Title: Non-vanishing of quantum geometric Whittaker coefficients

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Collection/Series: Mathematical Physics

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Abstract:

We will discuss the functor of geometric Whittaker coefficients in the context of quantum geometric Langlands. We will prove that tempered twisted D-modules on the stack of G-bundles on a smooth projective curve have non-vanishing Whittaker coefficients. Roughly, this means that a certain natural subcategory of twisted D-modules on the stack of G-bundles can be controlled by the category of twisted D-modules on the Beilinson-Drinfeld affine Grassmannian. The proof will combine generalizations of representation-theoretic and microlocal methods from the preceding works of Faergeman-Raskin and Nadler-Taylor respectively.

X - proj. smooth conn curve / $\mathbb{C}$

G - simple red gp $\check{G}$ - Langlands dual gp

Bun_G - stack of G-bundles on X

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Conj. (quantum GLC)

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$$D_\kappa(\mathrm{Bun}_G) \simeq D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}})$$

where κ and $\check{\kappa} \in \mathrm{Tw}(\mathrm{Bun}_G)$ coming from G-inv bil f

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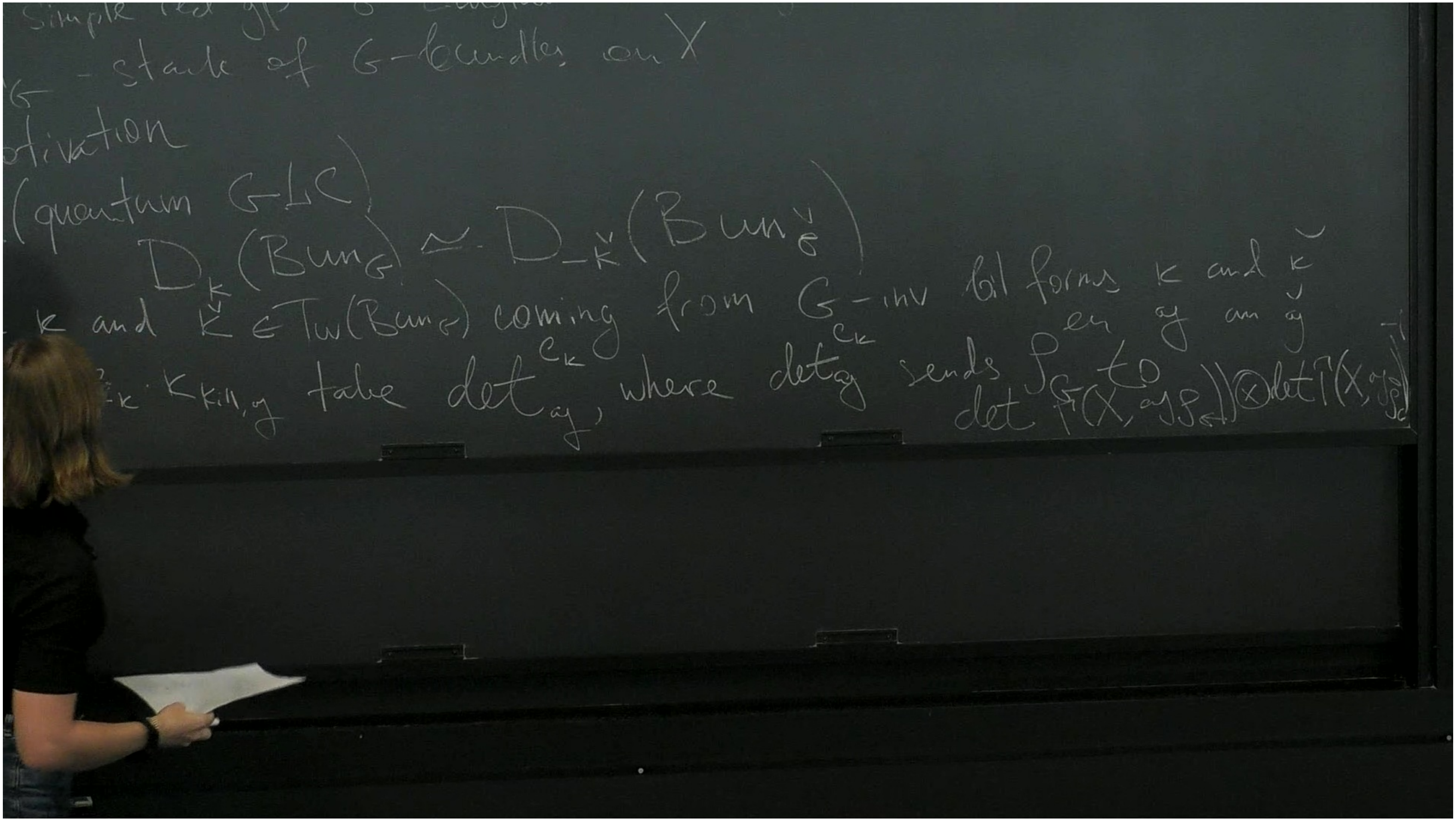
where κ and $\check{\kappa} \in \mathrm{Tw}(\mathrm{Bun}_G)$ coming from G-inv bil forms κ and $\check{\kappa}$ on $\mathfrak{g}$ and $\check{\mathfrak{g}}$

ie $\kappa = c_\kappa \cdot \kappa_{\mathrm{Kill},\mathfrak{g}}$ take $\det_{\mathfrak{g}}^{c_\kappa}$,


stack of G-bundles on X
Motivation
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$D_\kappa(\mathrm{Bun}_G) \simeq D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}})$
κ and $\check{\kappa} \in \mathrm{Tw}(\mathrm{Bun}_G)$ coming from G-inv bil forms κ and $\check{\kappa}$ on $\mathfrak{g}$ and $\check{\mathfrak{g}}$
$\kappa_{Kill,\mathfrak{g}}$ take $\det_{\mathfrak{g}}^{c_\kappa}$, where $\det_{\mathfrak{g}}$ sends P to $\det \Gamma(X, \mathfrak{g}_P) \otimes \det^{-1}(X, \mathfrak{g}_P)$

i.e. $\kappa = c_\kappa \cdot \kappa_{Kill,\mathfrak{g}}$ take $\det_{\mathfrak{g}}^{c_\kappa}$, where $\det_{\mathfrak{g}}^{c_\kappa}$ sends $\mathcal{P}_G$ to $\det\Gamma(X, \mathfrak{g}_{\mathcal{P}_G})) \otimes \det$

For us: κ rational, i.e. $c_\kappa \in \mathbb{Q}$, $\kappa - \underbrace{(-\tfrac{1}{2})\mathrm{Kill}_{\mathfrak{g}}}_{\kappa_{crit}}$ are non-deg.

Rem (relation between κ and $\check{\kappa}$) restr of $\kappa - \kappa_{crit}$ to $\mathfrak{t}$ and $\check{\kappa} - \check{\kappa}_{crit}$ to $\check{\mathfrak{t}} = \mathfrak{t}^*$ are dual bilinear forms.

For $\kappa = \kappa_{crit}$

Thm $D(\mathrm{Bun}_G) \overset{\kappa_{crit}}{\text{"}\simeq\text{"}} \mathrm{QCoh}(LS_{\check{G}}(X))$

Q What properties fix $\mathbb{L}_\kappa$?

· For κ_{crit} the main axiom is the Hecke eigenproperty

i.e. $\mathcal{L}^+G$ $D(Gr_G) \curvearrowright D(Bun_G)$, $\mathbb{L}_{crit}$ intertwines this action w/ the appropr. action on the RHS

Rem

• For κ_{crit} ...

i.e. $D(Gr_G)^{\mathcal{L}^+G} \curvearrowright D(Bun_G)$, $\mathbb{L}_{crit}$ intertwines this action w/ the appropri action on the RHS

Rem For κ rational $D_\kappa(Gr_G)^{\mathcal{L}^+G}$ is smaller.

i.e.

$$\begin{tikzcd}
D(\mathrm{Gr}_G)^{\mathcal{L}N,\chi} \arrow[r, "\sim"] & \mathrm{Rep}(\check{G}) \\
D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}^{\mathrm{loc}}"] \arrow[r, "\mathbb{L}_{\kappa_{crit}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \arrow[u]
\end{tikzcd}$$

- κ_{crit}

Generalizes to rational κ

$$\begin{tikzcd}
 & D_\kappa(\mathrm{Gr}_G)^{\mathcal{L}N,\chi} \arrow[r, "\sim"] & \mathrm{KL}(\check{G})_{-\check{\kappa}} \\
D_\kappa(\mathrm{Bun}_G)^{temp} \arrow[ur] & D_\kappa(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}^{\mathrm{loc}}"] \arrow[r] & D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}}) \arrow[u]
\end{tikzcd}$$

thm

this is conserv.

$\ldots G)^{\mathcal{L}N,\chi} \xrightarrow[\sim]{} \mathrm{Rep}(\check{G})$

$\ldots \mathrm{ff}^{!\mathrm{loc}}$

$\ldots \mathrm{un}_G) \xrightarrow{\mathbb{L}_{\mathrm{Kont}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$

$\mathbb{K}$

rational

$\ldots \mathrm{Gr}_G)^{\mathcal{L}N,\chi} \xrightarrow{\sim} \mathrm{KL}(\check{G})_{-\check{\kappa}}$

$\mathrm{coeff}^{!\mathrm{loc}}$

$\mathrm{Bun}_G) \longrightarrow D_{-\check{\kappa}}(\mathrm{Bun}_{\check{G}})$

- Eisenstein series ... which takes care of the rest of $D_\kappa(\mathrm{Bun}_G)$)

2 Whittaker coeffs.

Rem Compact gen of $D_k(Gr_G)^{\mathcal{L}N, \chi}$ are numbered by $\check{\Lambda}^+$-val div on X

Main thm $\Leftarrow$ $\Rightarrow$

Thm 1 $\mathcal{F} \in D_k(Bun_G)^{temp}$ $\exists$ $\check{\Lambda}^+$-val. div D s.t. $Hom(V_D, -) \circ coeff^{loc}(\mathcal{F}) \neq 0$

...definitive coeff$_D$

$$\mathrm{Bun}_N^{\omega(-D)} \longrightarrow \mathrm{Bun}_G$$

...der $\mathrm{Bun}_N^{\omega(-D)} := \mathrm{Bun}_B \times_{\mathrm{Bun}_T} \{\omega(-D)\}$

where $\mathrm{Bun}_T^{\omega(-D)} \to \mathrm{Bun}_T \times \mathrm{Bun}_T$, $\{\check{\rho}(\omega),\ \mathcal{O}_X(-D)\}$

...ed to have a canon char.

i.e. $\forall\, \lambda : T \to \mathbb{G}_m$, $\lambda(\mathcal{O}_X(-D)) = \mathcal{O}(-\lambda(D))$

$$\psi_D : \mathrm{Bun}_N^{\omega(-D)} \longrightarrow \mathbb{G}_a$$

$\mathcal{F} \in D_k(\mathrm{Bun}_G)$

$$\mathrm{coeff}_D(\mathcal{F}) = C_{dR}\left(\mathrm{Bun}_N^{\omega(-D)}, p^!(\mathcal{F}) \otimes \psi_D^!(\exp)\right)\left[\dim \mathrm{Bun}_G - \dim \mathrm{Bun}_N^{\omega(-D)}\right]$$

3. the statement: $Shv_{\kappa, Nilp}(Bun_G)$

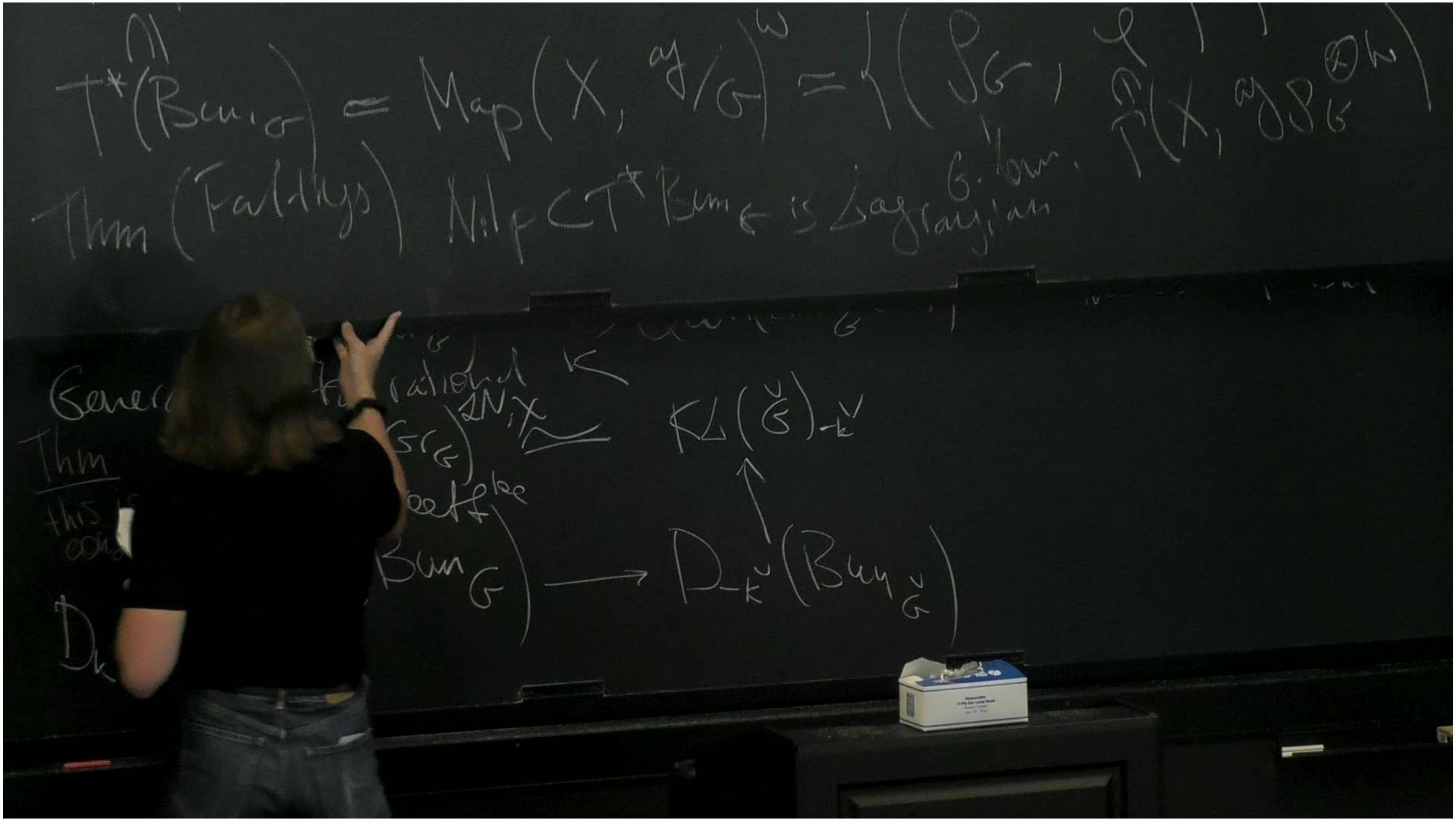
Shv_κ = regular hol twisted-D-mod

$$
\begin{array}{c}
Nilp \\
\cap \\
T^*(Bun_G)
\end{array}
= Map(X, \mathfrak{g}/G)^{\omega} = \{(\underset{G\text{-bun.}}{P_G}, \varphi)\}, \quad \varphi \in \Gamma(X, \mathfrak{g}_{P_G} \otimes \omega)
$$

3. the statement: $\mathrm{Shv}_{k,\mathrm{Nilp}}(\mathrm{Bun}_G)$

Shv_k = regular hol twisted-D-mod

$$\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G) = \mathrm{Map}(X, \mathfrak{g}/G)^{\omega} = \{(\underset{G\text{-bun.}}{P_G}, \varphi \in \Gamma(X, \mathfrak{g}_{P_G} \otimes \omega))\}$$


Thm (Faltings) Nilp ⊂ T* Bun is Lagrangian
KL
Bun

Thm (Faltings) $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$ is Lagrangian

$\exists$ $\check{\Lambda}^+$-val div D s.t. $\mathrm{coeff}_D(\mathcal{F}) \neq 0$

$\Lambda \subset T^*\mathrm{Bun}_G$ closed conic Lagrang.

Thm3 Suppose $T^*_{\mathrm{Bun}_N^{\omega(-D)}}\mathrm{Bun}_G + d\psi_D$ intersect Λ transv at a smooth pt $\{\lambda_D\}$. Then $\mathrm{coeff}_D|_{\mathrm{Shv}_{N,\Lambda}}$ calculates (twisted) microstalk at $\{\lambda_D\}$

Cor $\mathrm{coeff}_D|_{\mathrm{Shv}, \kappa, \Lambda}$ is t-exact and commutes w/ Verdier dual

$\mathcal{F} \in \mathrm{Shv}_{\kappa,\Lambda}(\mathrm{Bun}_G)$ $\quad CC(\mathcal{F}) = \sum_{\beta \in \mathrm{Irr}(\Lambda)} c_{\beta;\mathcal{F}} [\beta]$ $\quad \lambda_D \in \beta_D$

$\chi(\mathrm{coeff}_{D,\kappa}(\mathcal{F})) = c_{\beta_D, \mathcal{F}}$

Ex $G = SL_2$ $\qquad$ $\mathrm{Nilp}^{reg} \subset \mathrm{Nilp}$

- $T^*\mathrm{Bun}_G + d\psi_D =: \mathrm{Kos}_D$

$\mathrm{Bun}_N^{\omega(-D)}$
$\quad\; \|$

2-dim v.b. $\mathcal{E}$ $\qquad \det\mathcal{E} = 0$

$+\ \omega^{1/2}(-D) \to \mathcal{E} \to \omega^{-\frac{1}{2}}(D)$

$+\ \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega$

s.t. $\omega^{1/2}(-D) \to \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega \to \omega^{1/2}(D)$ is the can. map

$G = SL_2$

$\mathrm{Bun}_G + d_{\varphi D} =: \mathrm{Kos}_D$

$\omega(-D)$

$N \quad \Vert \quad$ v.b $\mathcal{E}$ $\quad \det \mathcal{E} = 0$

-dim

$\omega^{1/2}(-D) \to \mathcal{E} \to \omega^{-\frac{1}{2}}(D)$

$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega$

$^{1/2}(-D) \to \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega \to \omega^{1/2}(D)$ is the can. map

$\mathrm{Nilp}^{reg} \quad \mathcal{C} \to_{\omega} \mathrm{Nilp}$

$\Vert$

$\mathrm{Map}_{gen}\left(X, \frac{\mathcal{N}}{G}, \frac{\mathcal{N}^{reg}}{G}\right)_{red}$

$\frac{\mathrm{Rom}}{\mathrm{Kos}_D} \cap (\mathrm{Nilp} \setminus \mathrm{Nilp}^{reg})$

$\Vert$

$\emptyset$

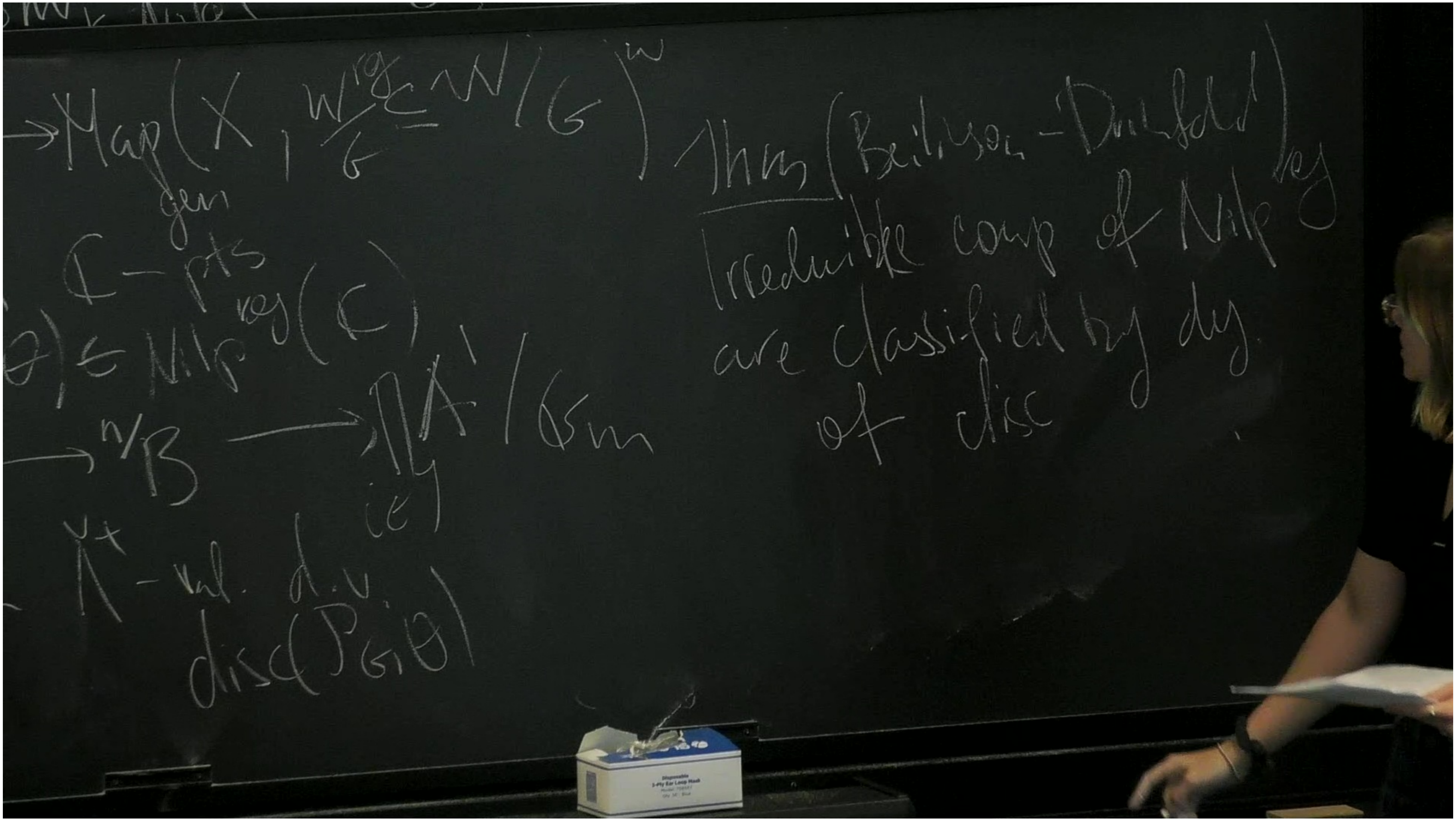
Thm2 ...

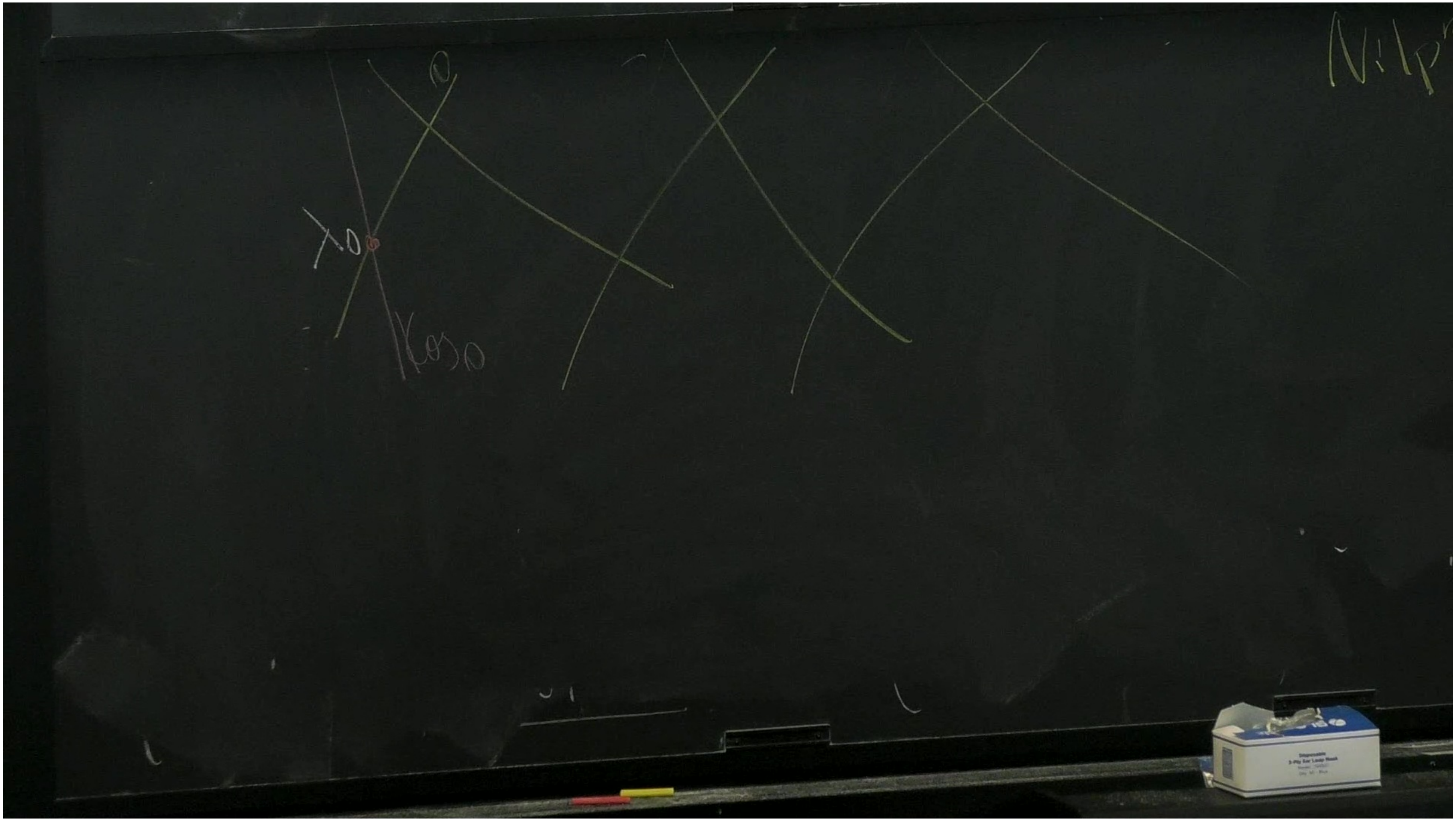
Rem $\mathrm{Map}_{gen}\left(X, \frac{n^{reg}}{B} \subseteq \frac{n}{B}\right)^{w} \longrightarrow \mathrm{Map}_{gen}\left(X, \frac{W^{reg}}{G} \subseteq W/G\right)^{w}$

bij on $\mathbb{C}$-pts

$\Rightarrow$ elt $(P_G, \theta) \in \mathrm{Nilp}^{reg}(\mathbb{C})$

g. reg $X \longrightarrow n/B \longrightarrow \prod_{i \in I} \mathbb{A}^1/\mathbb{G}_m$


Thm (Beilinson - Drinfeld)
Irreducible comp of Nilp reg
are classified by dig.
of disc



$G = SL_2$

$T^*\mathrm{Bun}_G + d\psi_D =: \mathrm{Kos}_D$

$\mathrm{Bun}_N^{\omega(-D)} \;\; \| \;\;$ 2-dim v.b. $\mathcal{E}$ $\quad \det \mathcal{E} = 0$

$+ \; \omega^{1/2}(-D) \to \mathcal{E} \to \omega^{-\frac{1}{2}}(D)$

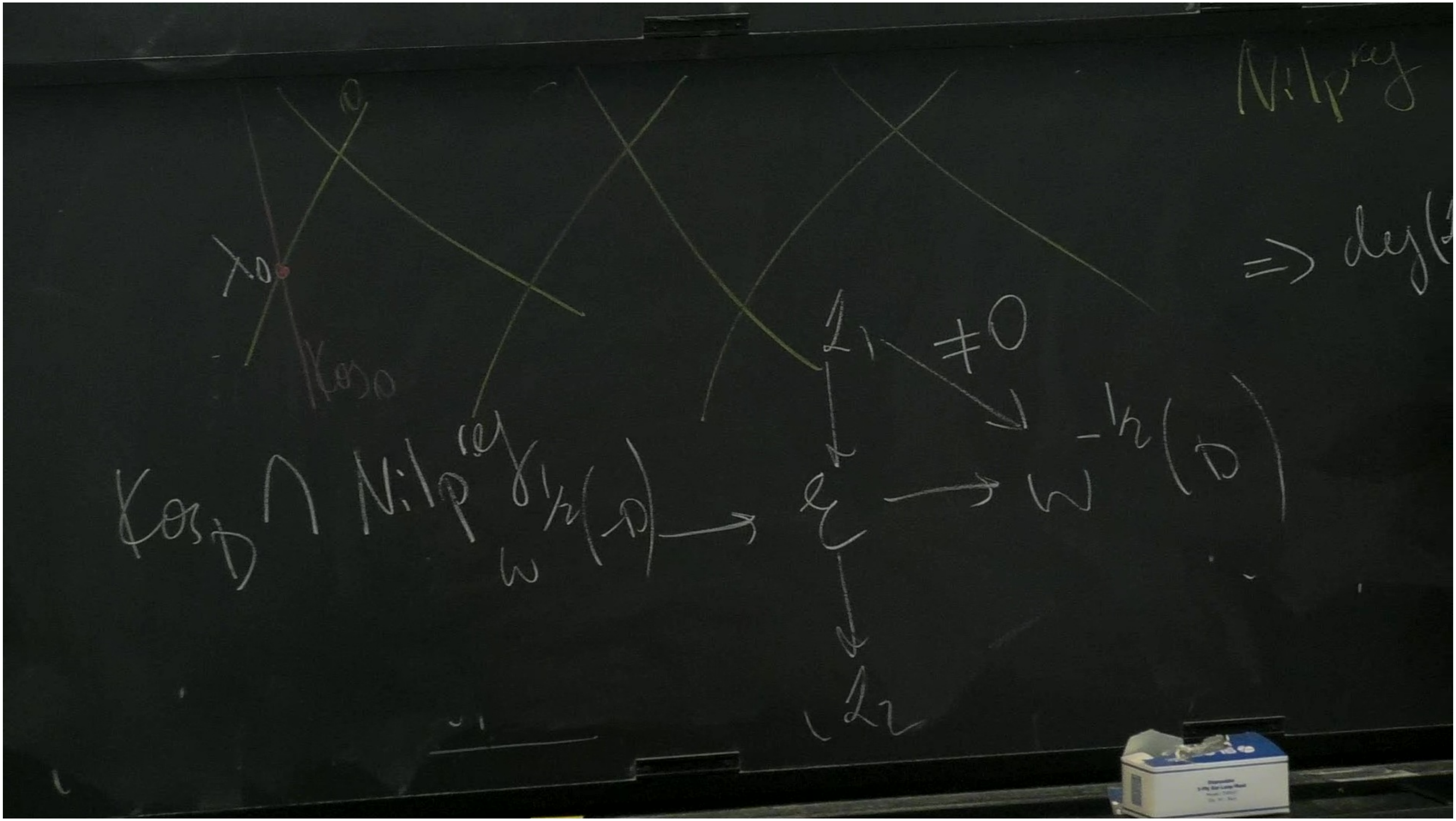
$+ \; \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega$

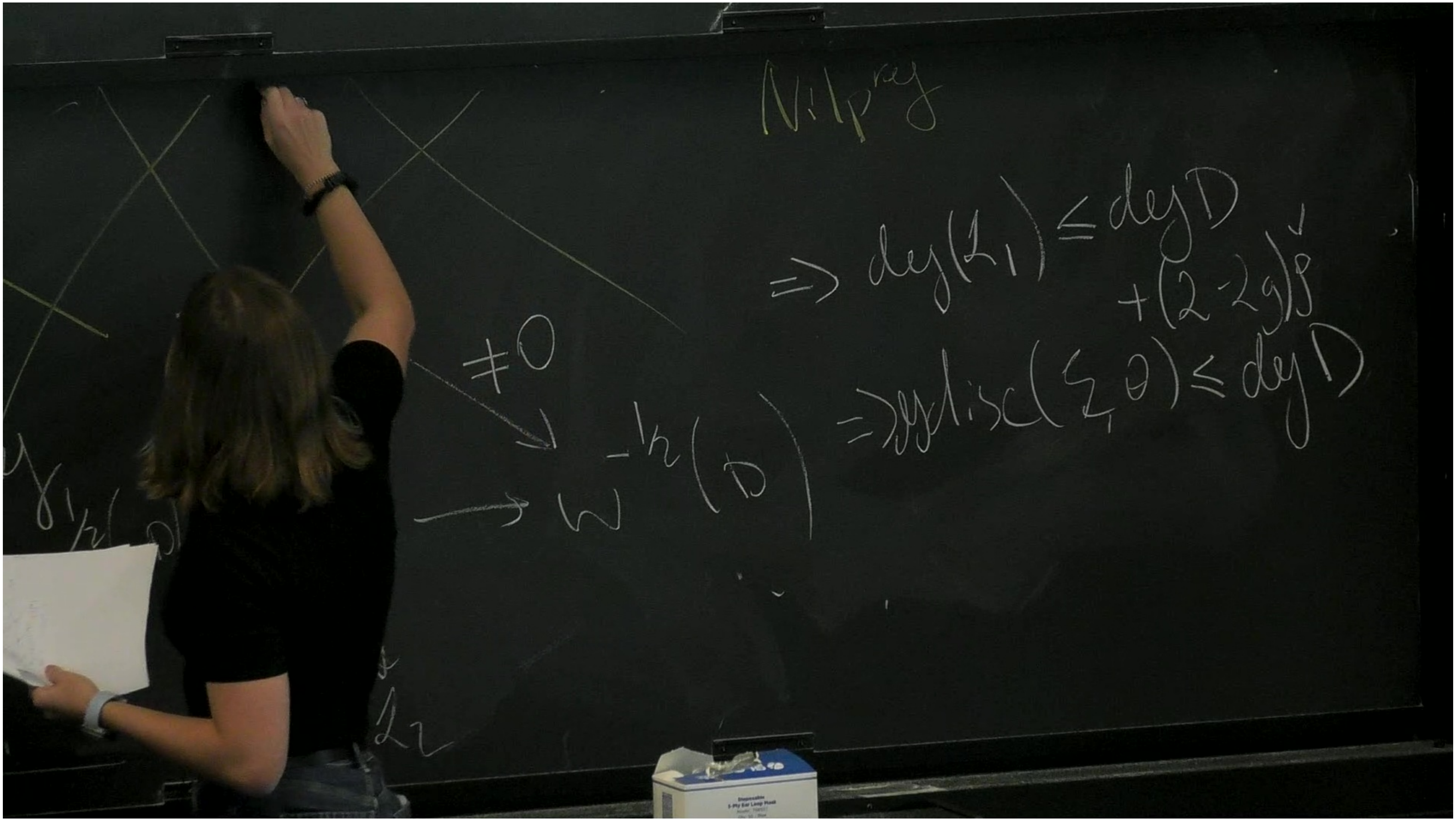
s.t. $\omega^{1/2}(-D) \to \mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega \to \omega^{1/2}(D)$ is the can. map

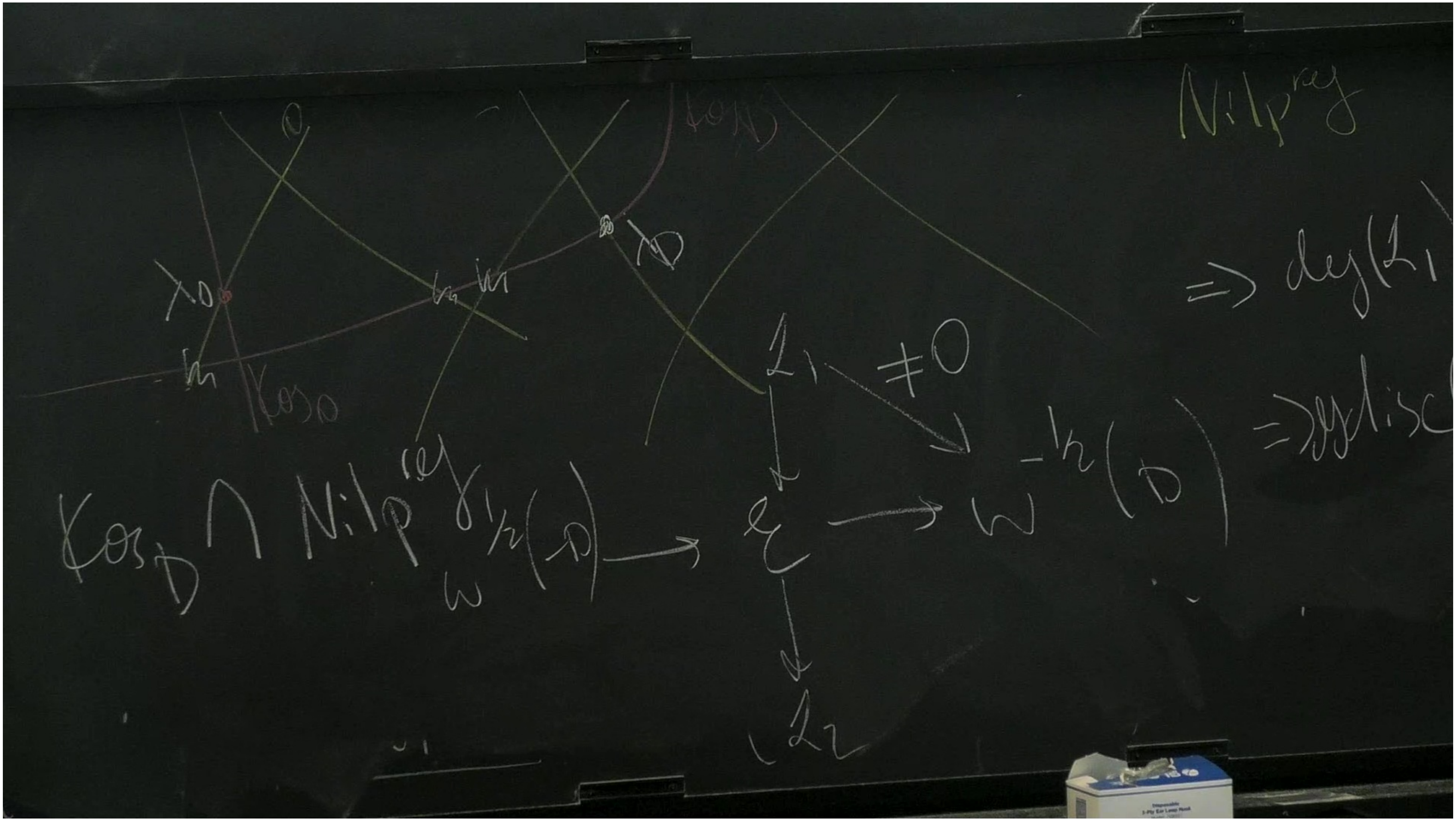
$\mathrm{Nilp}^{reg} \subset_{\omega} \mathrm{Nilp}$

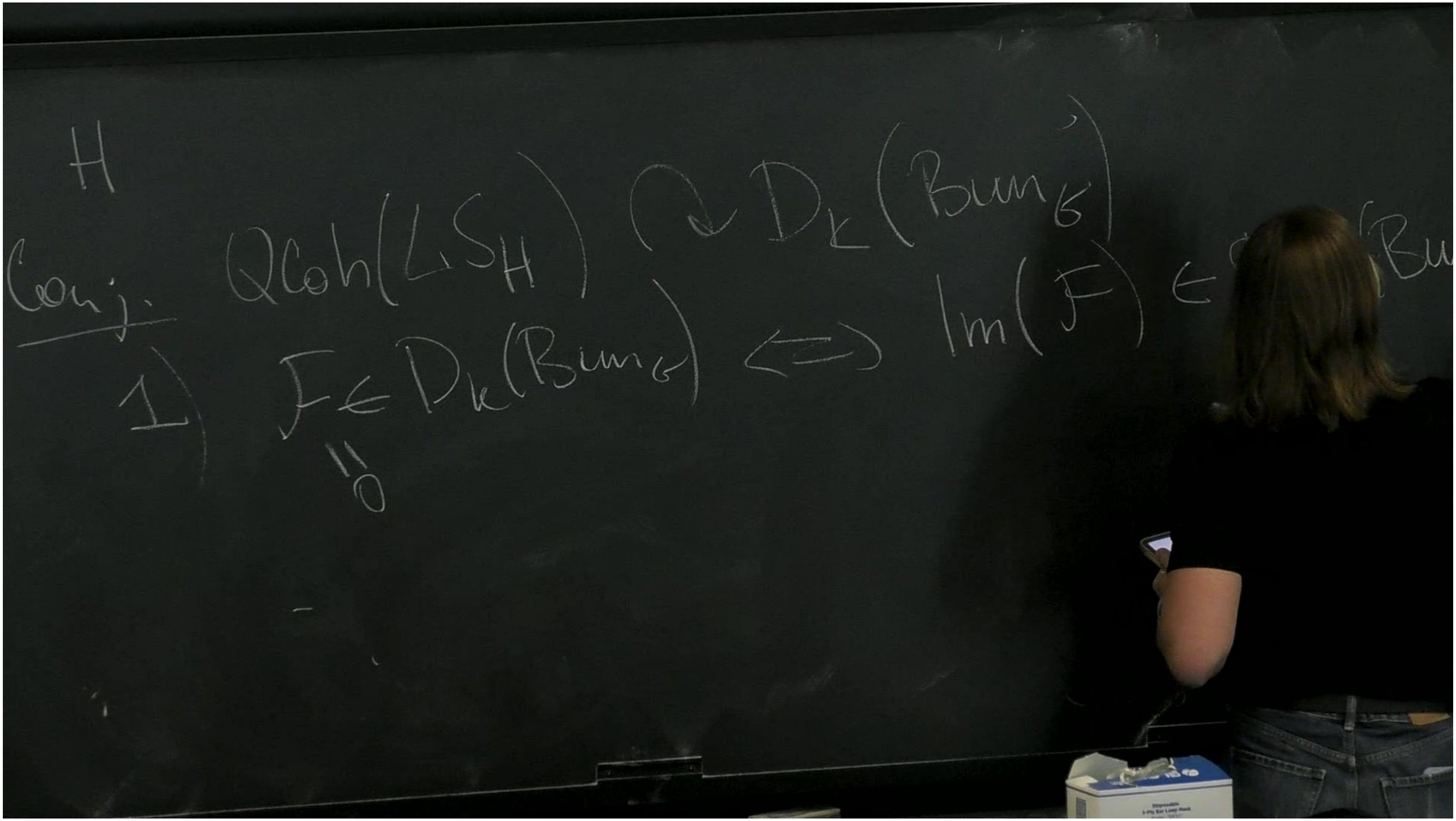
$\| \quad \mathrm{Map}_{gen}\left(X, \mathcal{N}/G\right)_{red}, \; \frac{\mathcal{N}^{reg}}{G}$

$\frac{\mathrm{Rom}}{\mathrm{Kos}_D} \cap \left(\mathrm{Nilp} \setminus \mathrm{Nilp}^{reg}\right) = \emptyset$


Nilp
⇒ deg(
Kos_D
L_1
≠0
ω^{-1/2}(D)
Kos_D ∩ Nilp ω^{1/2}(D) → E → ω^{-1/2}(D)
L_2


⇒ deg(L1) ≤ deg D + (2-2g)β
≠0
w^-1/2(D)
⇒ deg disc(ξ, θ) ≤ deg D


Nilpreg
⇒ deg(L1
≠0
Kos_D ∩ Nilp


H
Conj: QCoh(LS_H) ↻ D_K(Bun_G)
1) F ∈ D_K(Bun_G) ⟺ Im(F) ∈
= 0

H

Conj: $\mathrm{QCoh}(LS_H) \curvearrowright D_k(\mathrm{Bun}_{G'})$

1) $\mathcal{F} \in D_k(\mathrm{Bun}_G) \iff \mathrm{Im}(\mathcal{F}) \in D_k(\mathrm{Bu}$

$\overset{\shortparallel}{0} \qquad\qquad \overset{\shortparallel}{0} \quad \forall\, G_k'$

2) $D_k(\mathrm{Bun}_G)^{\mathrm{ten}} \underset{\mathrm{QCoh}(LS_H)}{\otimes} \mathrm{Vect}_{\sigma'} \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G$

$$\sigma_{k'} \in LS_{\check{H}}('k') \qquad k' \supset k$$

$$\ldots_H)) \curvearrowright D_k(\mathrm{Bun}_G)$$

$$\ldots_k(\mathrm{Bun}_G) \Longleftrightarrow \mathrm{Im}(\mathcal{F}) \neq 0 \in D_k(\mathrm{Bun}_G) \underset{QCoh(LS_{\check{H}})}{\otimes} \mathrm{Vect}_{\sigma_{k'}} \quad \forall\, \sigma_{k'}$$

$$\ldots\mathrm{un}_G)^{\mathrm{tem}} \underset{QCoh(LS_{\check{H}})}{\otimes} \mathrm{Vect}_{\sigma'} \simeq \mathrm{Shv}_{k,\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{temp}} \underset{QCoh(LS_{\check{H}})}{\otimes} \mathrm{Vect}_{\sigma_{k'}}$$