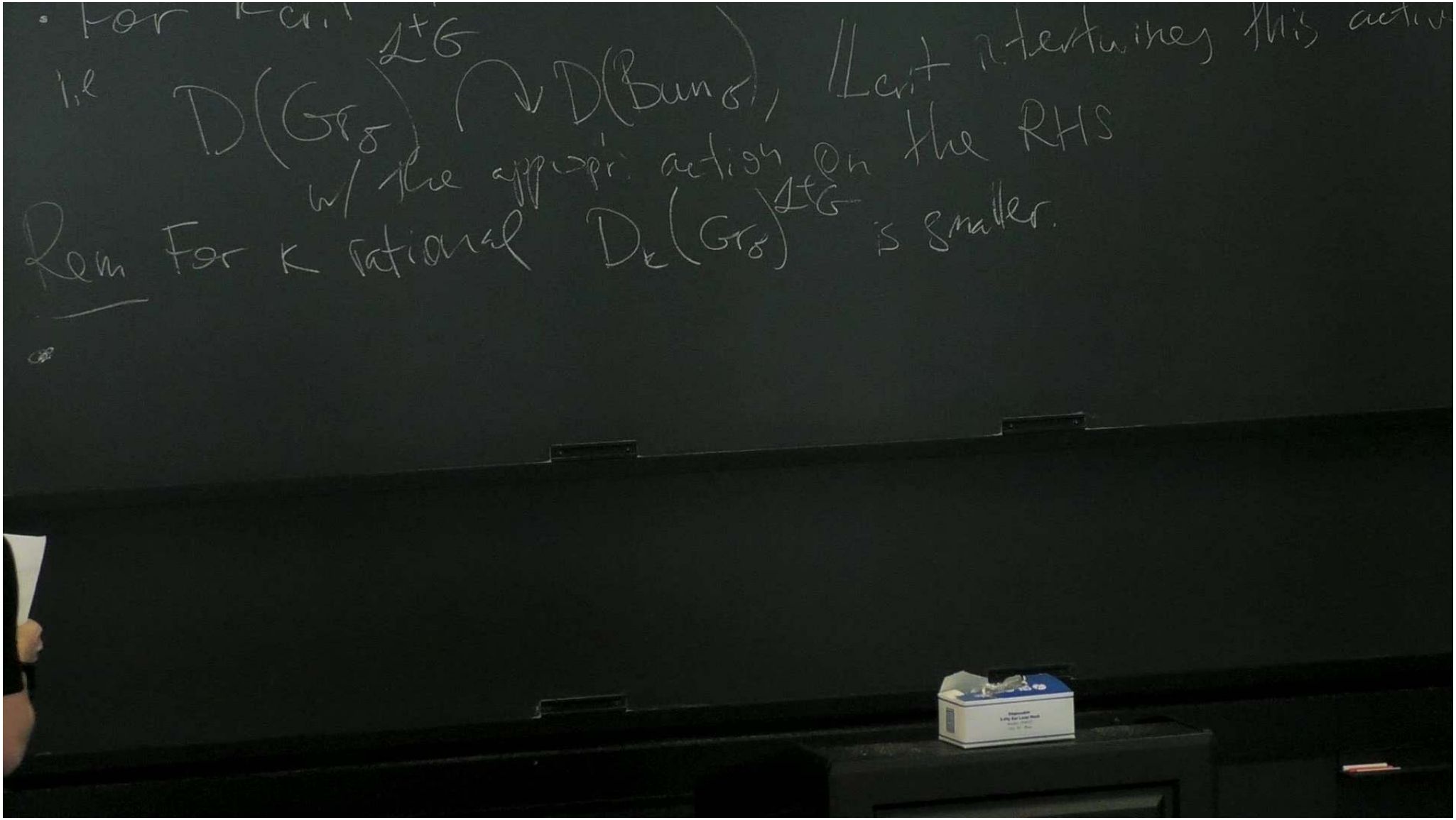
Q What properties fix $\mathbb{L}_\kappa$?

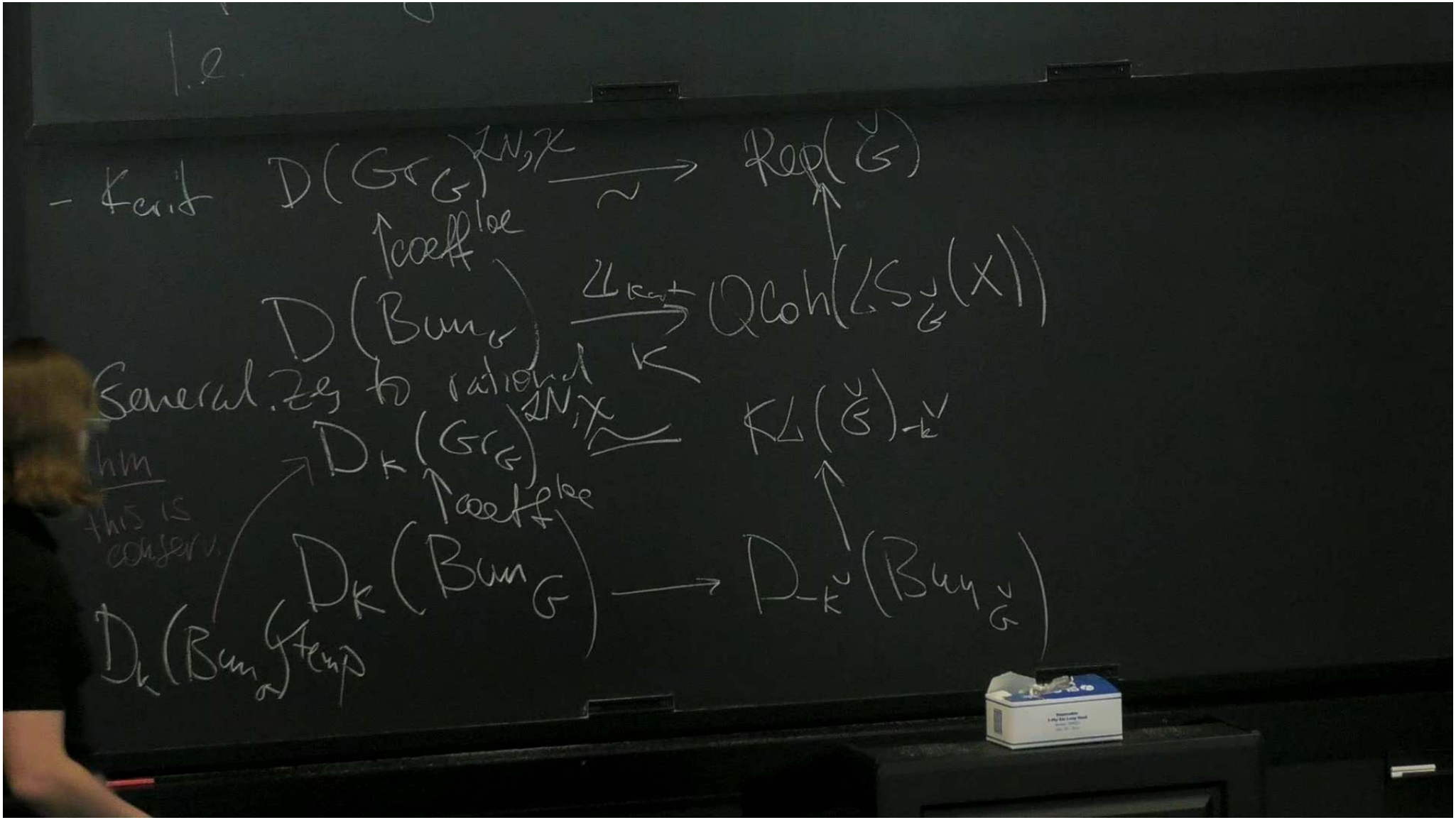
- For κ_{crit} the main axiom is the Hecke eigenproperty

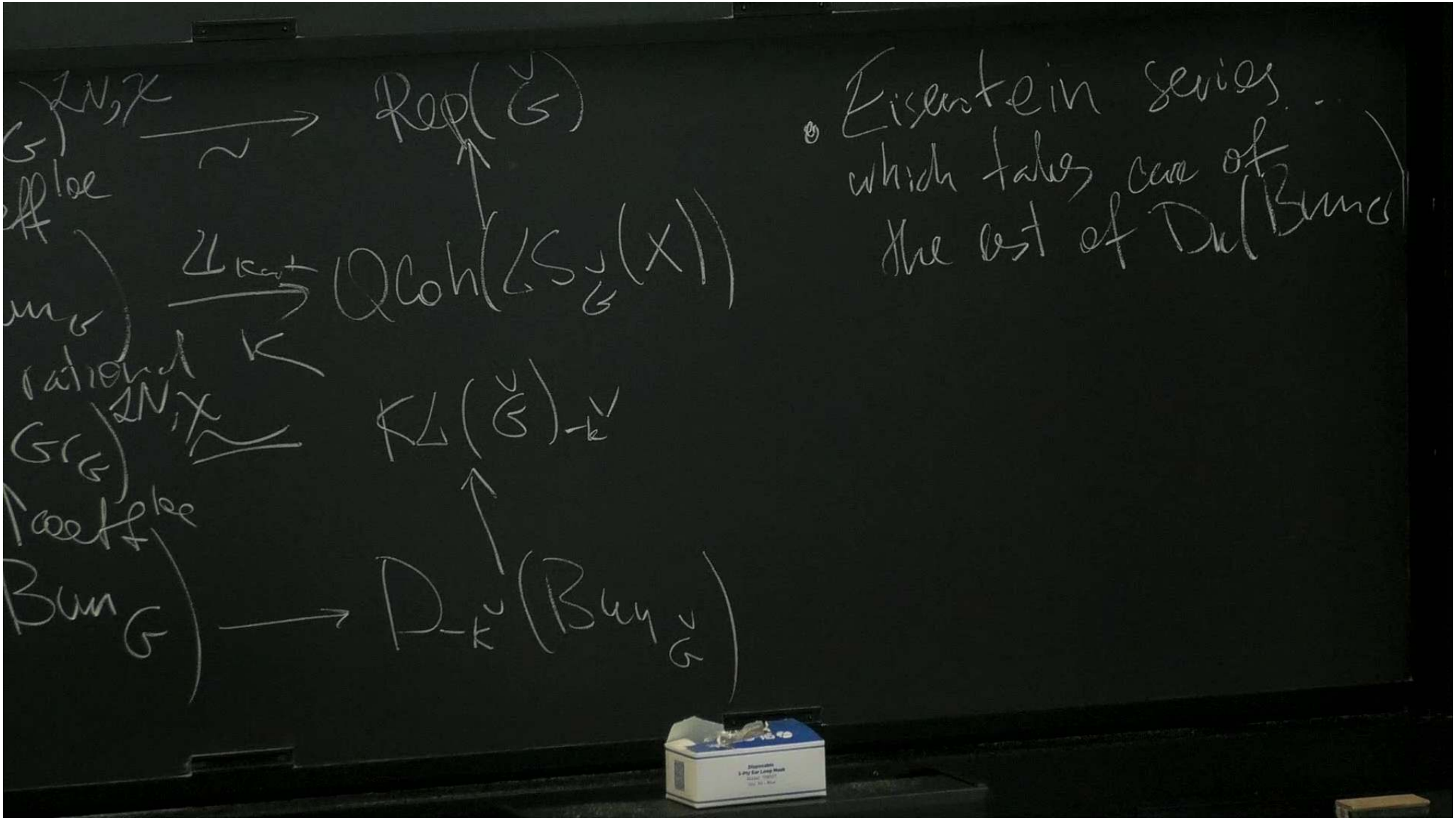
i.e $D(Gr_G)^{\mathcal{L}^+G} \curvearrowright D(Bun_G)$, $\mathbb{L}_{crit}$ intertwines this activ

w/ the appropr. action on the RHS

Rem


• For κ_{crit}
i.e $D(Gr_\sigma)^{\mathcal{L}^+G} \curvearrowright D(Bun_\sigma)$, $\mathbb{L}_{crit}$ intertwines this activ
w/ the appropr. action on the RHS
Rem For κ rational $D_\kappa(Gr_\sigma)^{\mathcal{L}^+G}$ is smaller.


i.e.
- Kcrit
D(Gr_G)^{LN,χ} ~ Rep(Ǧ)
coeff^loc
D(Bun_G) → QCoh(LS_Ǧ(X))
Generalizes to rational K
D_K(Gr_G)^{LN,χ} ≃ KL(Ǧ)_{-K^∨}
coeff^loc
D_K(Bun_G) → D_{-K^∨}(Bun_Ǧ)
D_K(Bun_G)^{temp}
this is conserv.


Rep(Ǧ)
QCoh(LS_Ǧ(X))
KL(Ǧ)_{-k^∨}
D_{-k^∨}(Bun_Ǧ)
• Eisenstein series
which takes care of
the rest of D_k(Bun_G)

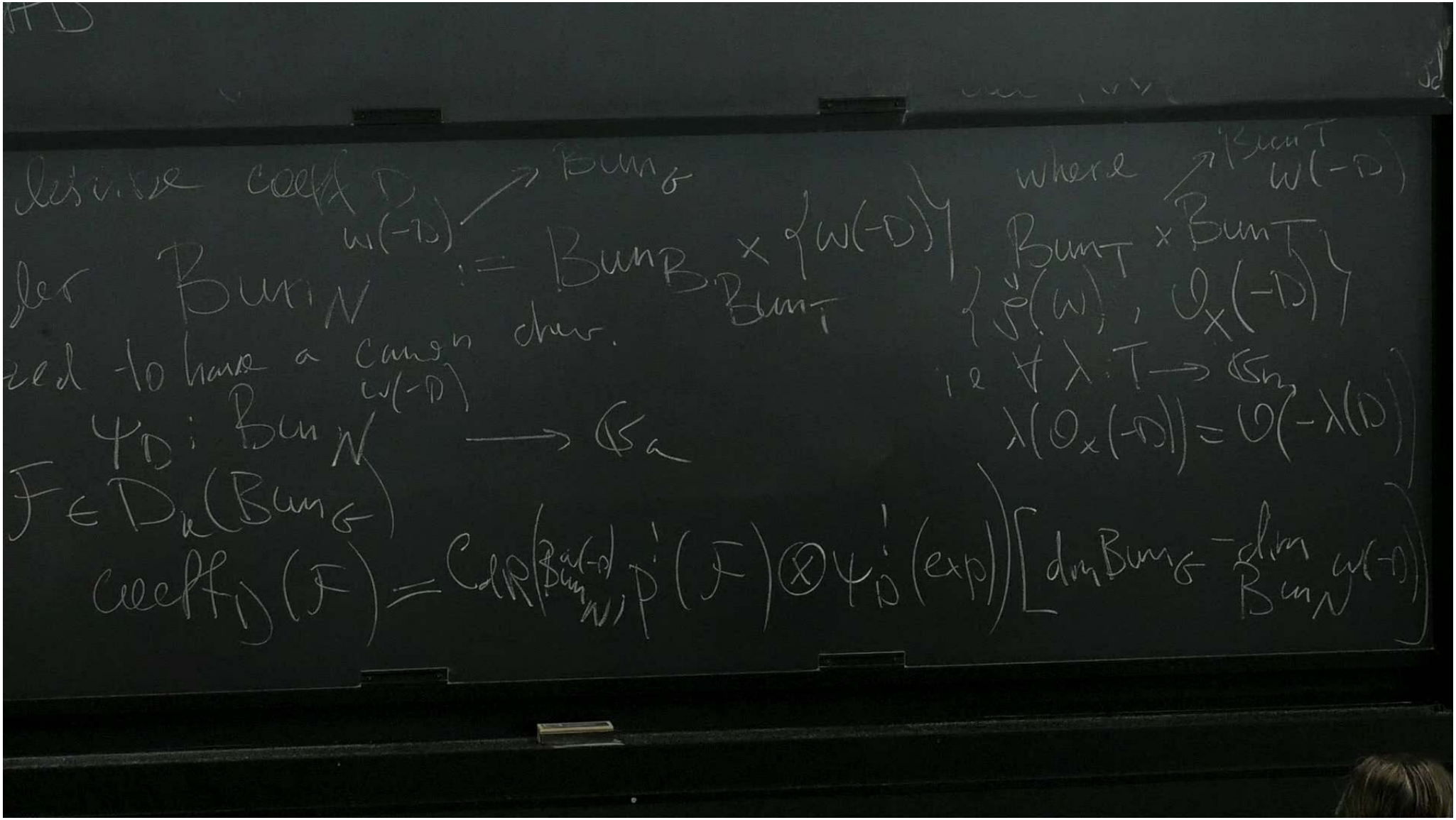
2 Whittaker cases.

Rem Compact gen of $D_k(Gr_G)^{\lambda_N, \chi}$ are numbered by $\check{\Lambda}^+$-val div on X

Main thm $\Leftrightarrow$

Thm 1 $F \in D_k(Bun_G)^{temp}$ $\exists$ $\check{\Lambda}^+$-val. div D

s.t. $Hom(V_D, -) \circ coeff^{loc}(F) \neq 0$



describe coeff$_D$ $\to \mathrm{Bun}_G$

der $\mathrm{Bun}_N^{w(-D)} := \mathrm{Bun}_B \times_{\mathrm{Bun}_T} \{w(-D)\}$

where $\mathrm{Bun}_T \times \mathrm{Bun}_T$ $\to \mathrm{Bun}_T$ $w(-D)$

$\{\check{\rho}(w), \mathcal{O}_X(-D)\}$

ized to have a canon char.

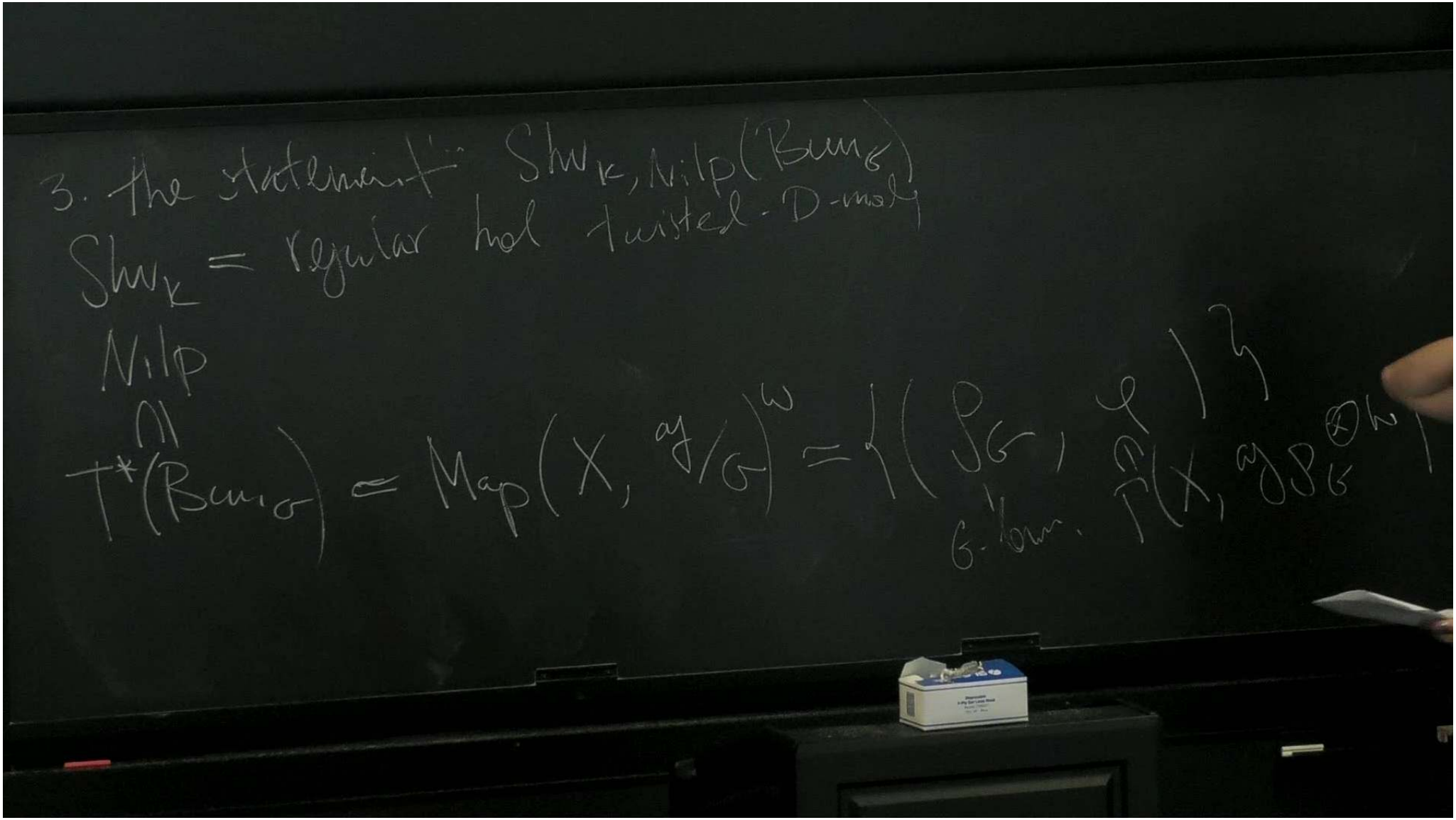
i.e. $\forall \lambda : T \to \mathbb{G}_m$

$\lambda(\mathcal{O}_X(-D)) = \mathcal{O}(-\lambda(D))$

$\psi_D : \mathrm{Bun}_N^{w(-D)} \longrightarrow \mathbb{G}_a$

$\mathcal{F} \in D_k(\mathrm{Bun}_G)$

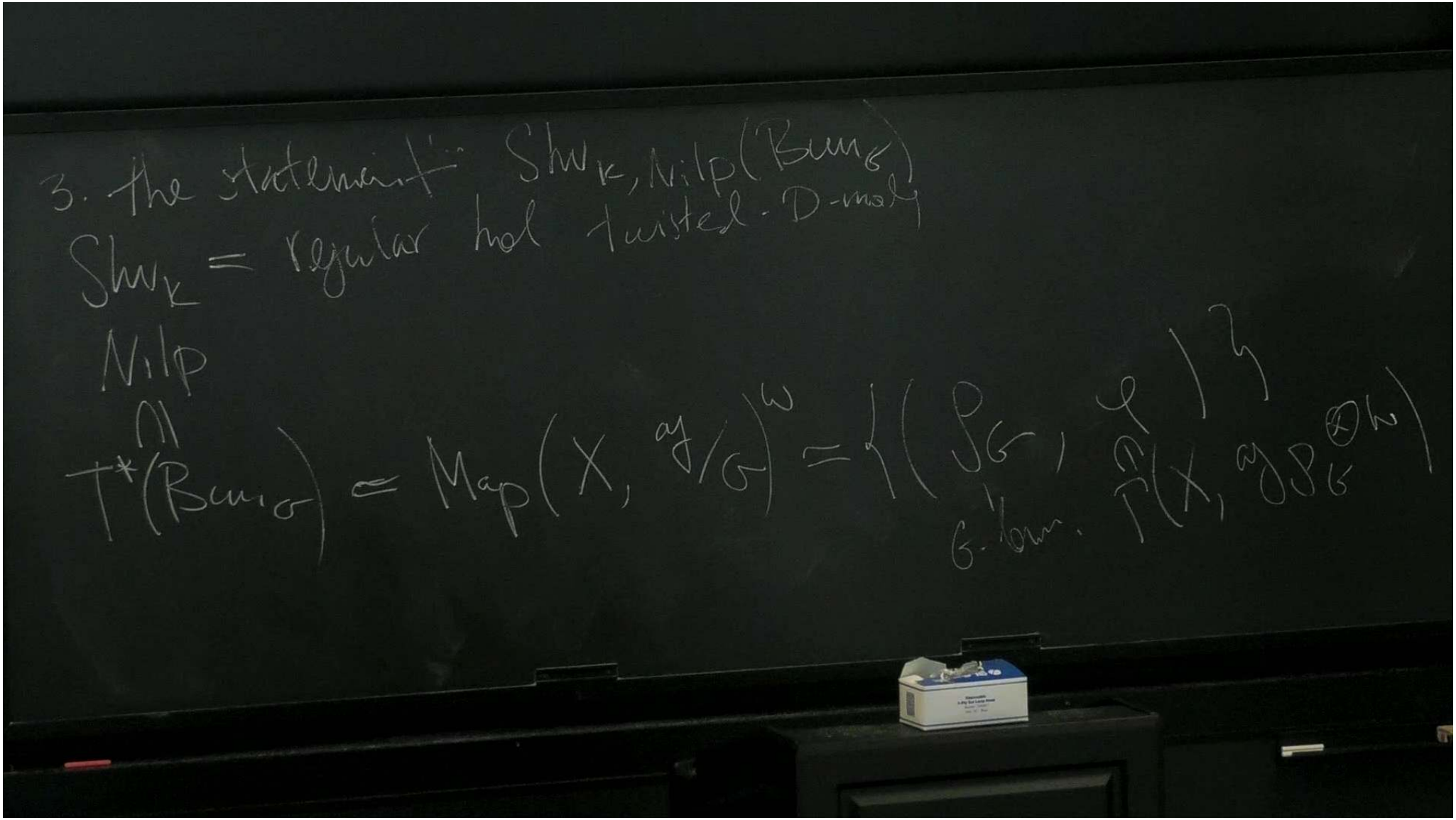
$$\mathrm{coeff}_D(\mathcal{F}) = C_{dR}\left(\mathrm{Bun}_N^{w(-D)}, p^!(\mathcal{F}) \otimes \psi_D^!(\exp)\right)\left[\dim \mathrm{Bun}_G - \dim \mathrm{Bun}_N^{w(-D)}\right]$$



3. the statement: $Shv_{K,Nilp}(Bun_G)$

Shv_K = regular hol twisted-D-mod

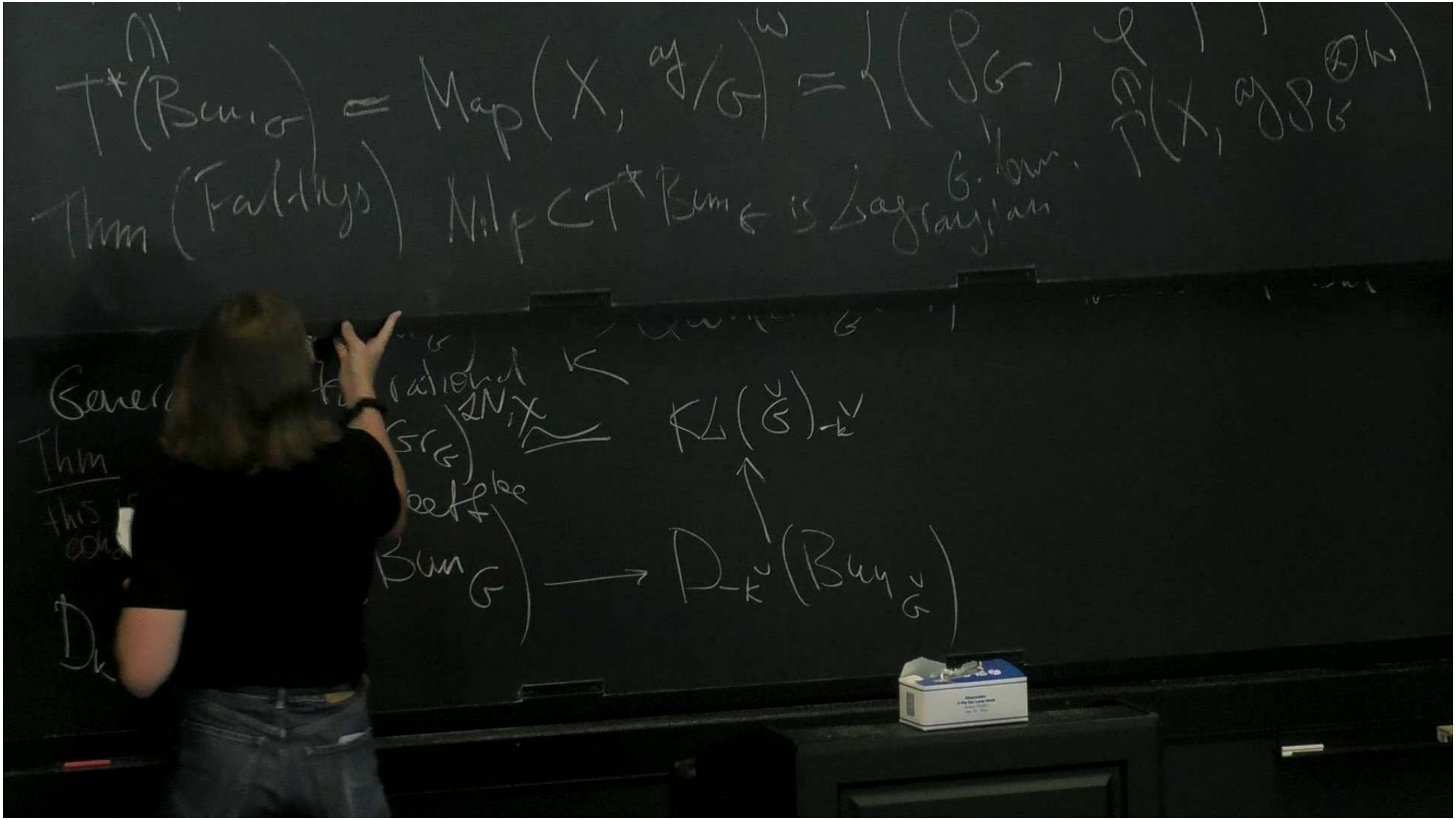
$$\begin{array}{c} Nilp \\ \cap \\ T^*(Bun_G) \simeq Map(X, \mathfrak{g}/G)^{\omega} \simeq \{(\underset{G\text{-bun.}}{\rho_G}, \underset{\Gamma(X, \mathfrak{g}_{\rho_G}^{\otimes \omega})}{\varphi})\} \end{array}$$

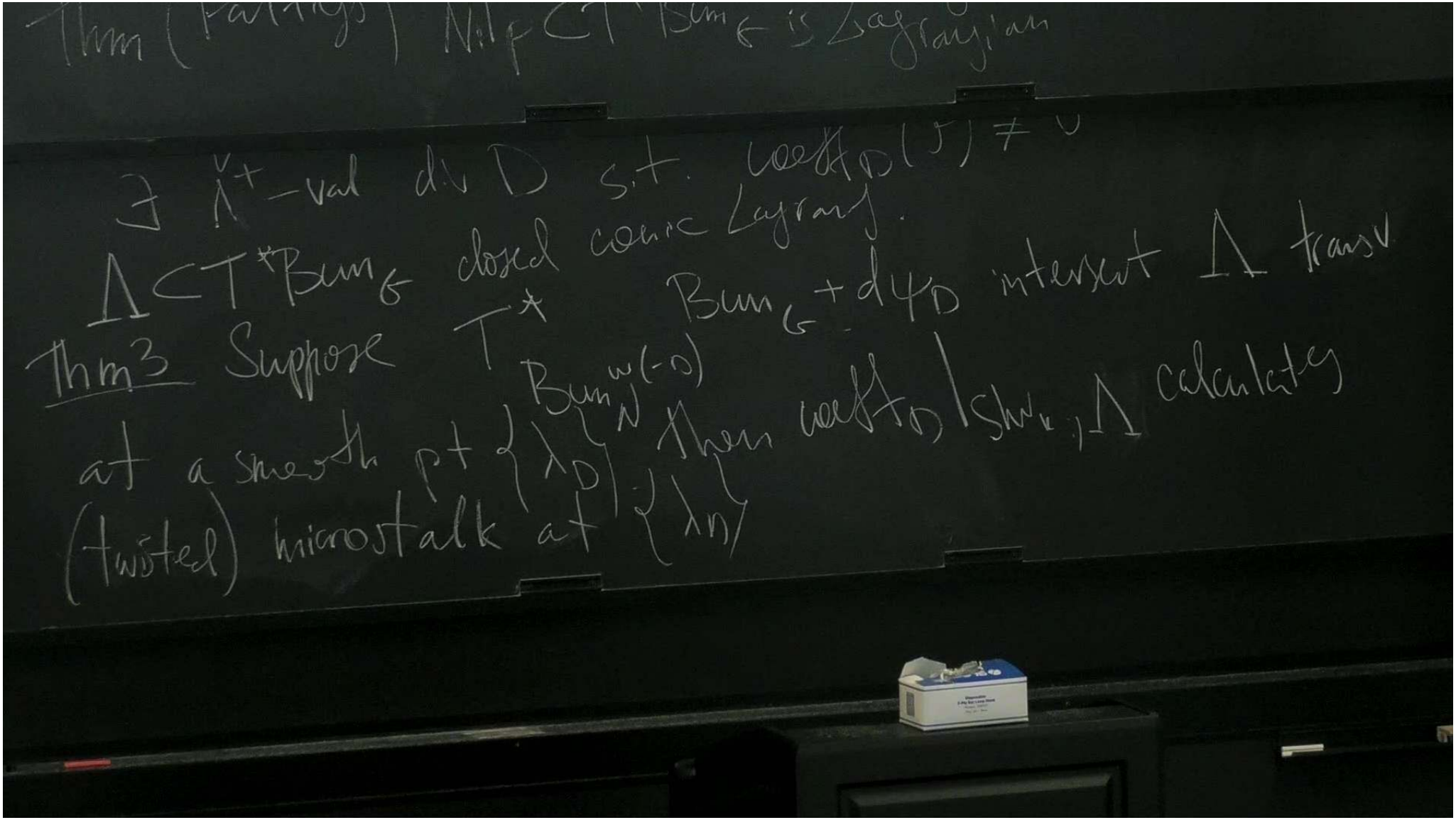


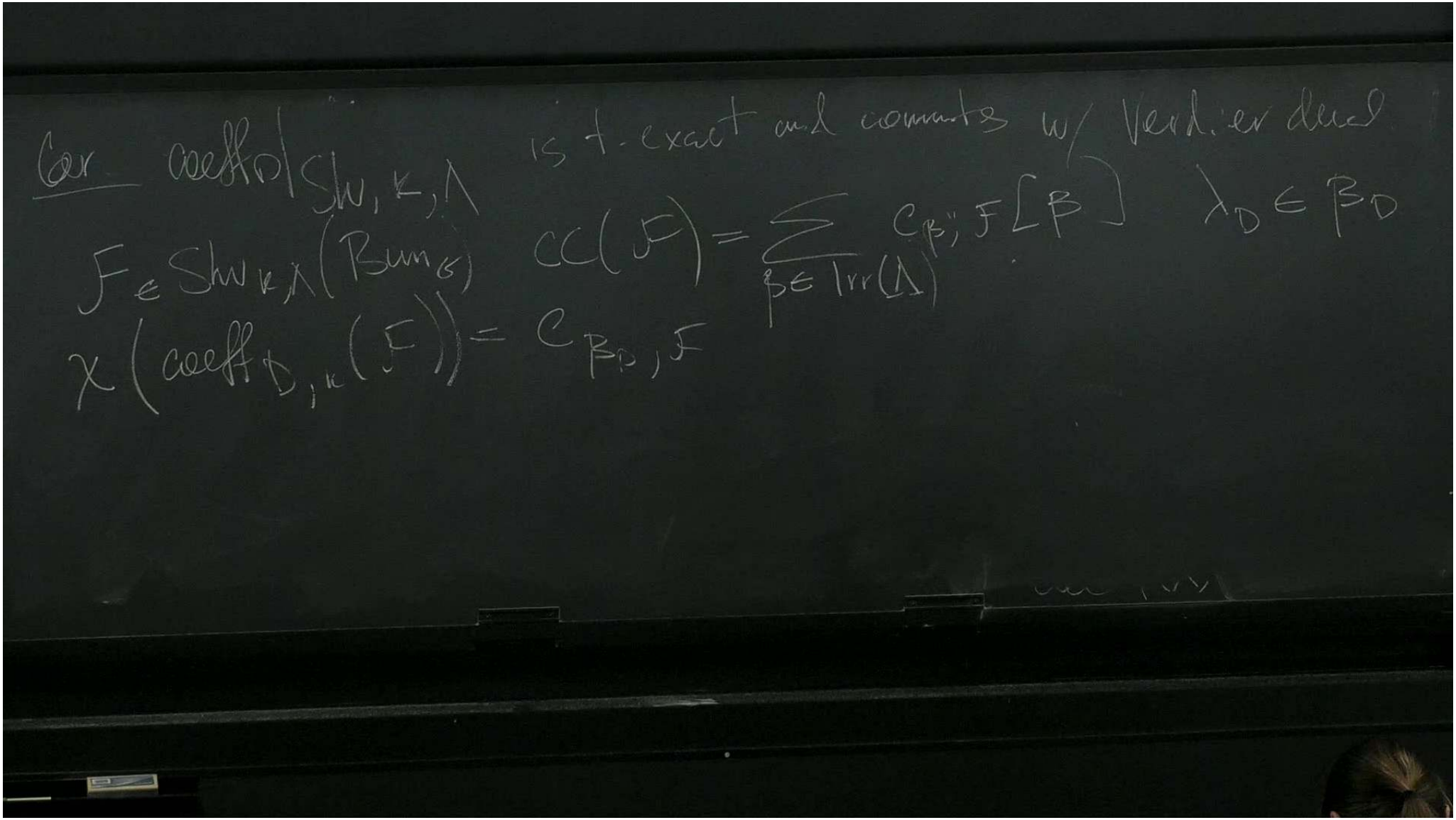
3. the statement: $Shv_{\kappa, Nilp}(Bun_G)$

Shv_κ = regular hol twisted-D-mod

$$\begin{array}{c} Nilp \\ \cap \\ T^*(Bun_G) \simeq Map(X, \mathfrak{g}/G)^\omega \simeq \{(\underset{G\text{-bun.}}{\rho_G}, \underset{\Gamma(X, \mathfrak{g}_{\rho_G}^{\otimes \omega})}{\varphi})\} \end{array}$$


Thm (Faltings) Nilp ⊂ T* Bun_G is Lagrangian
G-bun.
Thm
KL(Ǧ)
D_{-k^∨}(Bun_{Ǧ})


Thm (Partial)
$\exists$ $\check{\lambda}^+$-val div D s.t. $\mathrm{coeff}_D(J) \neq 0$
$\Lambda \subset T^*\mathrm{Bun}_G$ closed conic Lagrang.
Thm3 Suppose $T^*_{\mathrm{Bun}^{w(-0)}_N}\mathrm{Bun}_G + d\psi_D$ intersect Λ transv
at a smooth pt $\{\lambda_D\}$. Then $\mathrm{coeff}_D|_{\mathrm{Sh}_{\mathcal{L}},\Lambda}$ calculates
(twisted) microstalk at $\{\lambda_D\}$


Cor coeff_D|_{Sh_{N,K,Λ}} is t-exact and commutes w/ Verdier duals
F ∈ Sh_{N,K,Λ}(Bun_G)
CC(F) = Σ_{β∈Irr(Λ)} c_{β;F}[β]
λ_D ∈ B_D
χ(coeff_{D,K}(F)) = c_{B_D,F}

Ex $G = SL_2$ $\qquad$ $\mathrm{Nilp}^{reg} \hookrightarrow \mathrm{Nilp}$

$\cdot\ T^*\mathrm{Bun}_G + d\psi_D =: \mathrm{Kos}_D$

$\mathrm{Bun}_N^{\omega(-D)}$

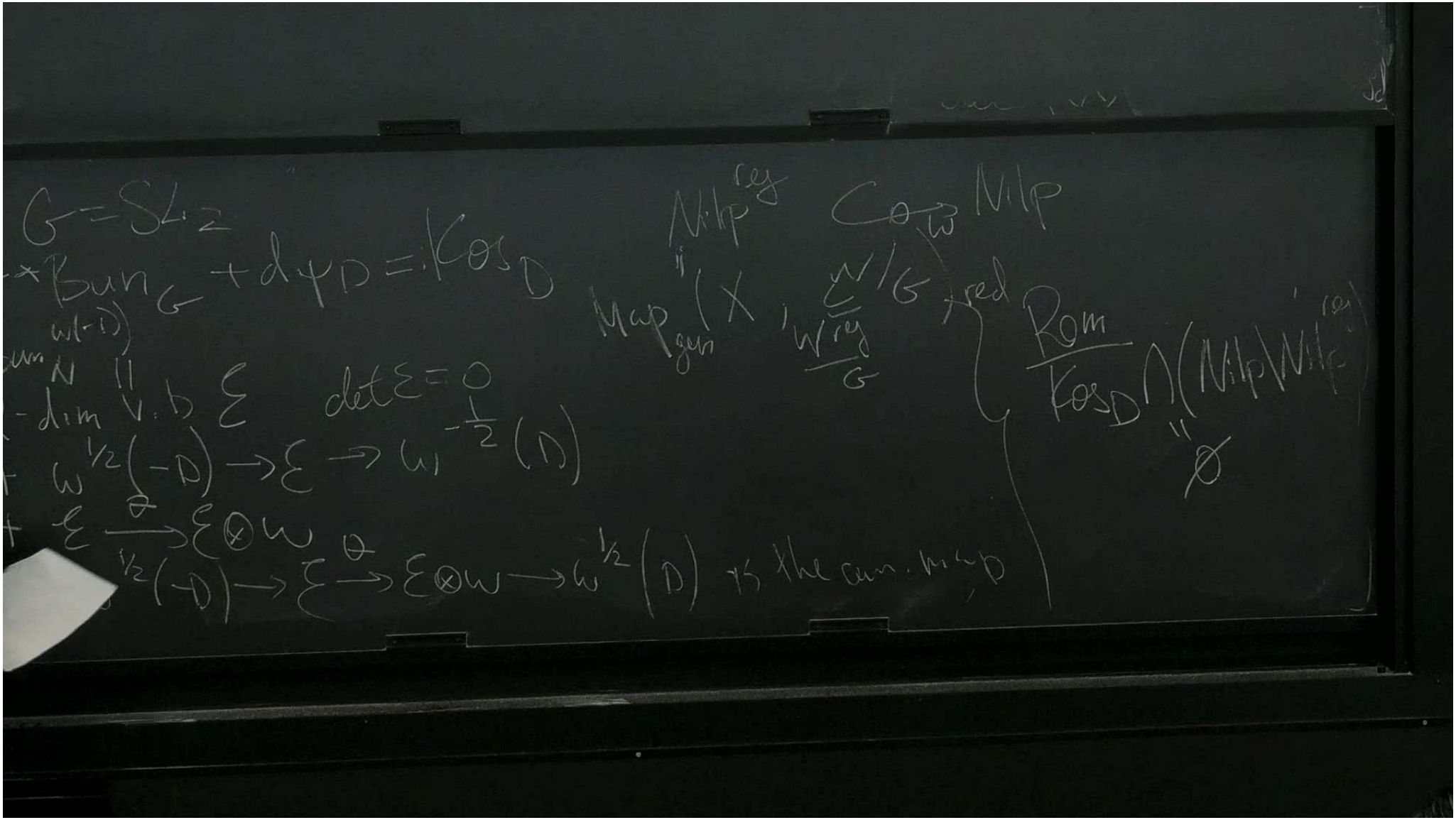
$\|$

2-dim v.b $\mathcal{E}$ $\quad \det \mathcal{E} = 0$

$+\ \omega^{1/2}(-D) \to \mathcal{E} \to \omega^{-\frac{1}{2}}(D)$

$+\ \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega$

s.t. $\omega^{1/2}(-D) \to \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega \to \omega^{1/2}(D)$ is the can. map



$G = SL_2$

$\ast \mathrm{Bun}_G + d\psi_D =: \mathrm{Kos}_D$

$\mathrm{Nilp}^{reg} \subset \mathrm{Nilp}$

$\mathrm{Map}_{gen}(X, \underset{\underset{G}{\underline{W^{reg}}}}{\overset{W/G}{\subseteq}})^{red}$

$\mathrm{Rom}_{\mathrm{Kos}_D} \cap (\mathrm{Nilp} \setminus \mathrm{Nilp}^{reg}) = \emptyset$

$\mathrm{Sum}_N^{\omega(-D)}$

$\|$

2-dim v.b. $\mathcal{E}$ $\quad \det \mathcal{E} = 0$

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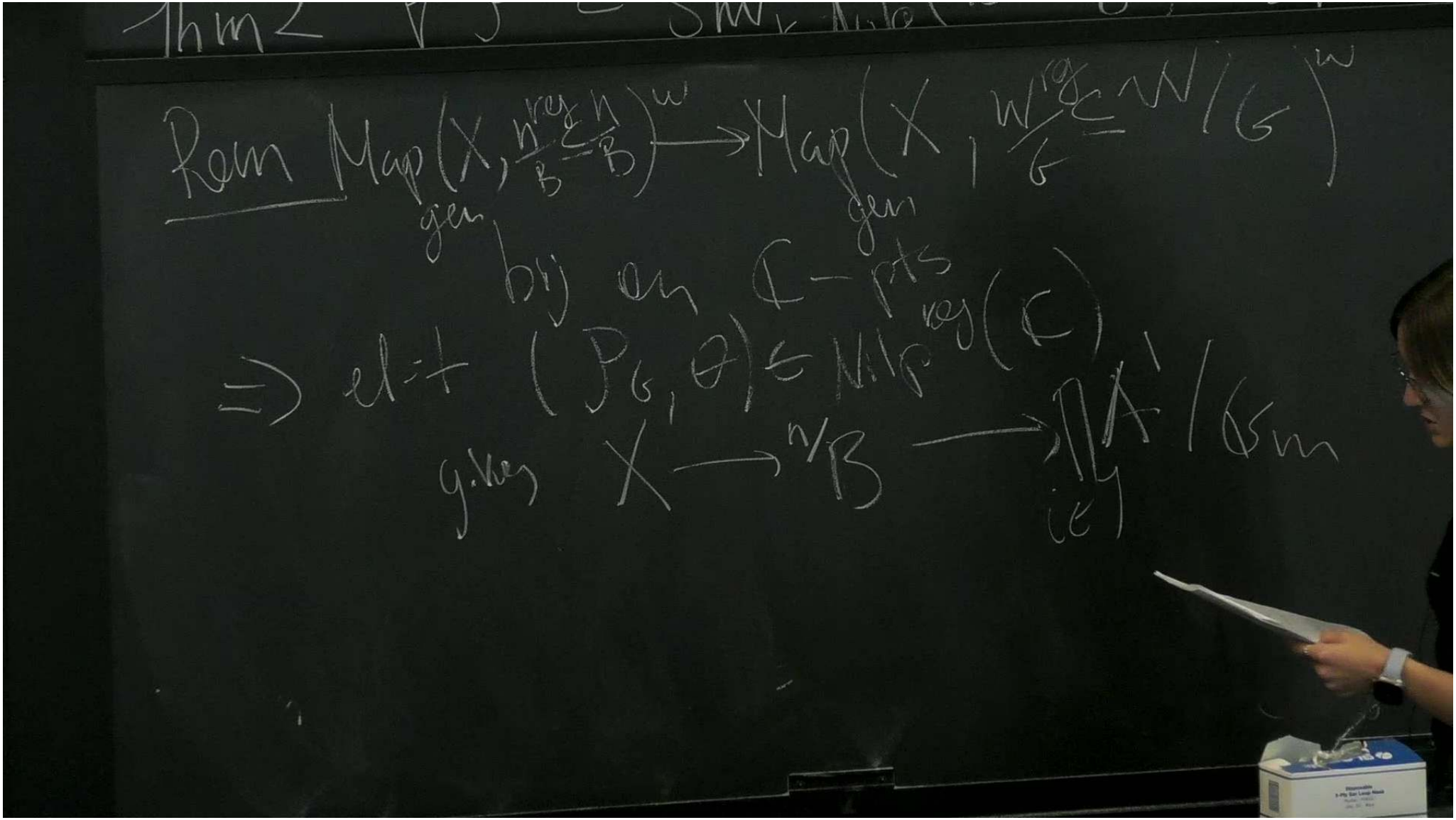


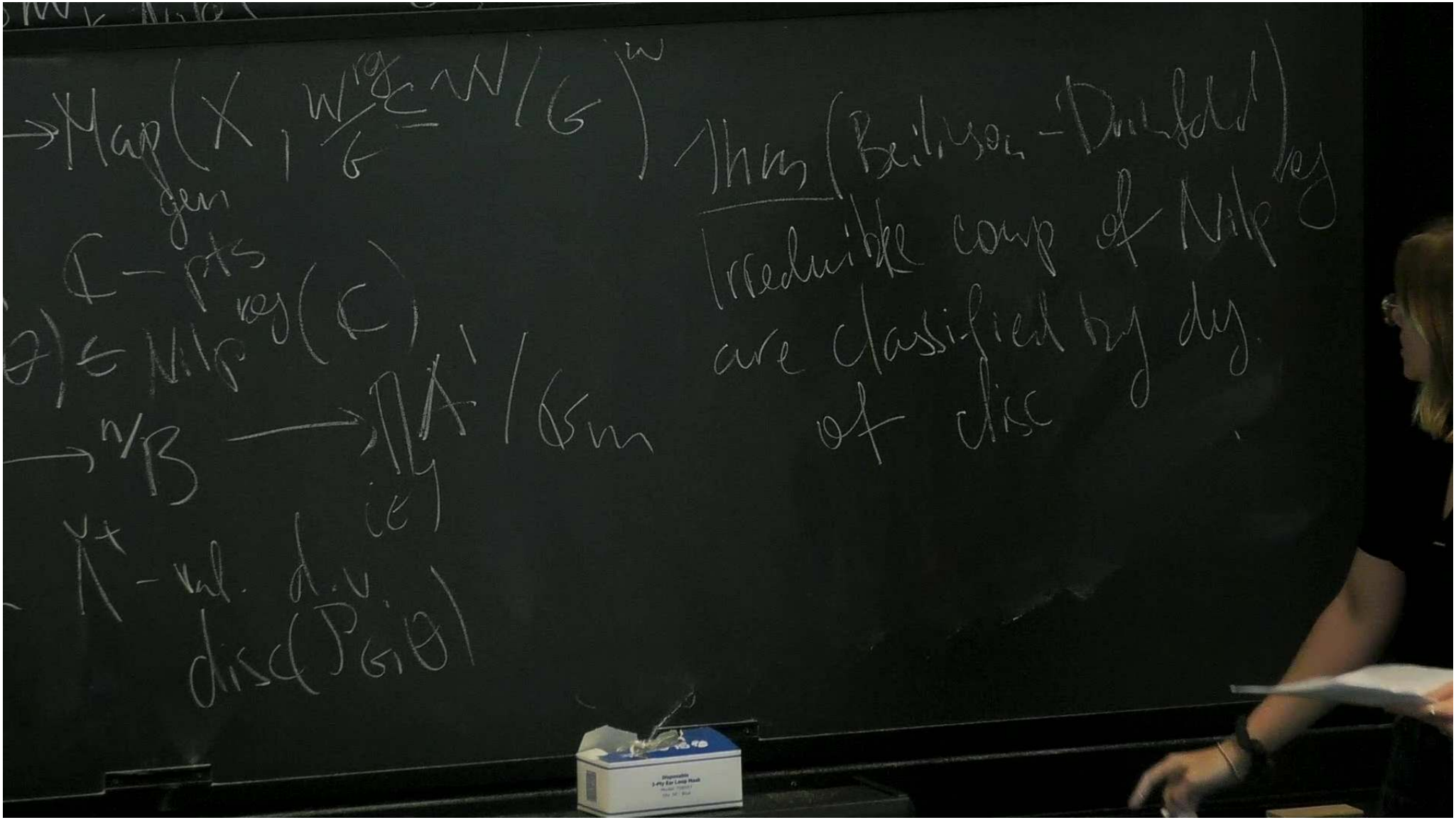
Thm 2 $\forall J \subset SW_k$ ...

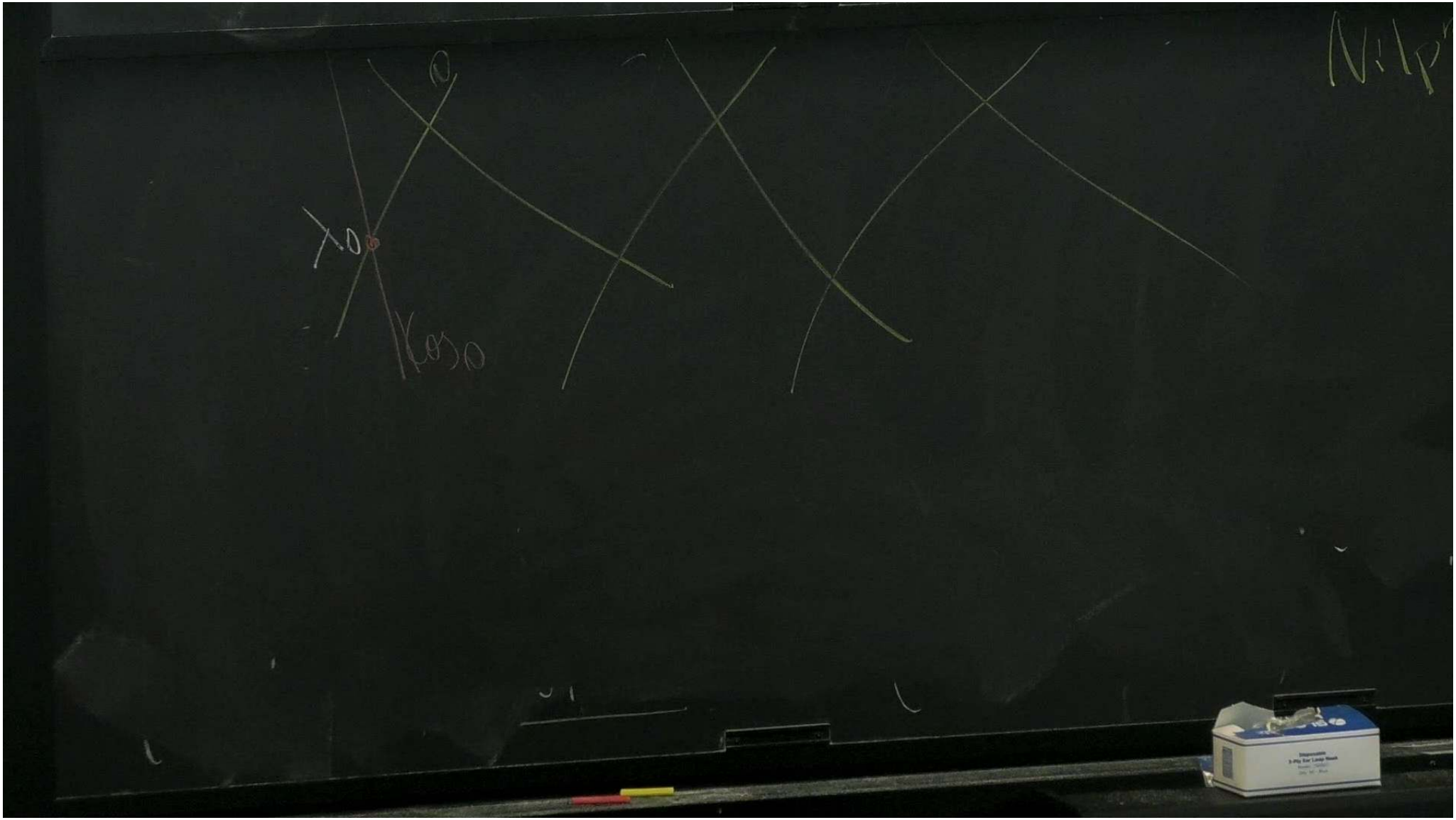
Rem $\mathrm{Map}_{gen}\left(X, \frac{n^{reg}}{B} \subseteq \frac{n}{B}\right)^{w} \longrightarrow \mathrm{Map}_{gen}\left(X, \frac{W^{reg}}{G} \subseteq W/G\right)^{w}$

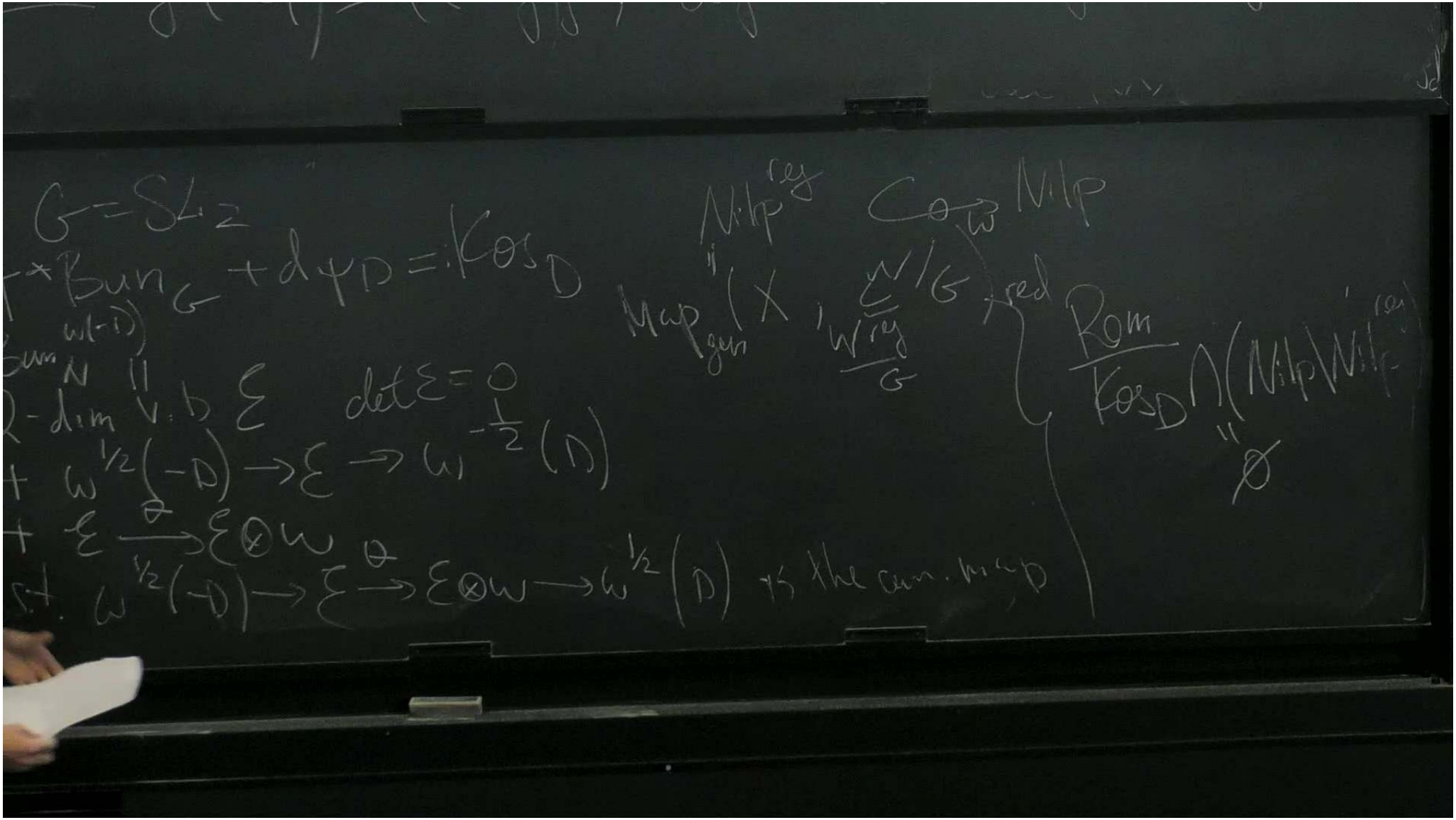
bij on $\mathbb{C}$-pts

$\Rightarrow$ el-t $(P_G, \theta) \in \mathrm{Nilp}^{reg}(\mathbb{C})$

gives $X \longrightarrow n/B \longrightarrow \prod_{i \in I} \mathbb{A}^1/\mathbb{G}_m$


Thm (Beilinson-Drinfeld)
Irreducible comp of Nilp^reg
are classified by dg.
of disc
gen
C-pts
disc(P_G, θ)





$G = SL_2$

$T^*\mathrm{Bun}_G + d\psi_D =: \mathrm{Kos}_D$

$\mathrm{Bun}_N^{\omega(-D)}$

$\|$

2-dim v.b. $\mathcal{E}$ $\quad \det \mathcal{E} = 0$

$+ \ \omega^{1/2}(-D) \to \mathcal{E} \to \omega^{-\frac{1}{2}}(D)$

$+ \ \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega$

s.t. $\omega^{1/2}(-D) \to \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega \to \omega^{1/2}(D)$ is the can. map

$\mathrm{Nilp}^{reg} \subset \mathrm{Nilp}$

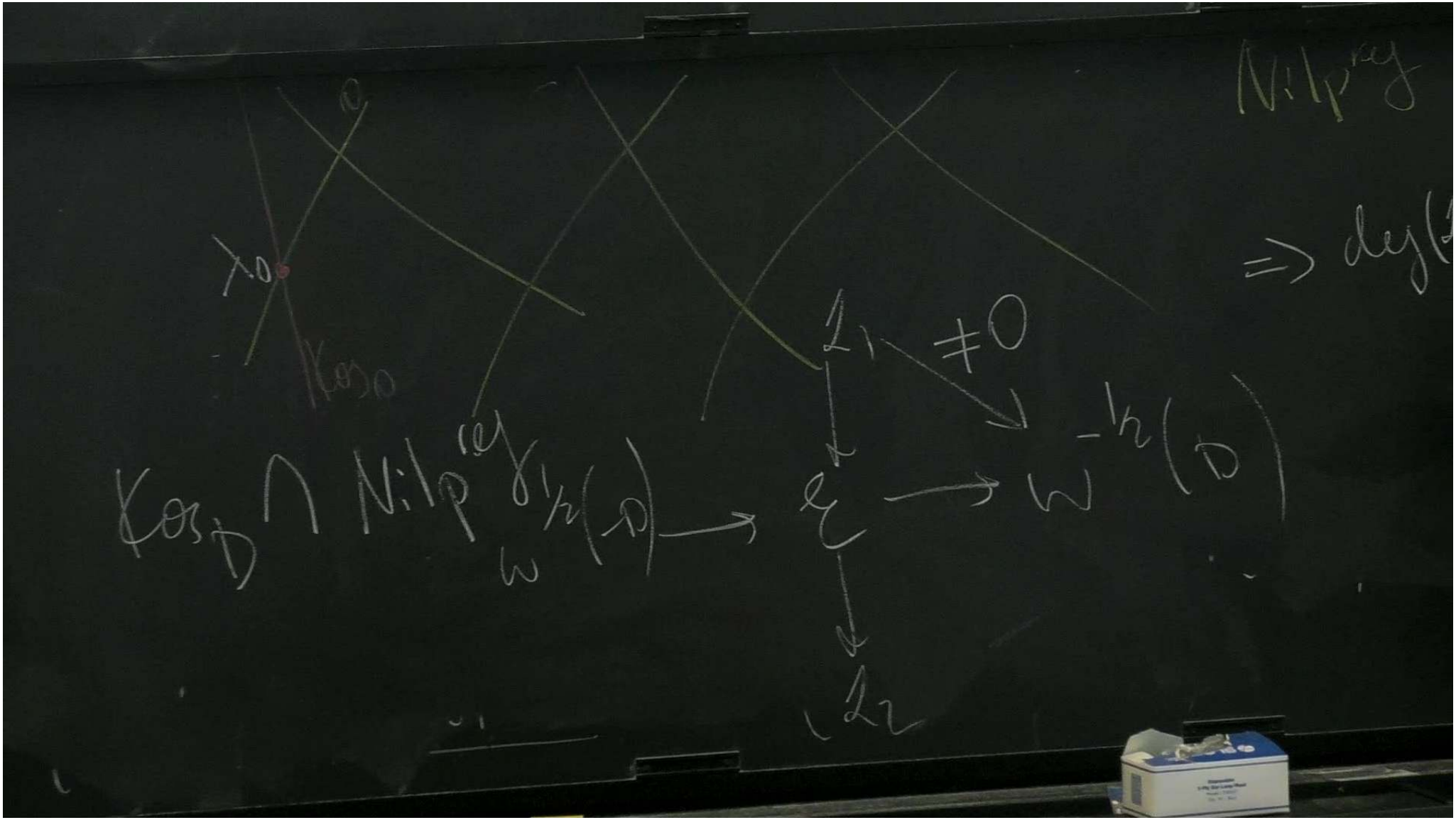
$\|$

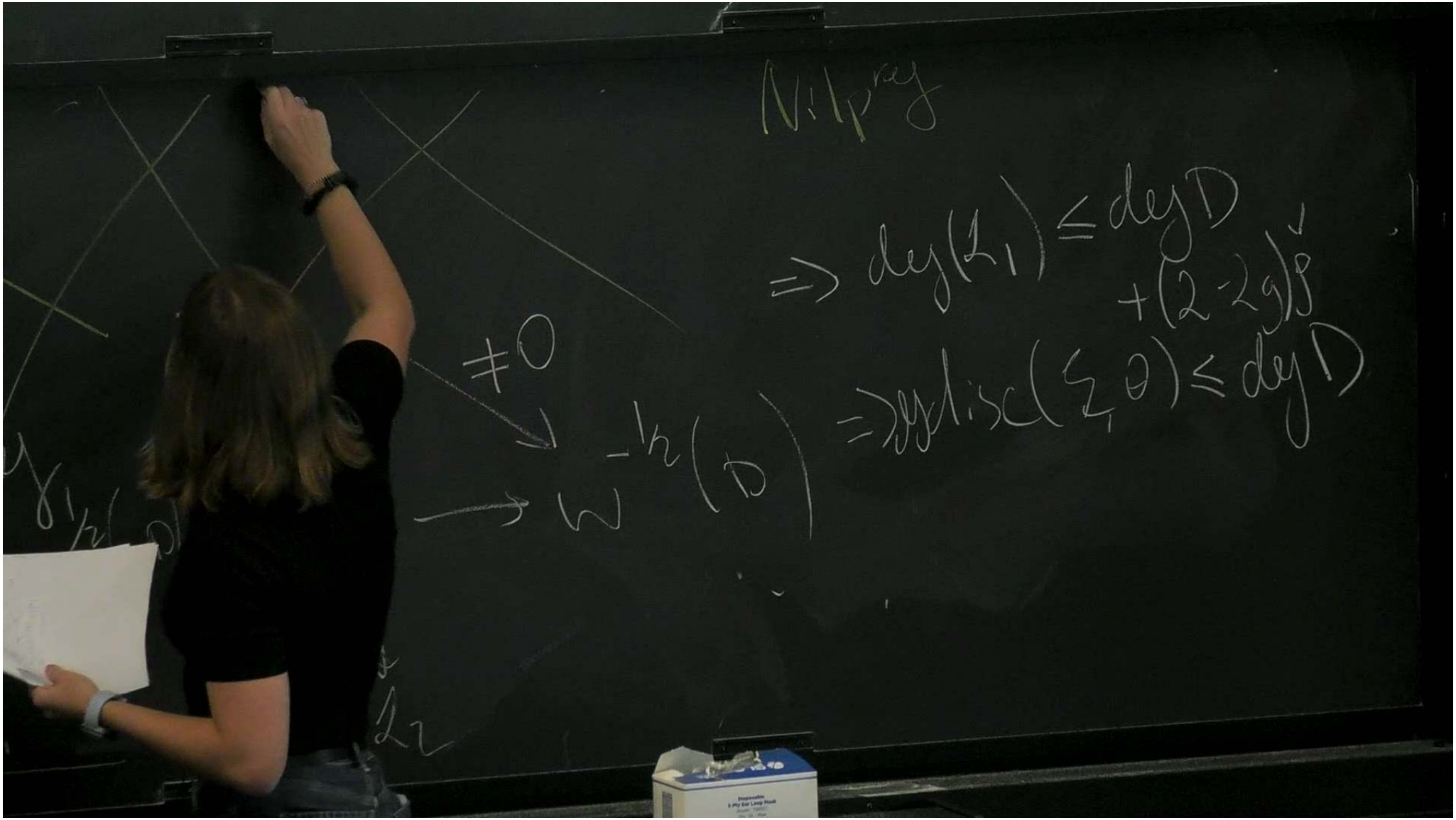
$\mathrm{Map}_{gen}(X, \underset{\underset{G}{\mathfrak{W}^{reg}}}{\mathfrak{W}/G})^{red}$

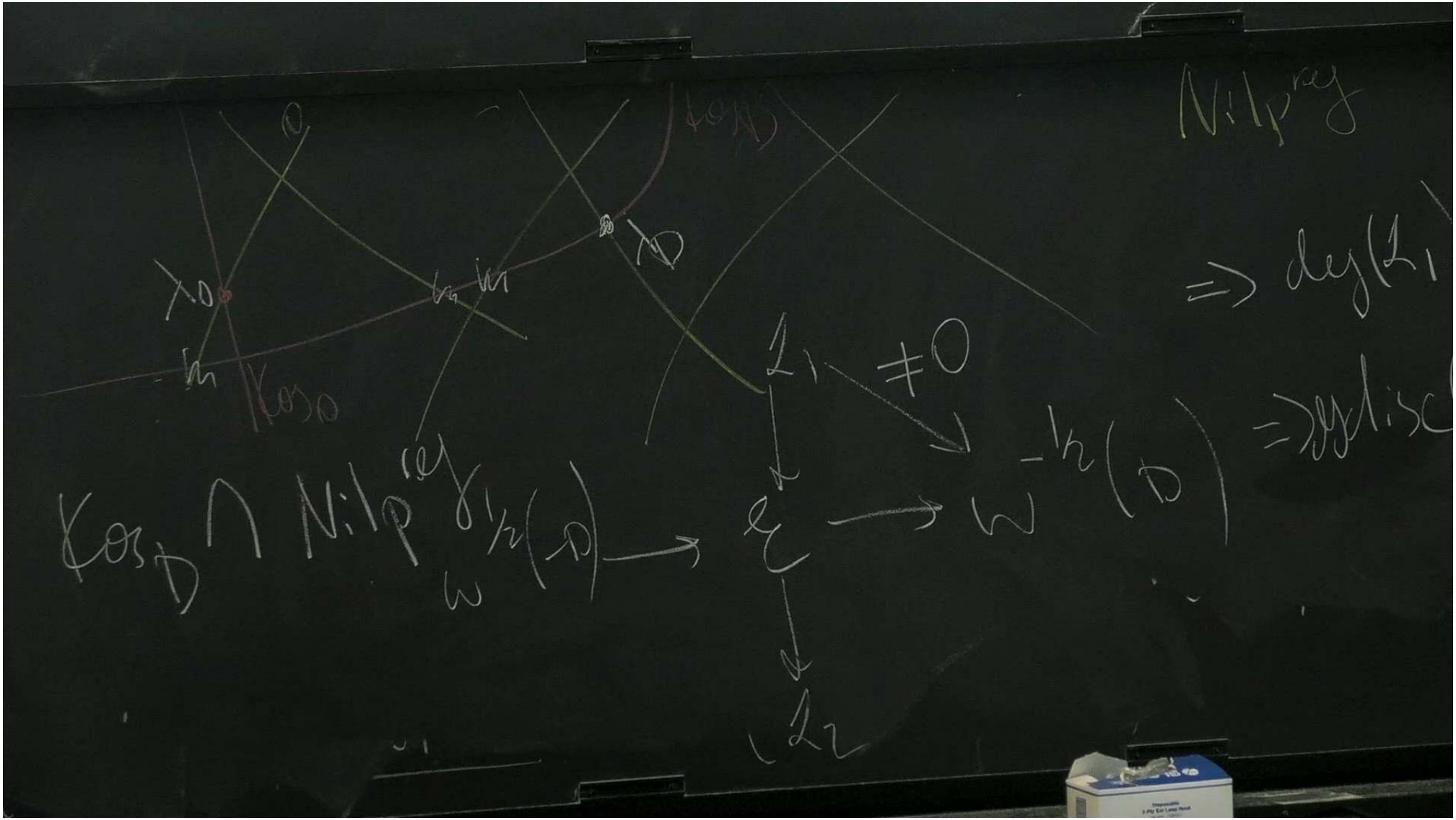
$\dfrac{\mathrm{Rom}}{\mathrm{Kos}_D} \cap (\mathrm{Nilp} \setminus \mathrm{Nilp}^{reg})$

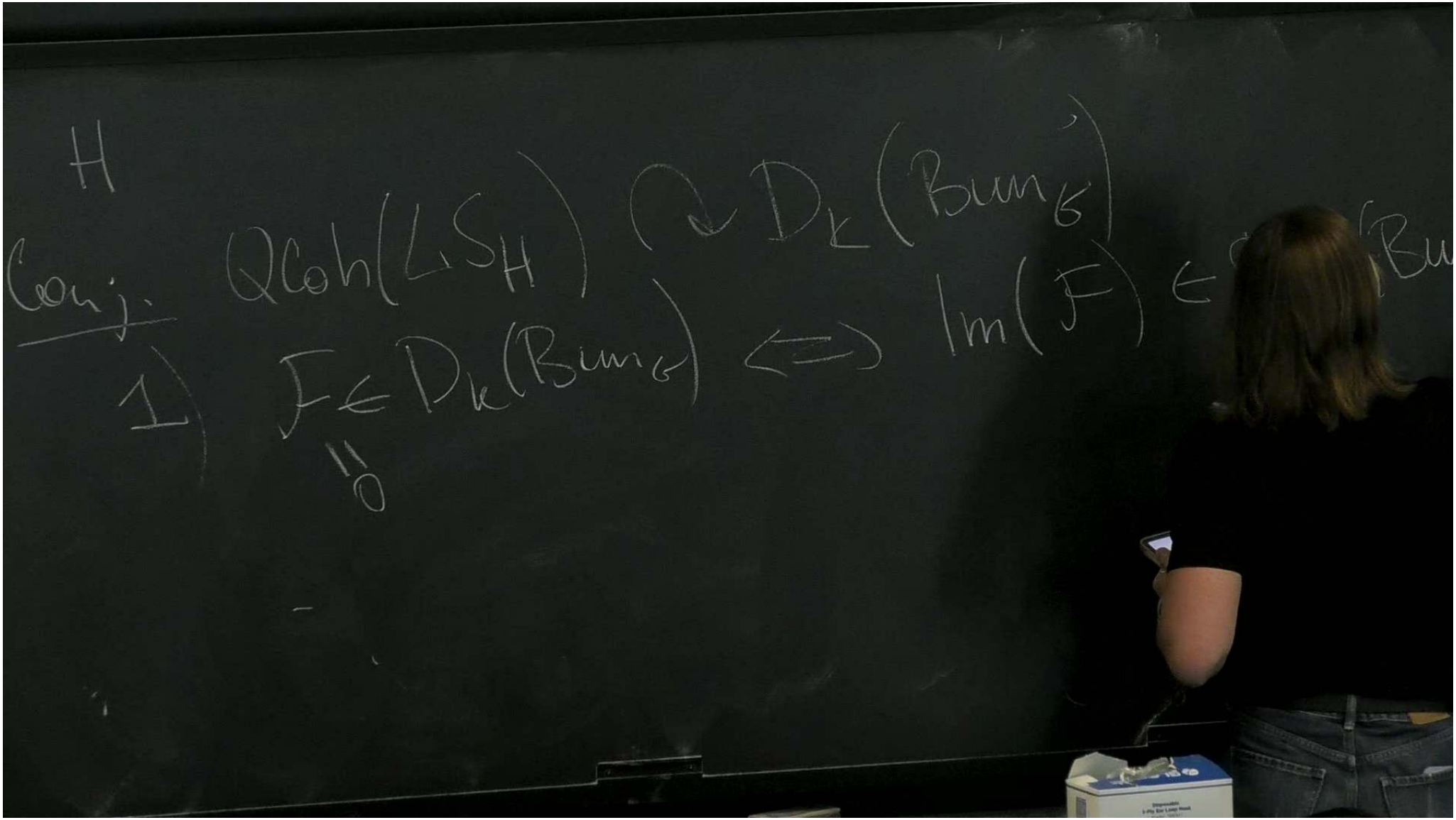
$\|$

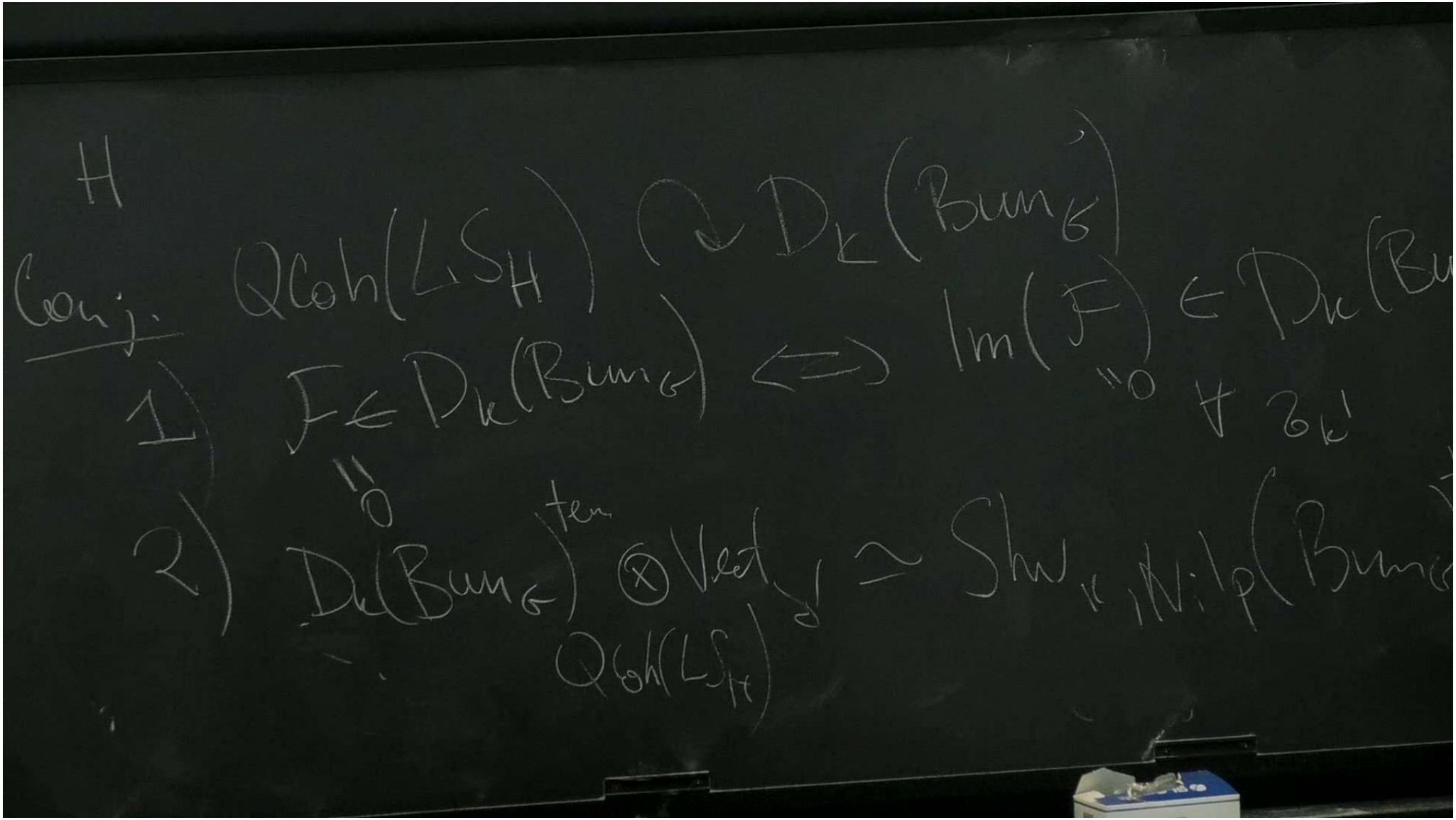
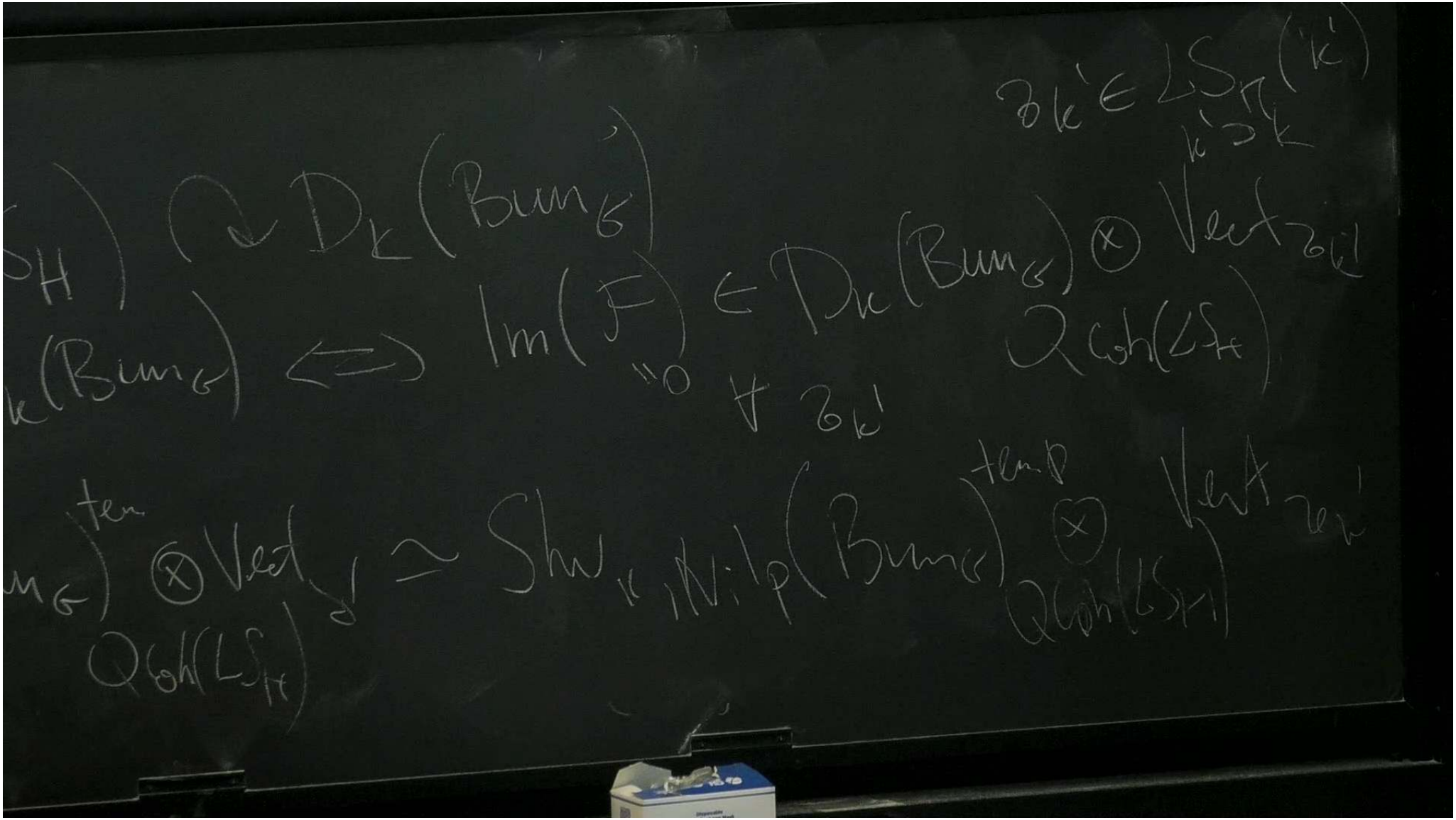
$\emptyset$


Nilp^reg
⇒ deg(
λ0
Kos_D
L1
≠0
Kos_D ∩ Nilp^reg
ω^{1/2}(D)
E
ω^{-1/2}(D)
L2


Nilpreg
$\Rightarrow \deg(\mathcal{L}_1) \leq \deg D + (2-2g)\check{\rho}$
$\neq 0$
$\Rightarrow \deg \mathrm{disc}(\Sigma, \theta) \leq \deg D$
$\longrightarrow W^{-1/2}(D)$
$\mathcal{L}_2$


Nilp^reg
⇒ deg(L1
⇒ eg disc
Kos_D ∩ Nilp^reg
ω^{1/2}(-D)
ℰ
ω^{-1/2}(D)
L1
L2
≠0
λ0
Kos_D


H
Conj:
1)
$QCoh(LS_H) \curvearrowright D_k(Bun_G)$
$I \in D_k(Bun_G) \iff Im(F) \in$
$= 0$


H
Conj: QCoh(LS_H) ≃ D_k(Bun_G)
1) F ∈ D_k(Bun_G) ⟺ Im(F) ≠ 0 ∀ 2k'
2) D_k(Bun_G) ⊗ Vect ≃ Shv
QCoh(LS_H)