Title: Lecture - Beautiful Papers

Speakers: Pedro Vieira

Collection/Series: Beautiful Papers - October 7, 2024 - January 31, 2025

Subject: Other

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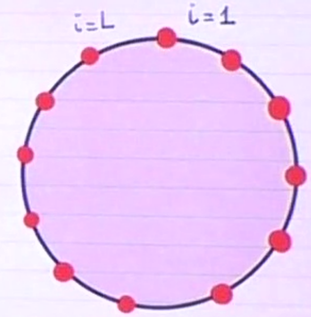
Majorana chain and Ising model – (non-invertible) translations, anomalies, and emanant symmetries
Nathan Seiberg[1] and Shu-Heng Shao[2]
also,
2401.12281 Seiberg, Seifnashri, Shao
2308.00747 TASI lectures by Shao

Quantum 1+1d Ising (Lattice) Model

One qubit @ each site

$\{|\uparrow\rangle, |\downarrow\rangle\}$ basis of $\mathcal{H}_i = \mathbb{C}^2$



$Z|\uparrow\rangle = |\uparrow\rangle \qquad Z|\downarrow\rangle = -|\downarrow\rangle$

$|\pm\rangle = |\uparrow\rangle \pm |\downarrow\rangle \; ; \quad X|\pm\rangle = \pm|+\rangle$

where

$$X_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X_j^2 = Z_j^2 = \mathbb{1}$, $\{X_j, Z_j\} = 0$

$$\mathcal{H} = \bigotimes_{i=1}^{L} \mathcal{H}_i$$

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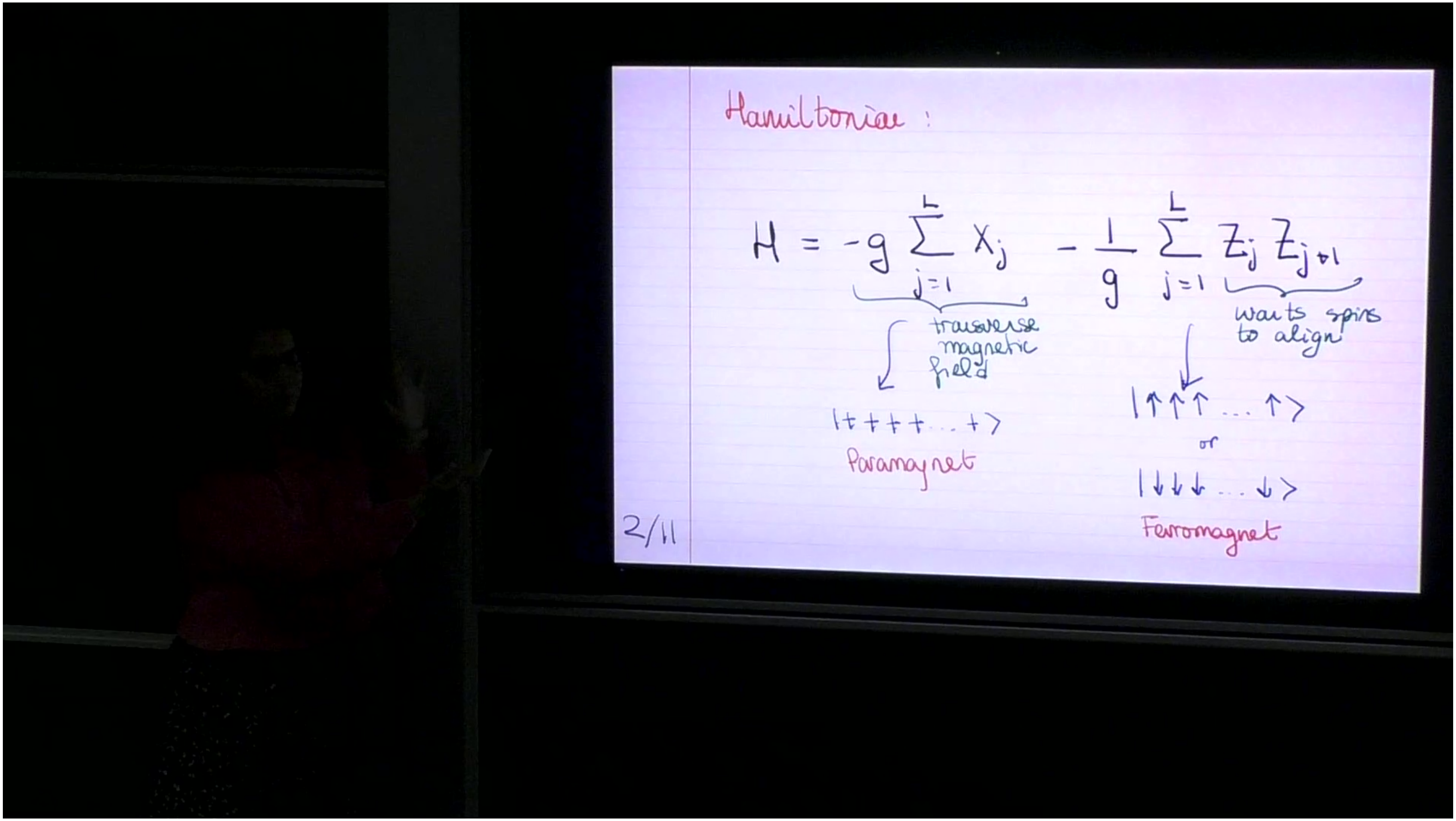
Hamiltonian :
$H = -g \sum_{j=1}^{L} X_j - \frac{1}{g} \sum_{j=1}^{L} Z_j Z_{j+1}$
transverse magnetic field
wants spins to align
$|{+}{+}{+}{+} \ldots {+}\rangle$
Paramagnet
$|\uparrow\uparrow\uparrow \ldots \uparrow\rangle$
or
$|\downarrow\downarrow\downarrow \ldots \downarrow\rangle$
Ferromagnet
2/11

$$H = \overbrace{-g \sum_{j=1}^{L} X_j}^{\text{transverse magnetic field}} - \frac{1}{g} \sum_{j=1}^{L} \underbrace{Z_j Z_{j+1}}_{\text{wants spins to align}}$$

Symmetries:

- Flipping all spins: $\mathbb{Z}_2$

$$\eta = \prod_{i=1}^{L} X_i \quad , \quad \eta^2 = \mathbb{1}$$

- Lattice translation: $\mathbb{Z}_L$

each term swaps sites $i, i+1$

$$T = \prod_{i=1}^{L-1} \frac{1}{2}\left(X_i X_{i+1} + Y_i Y_{i+1} + Z_i Z_{i+1} + \mathbb{1} \right) , \quad T^L = \mathbb{1}$$

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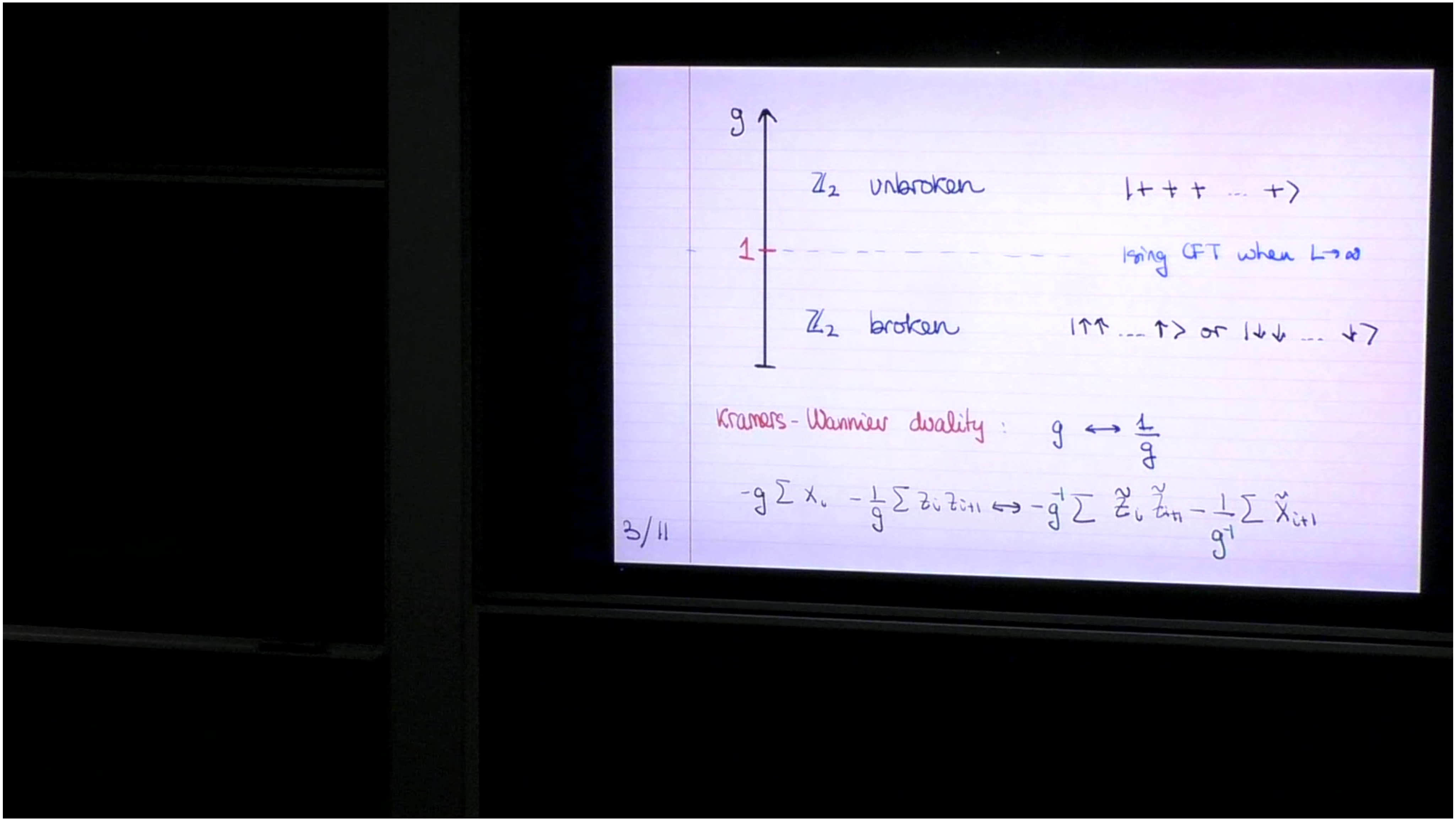
g
$\mathbb{Z}_2$ unbroken
$|+ + + \dots +\rangle$
1
Ising CFT when $L \to \infty$
$\mathbb{Z}_2$ broken
$|\uparrow\uparrow \dots \uparrow\rangle$ or $|\downarrow\downarrow \dots \downarrow\rangle$
Kramers-Wannier duality : $g \leftrightarrow \frac{1}{g}$
$-g \sum X_i - \frac{1}{g} \sum Z_i Z_{i+1} \leftrightarrow -g^{-1} \sum \tilde{Z}_i \tilde{Z}_{i+1} - \frac{1}{g^{-1}} \sum \tilde{X}_{i+1}$
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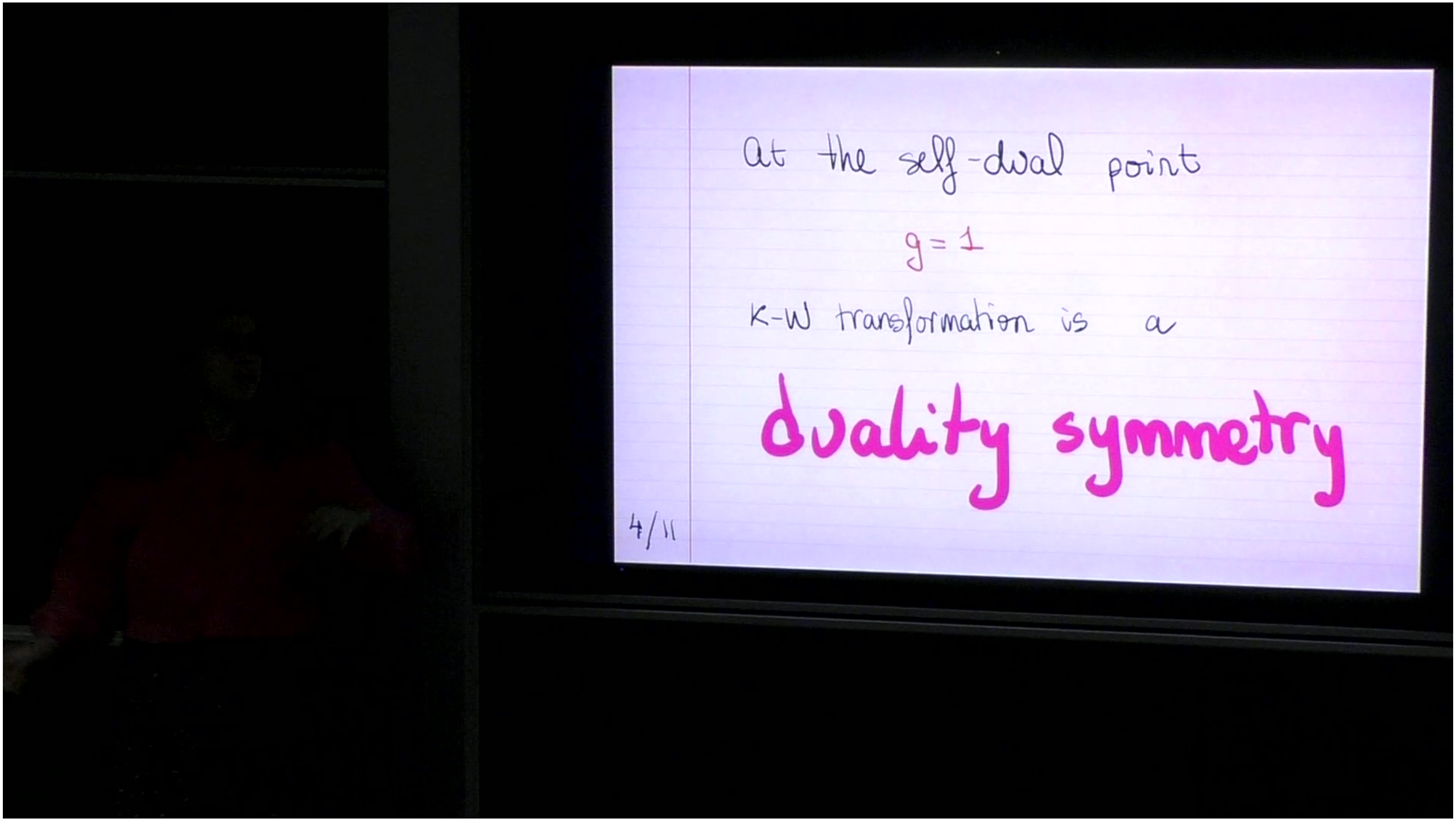
at the self-dual point
g = 1
K-W transformation is a
duality symmetry
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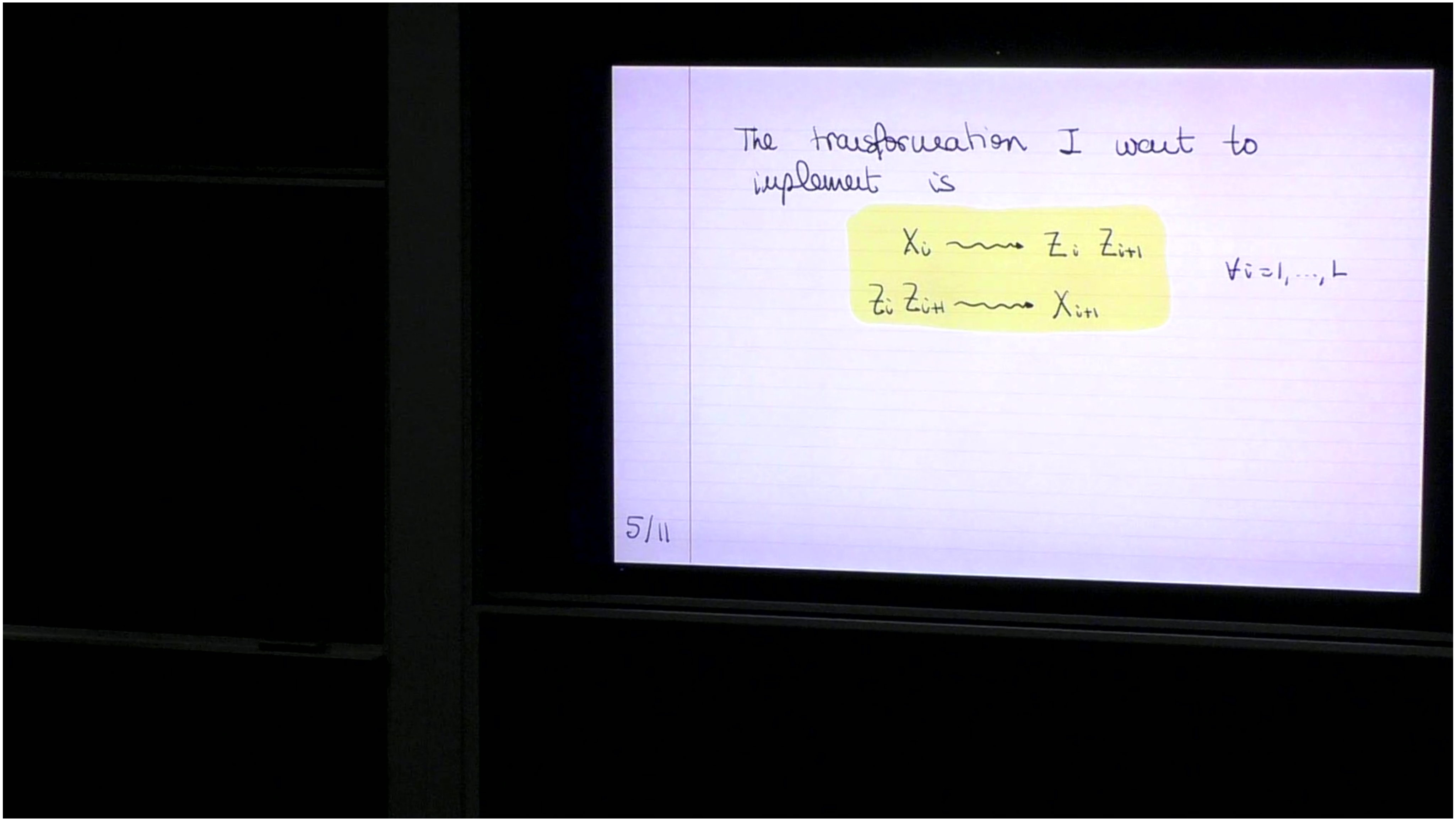
The transformation I want to implement is
$X_i \rightsquigarrow Z_i Z_{i+1}$
$Z_i Z_{i+1} \rightsquigarrow X_{i+1}$
$\forall i = 1, \dots, L$
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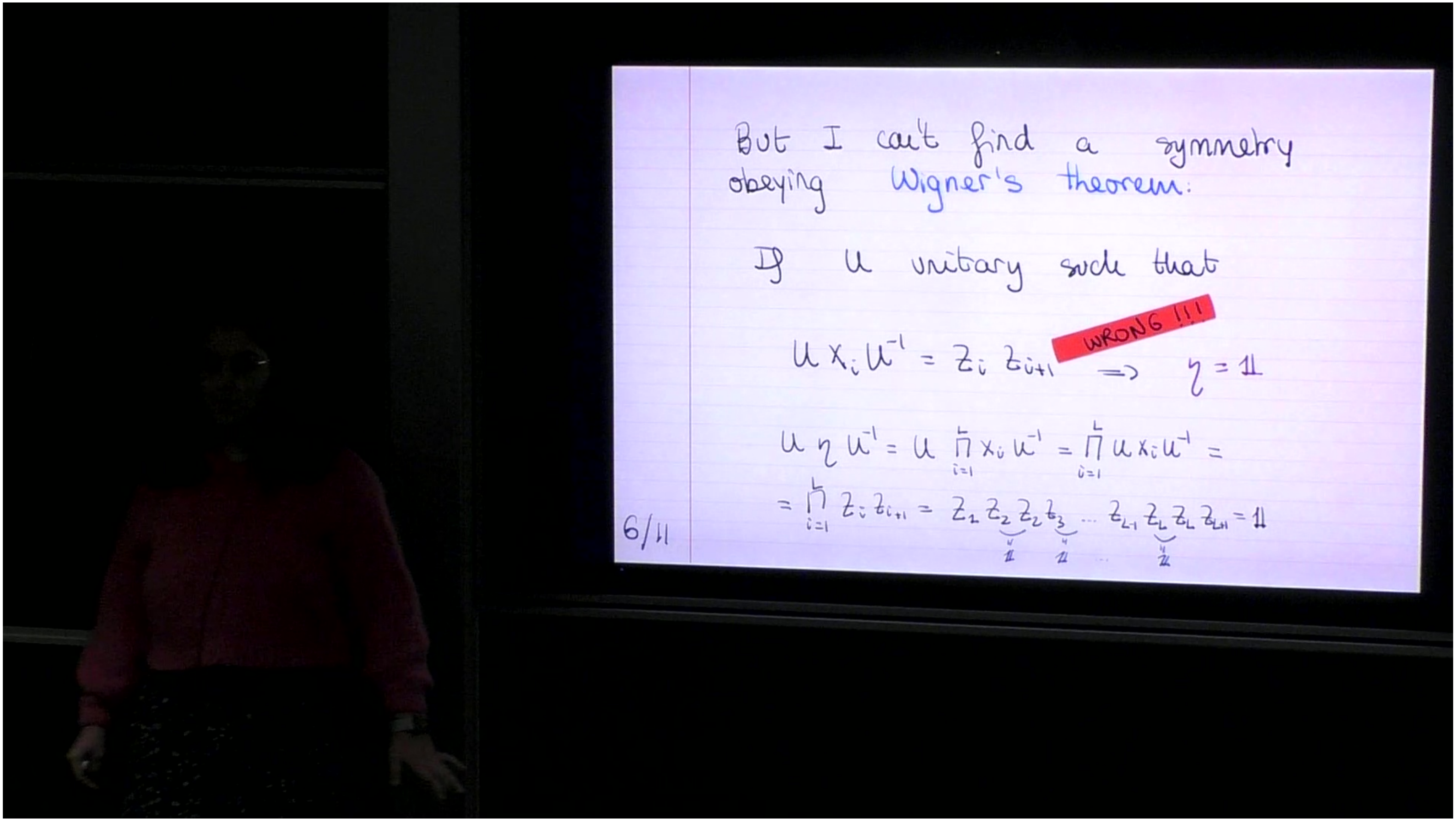
But I can't find a symmetry obeying Wigner's theorem:

If U unitary such that

WRONG !!!

$$U X_i U^{-1} = Z_i Z_{i+1} \implies \eta = \mathbb{1}$$

6/11


But I can't find a symmetry
obeying Wigner's theorem:
If U unitary such that
$U X_i U^{-1} = Z_i Z_{i+1}$
WRONG !!!
$\Longrightarrow \eta = \mathbb{1}$
$U \eta U^{-1} = U \prod_{i=1}^{L} X_i U^{-1} = \prod_{i=1}^{L} U X_i U^{-1} =$
$= \prod_{i=1}^{L} Z_i Z_{i+1} = Z_1 \underbrace{Z_2 Z_2}_{=\mathbb{1}} \underbrace{Z_3 \ldots}_{=\mathbb{1}} \ldots Z_{L-1} \underbrace{Z_L Z_L}_{=\mathbb{1}} Z_{L+1} = \mathbb{1}$
6/11

classical quantum

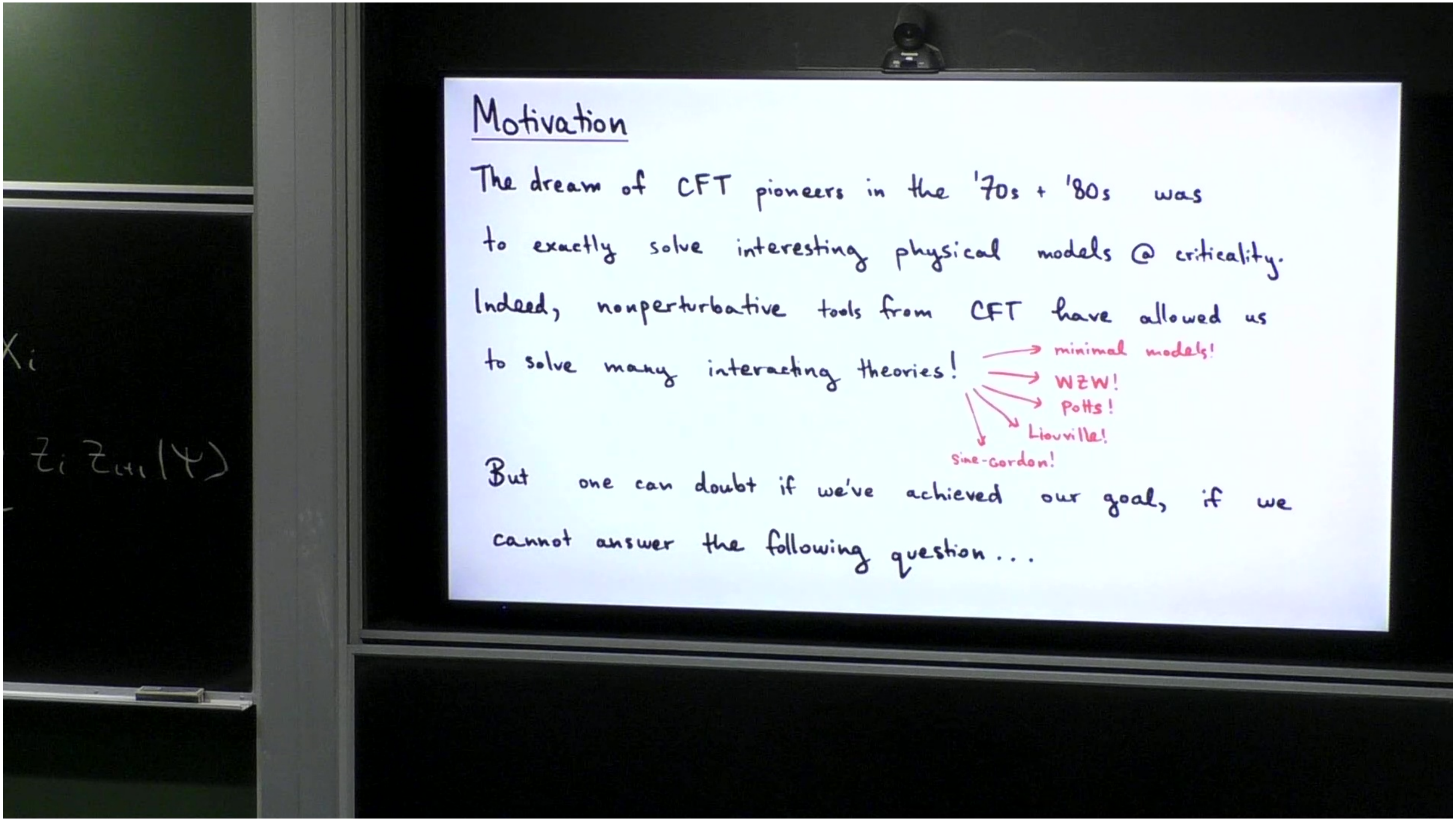
$d+1$ d

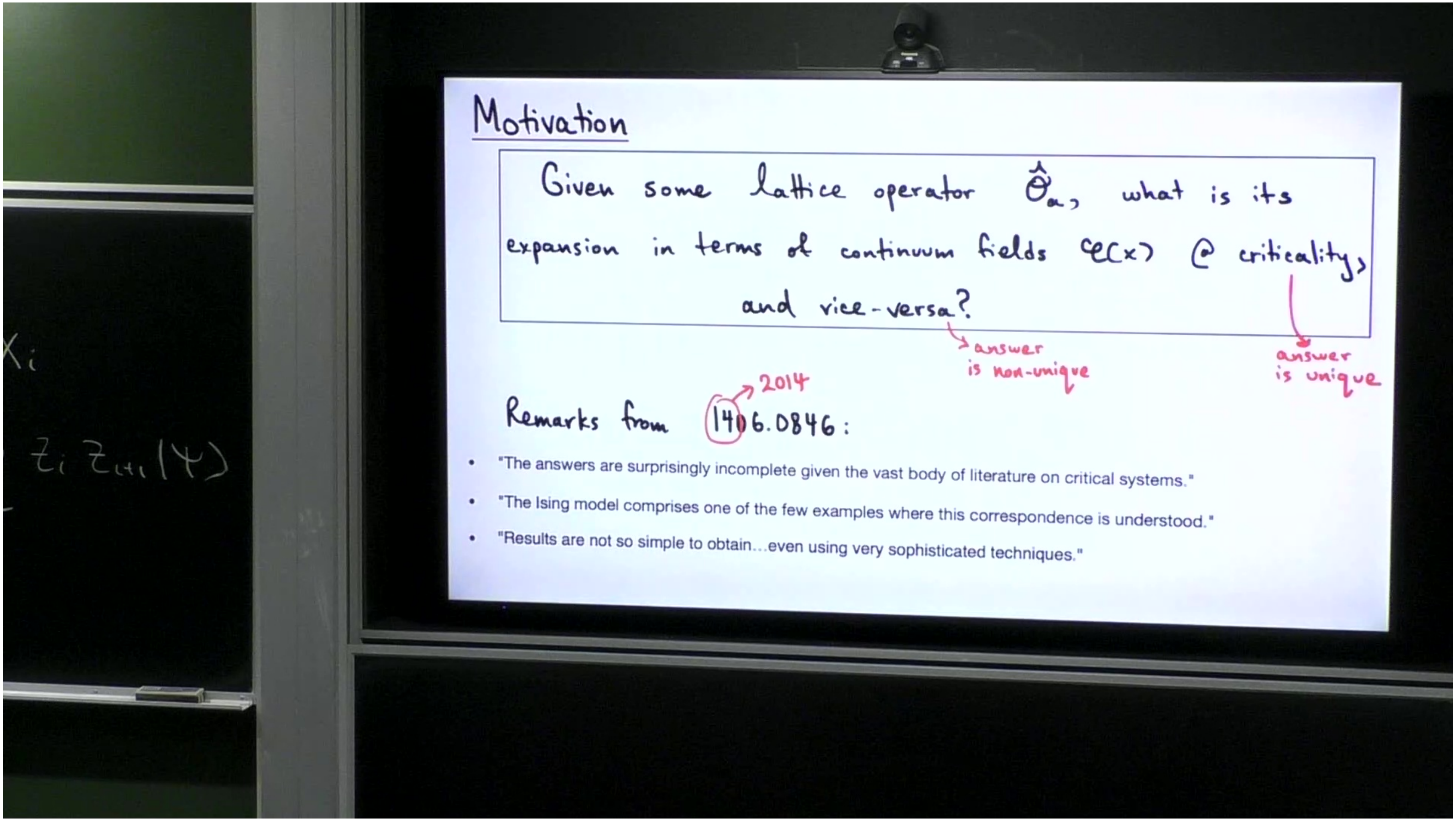
$$\eta = \prod_{i=1}^{L} X_i$$

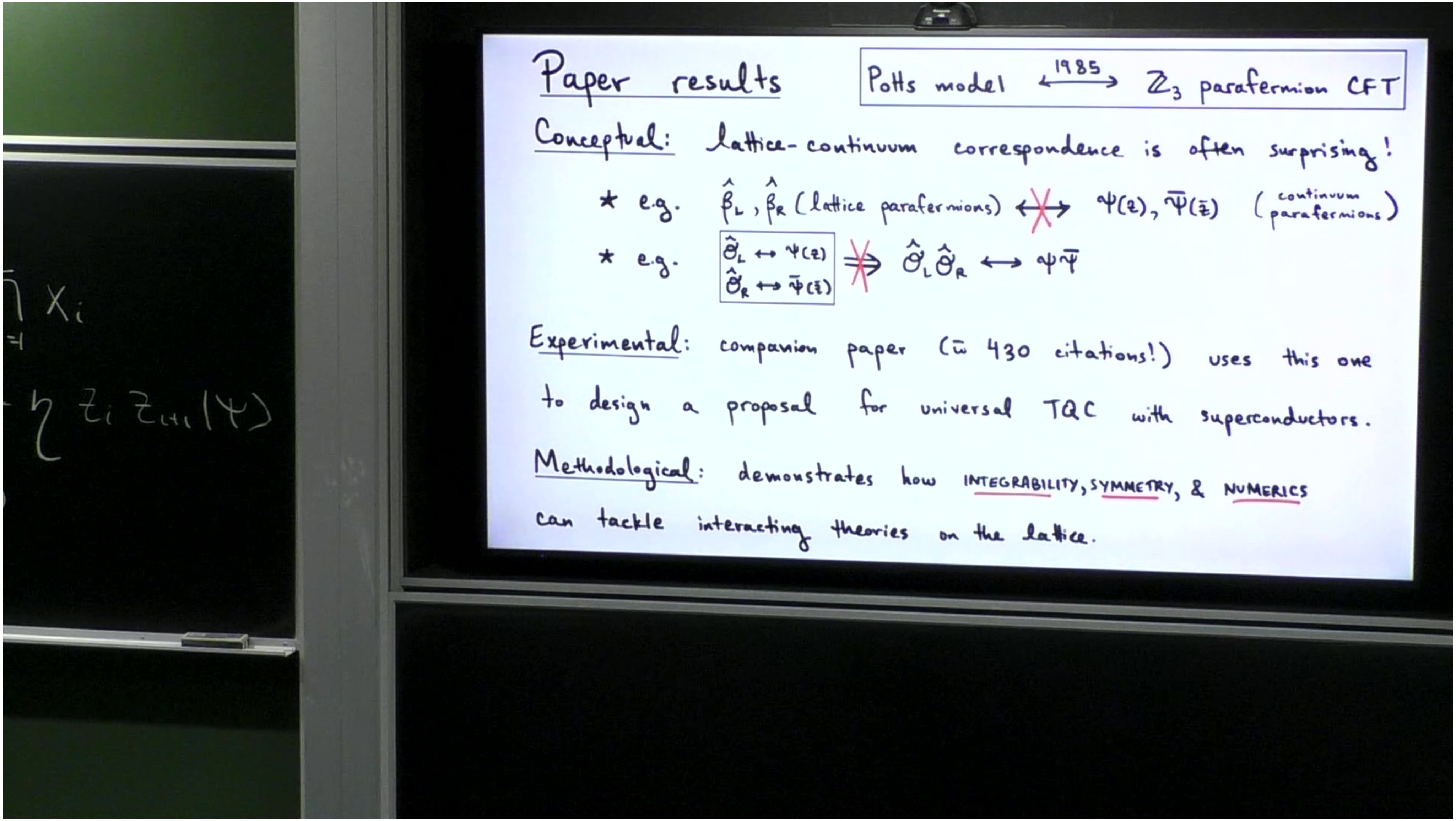
$$U X_i U^{-1} |\psi\rangle = \eta Z_i Z_{i+1} |\psi\rangle$$

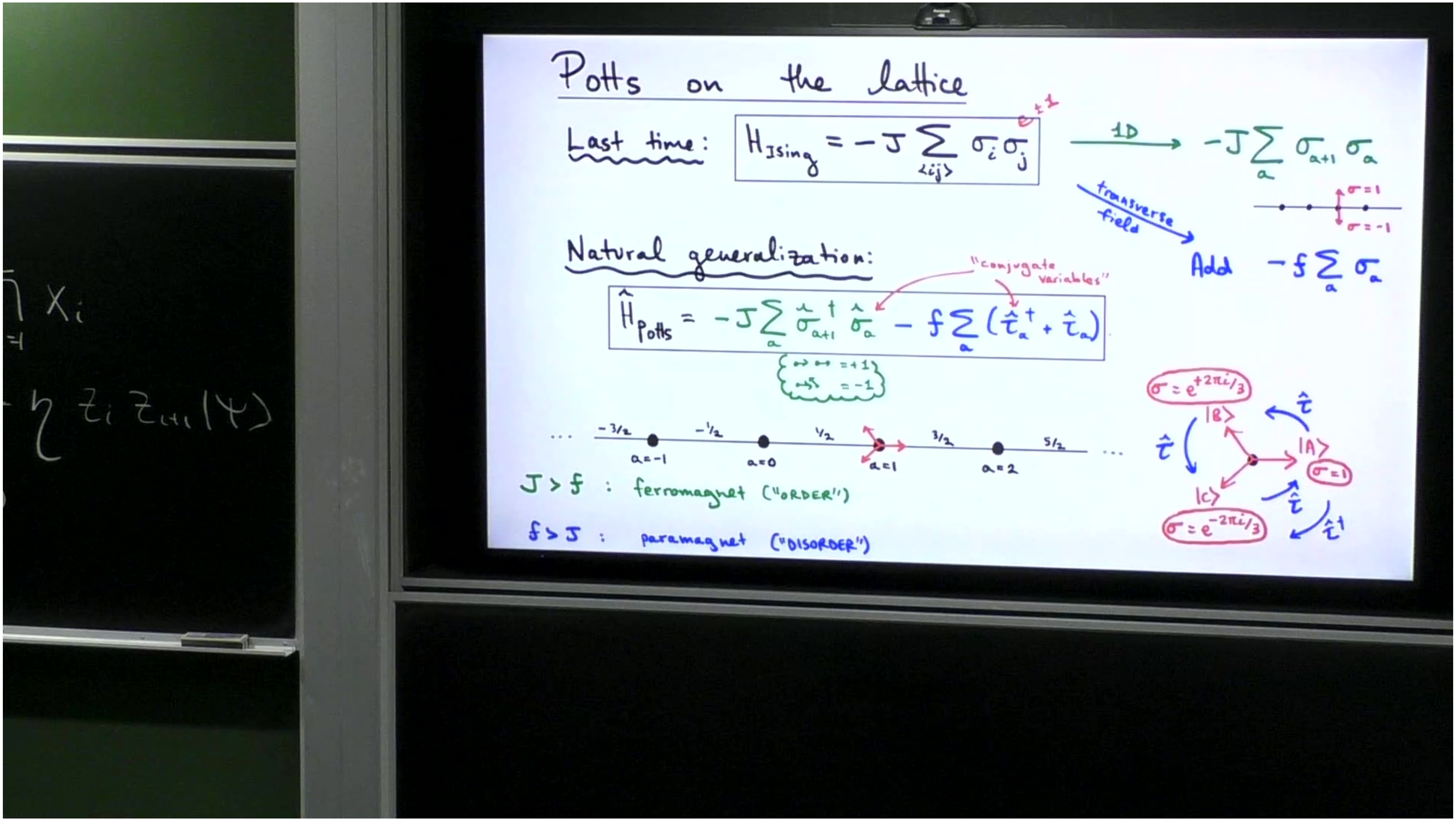
$$DH = HD$$

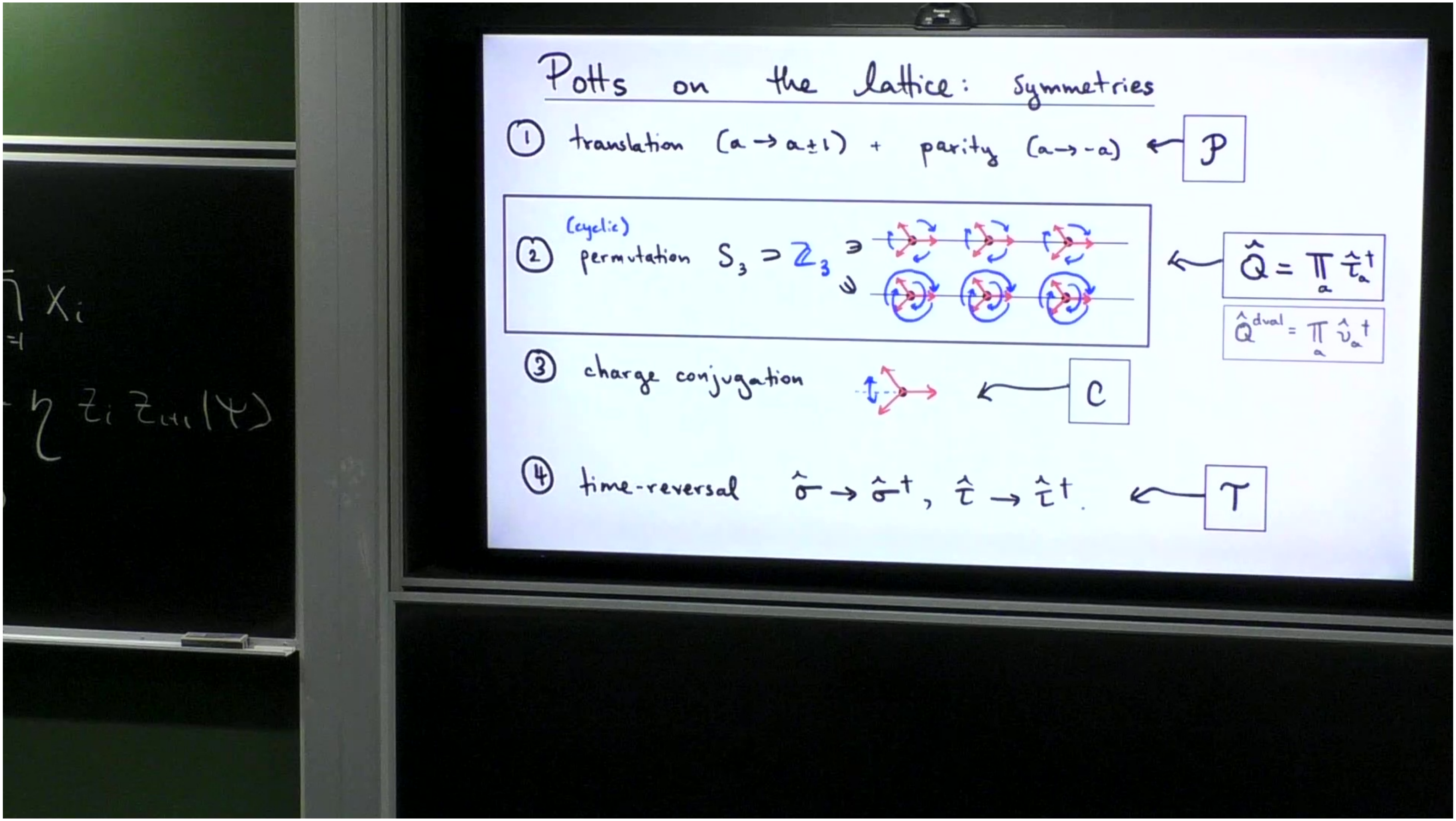
$$DHD^{-1} = H$$

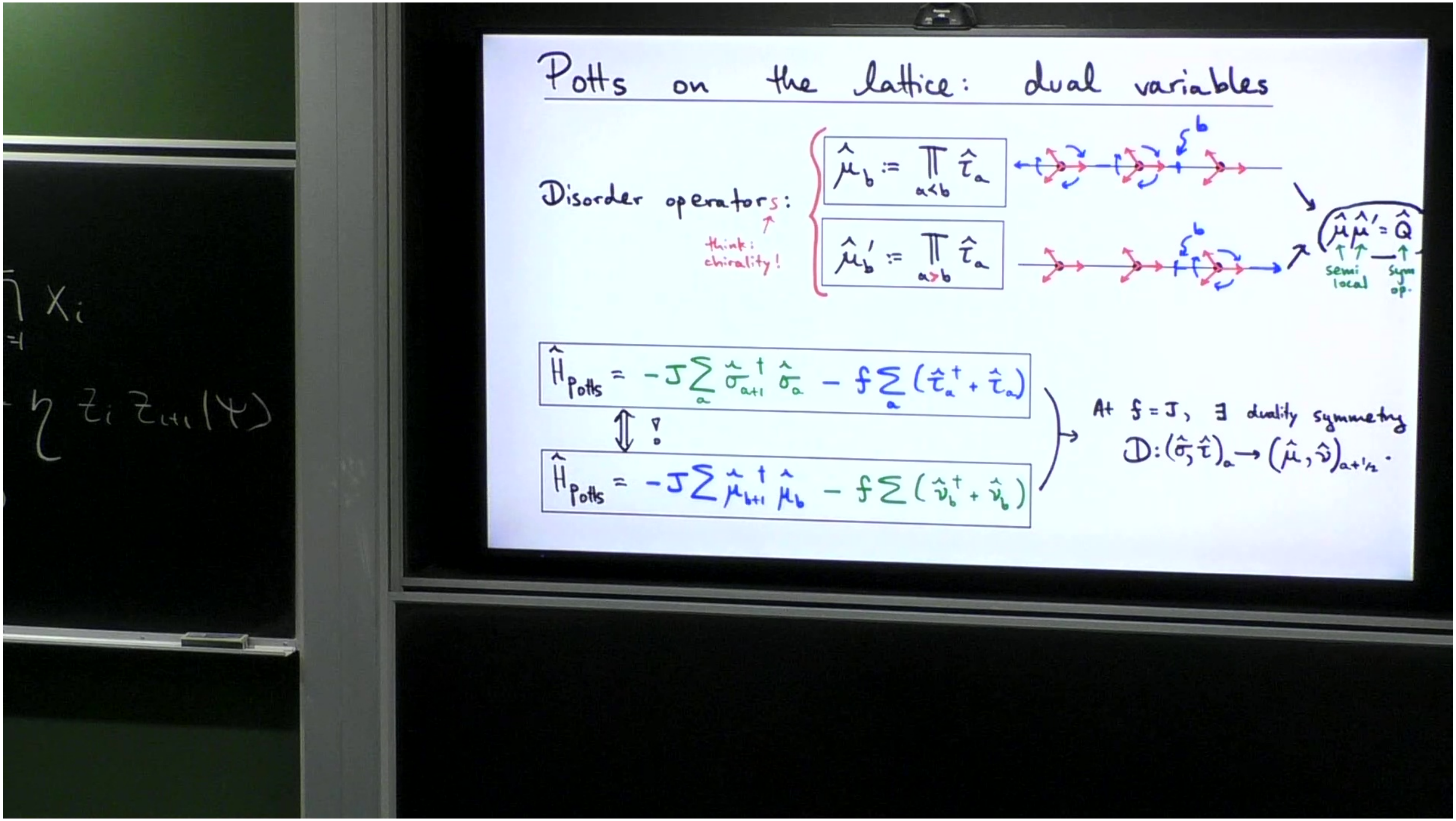

Motivation
The dream of CFT pioneers in the '70s + '80s was
to exactly solve interesting physical models @ criticality.
Indeed, nonperturbative tools from CFT have allowed us
to solve many interacting theories!
minimal models!
WZW!
Potts!
Liouville!
Sine-Gordon!
But one can doubt if we've achieved our goal, if we
cannot answer the following question...

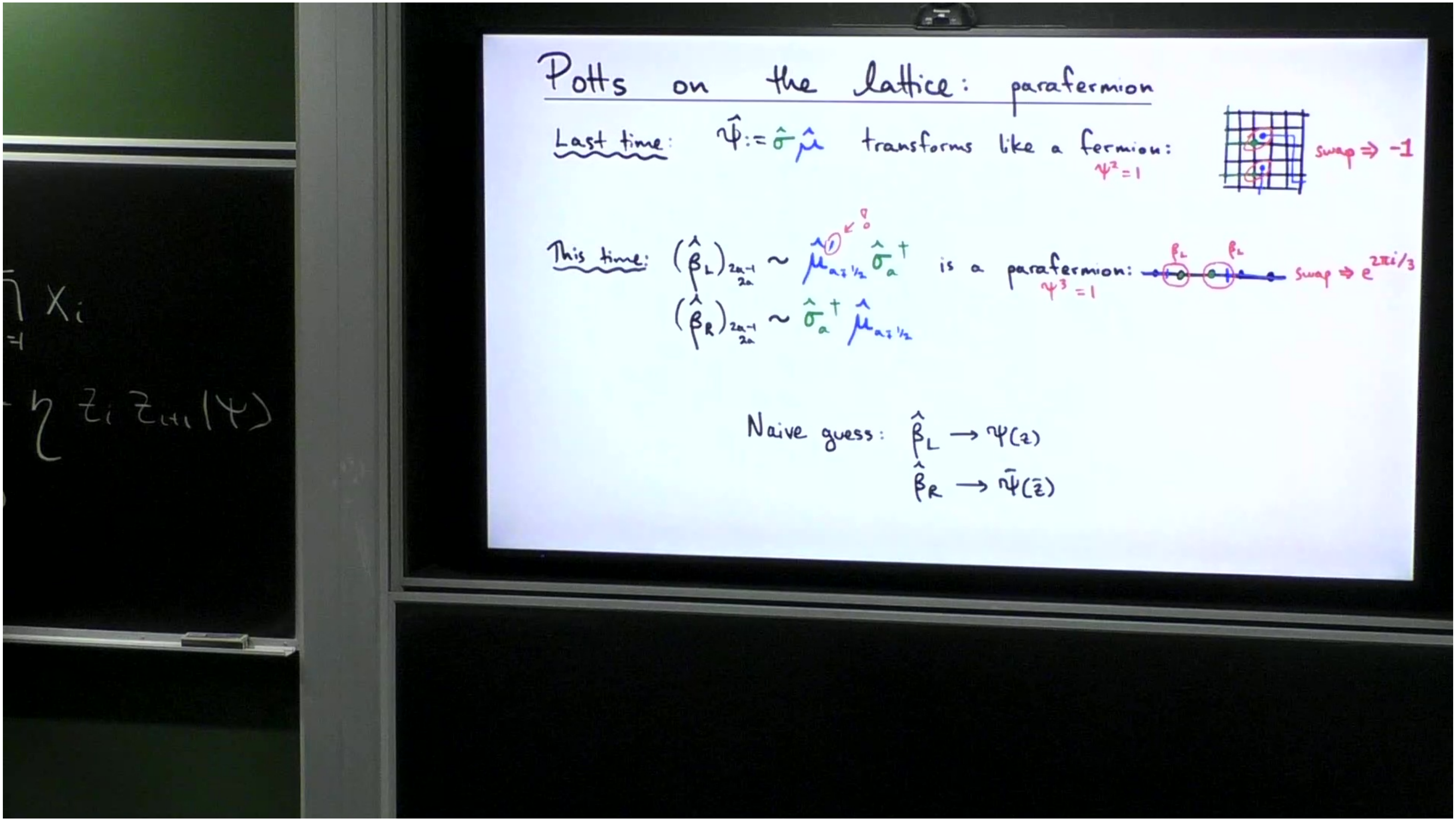

Motivation
Given some lattice operator $\hat{O}_a$, what is its expansion in terms of continuum fields $\varphi(x)$ @ criticality, and vice-versa?
answer is non-unique
answer is unique
2014
Remarks from 1406.0846:
• "The answers are surprisingly incomplete given the vast body of literature on critical systems."
• "The Ising model comprises one of the few examples where this correspondence is understood."
• "Results are not so simple to obtain…even using very sophisticated techniques."

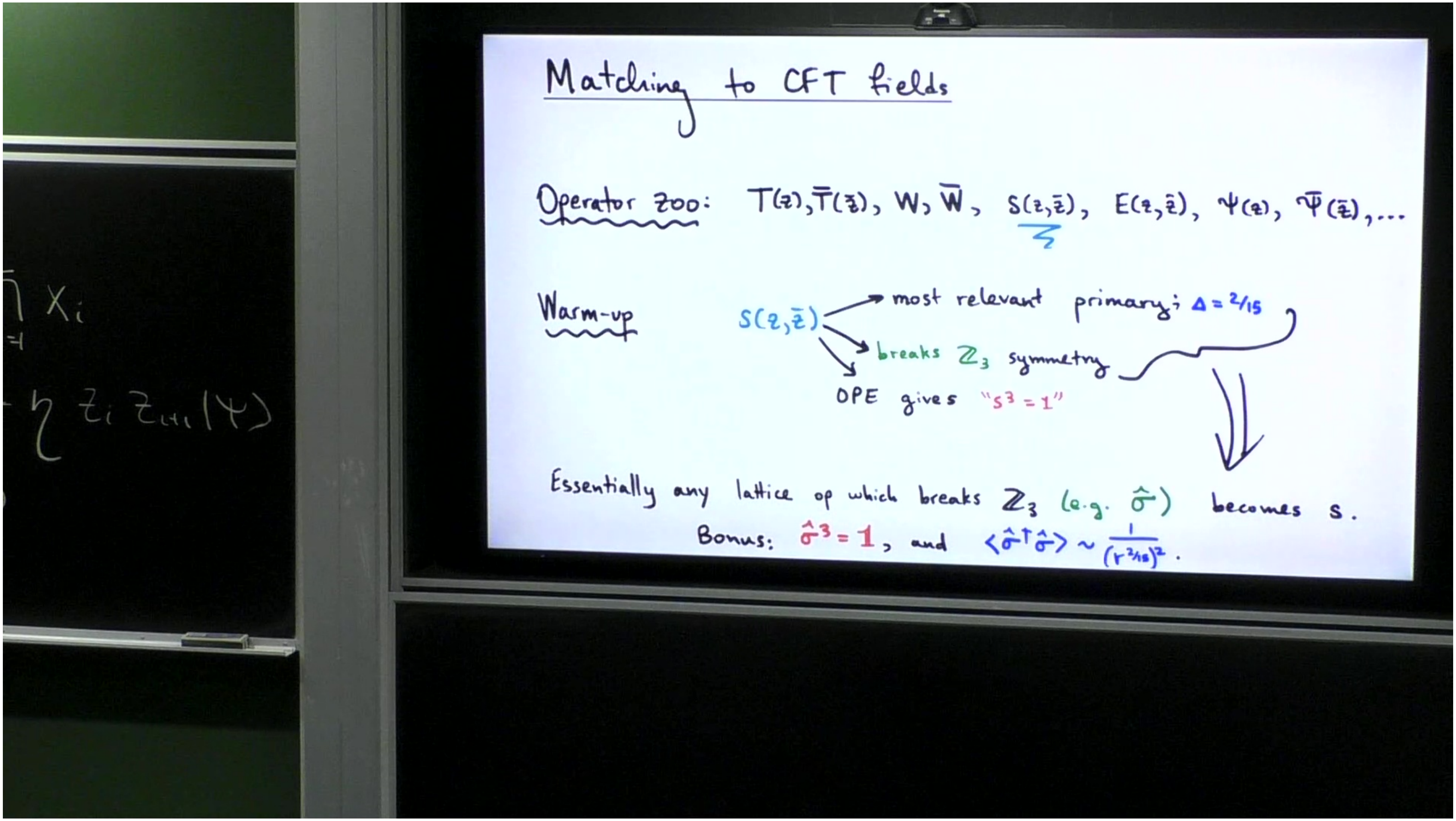

Paper results
Potts model $\xleftrightarrow{1985}$ $\mathbb{Z}_3$ parafermion CFT
Conceptual: lattice-continuum correspondence is often surprising!
* e.g. $\hat{\beta}_L, \hat{\beta}_R$ (lattice parafermions) $\not\leftrightarrow$ $\psi(z), \bar{\psi}(\bar{z})$ (continuum parafermions)
* e.g. $\boxed{\begin{array}{l}\hat{\theta}_L \leftrightarrow \psi(z) \\ \hat{\theta}_R \leftrightarrow \bar{\psi}(\bar{z})\end{array}}$ $\not\Rightarrow$ $\hat{\theta}_L \hat{\theta}_R \leftrightarrow \psi\bar{\psi}$
Experimental: companion paper ($\approx$ 430 citations!) uses this one to design a proposal for universal TQC with superconductors.
Methodological: demonstrates how INTEGRABILITY, SYMMETRY, & NUMERICS can tackle interacting theories on the lattice.


Potts on the lattice
Last time:
$H_{Ising} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$
± 1
1D
$-J \sum_a \sigma_{a+1} \sigma_a$
$\sigma = 1$
$\sigma = -1$
transverse field
Add $-f \sum_a \sigma_a$
Natural generalization:
"conjugate variables"
$\hat{H}_{Potts} = -J \sum_a \hat{\sigma}_{a+1}^\dagger \hat{\sigma}_a - f \sum_a (\hat{\tau}_a^\dagger + \hat{\tau}_a)$
$= +1$
$= -1$
$-3/2$ $-1/2$ $1/2$ $3/2$ $5/2$
$a=-1$ $a=0$ $a=1$ $a=2$
$\sigma = e^{+2\pi i/3}$
$|B\rangle$
$|A\rangle$
$\sigma = 1$
$|C\rangle$
$\sigma = e^{-2\pi i/3}$
$\hat{\tau}$
$\hat{\tau}^\dagger$
J > f : ferromagnet ("ORDER")
f > J : paramagnet ("DISORDER")


Potts on the lattice: symmetries
① translation (a → a±1) + parity (a → −a)
P
(cyclic)
② permutation S₃ ⊃ Z₃
Q̂ = ∏ₐ τ̂ₐ†
Q̂^dual = ∏ₐ ν̂ₐ†
③ charge conjugation
C
④ time-reversal σ̂ → σ̂†, τ̂ → τ̂†
T


Potts on the lattice: dual variables
Disorder operators:
think: chirality!
$\hat{\mu}_b := \prod_{a<b} \hat{\tau}_a$
$\hat{\mu}_b' := \prod_{a>b} \hat{\tau}_a$
$\hat{\mu}\hat{\mu}' = \hat{Q}$
semi local
sym op.
$\hat{H}_{Potts} = -J\sum_a \hat{\sigma}_{a+1}^\dagger \hat{\sigma}_a - f\sum_a (\hat{\tau}_a^\dagger + \hat{\tau}_a)$
$\hat{H}_{Potts} = -J\sum \hat{\mu}_{b+1}^\dagger \hat{\mu}_b - f\sum (\hat{\nu}_b^\dagger + \hat{\nu}_b)$
At $f = J$, $\exists$ duality symmetry
$\mathcal{D}: (\hat{\sigma}, \hat{\tau})_a \to (\hat{\mu}, \hat{\nu})_{a+1/2}$.


Potts on the lattice: parafermion
Last time: $\hat{\psi} := \hat{\sigma}\hat{\mu}$ transforms like a fermion:
$\psi^2 = 1$
swap ⇒ −1
This time: $(\hat{\beta}_L)_{\frac{2a-1}{2a}} \sim \hat{\mu}_{a\mp 1/2}\,\hat{\sigma}_a^\dagger$ is a parafermion:
$\psi^3 = 1$
β_L
β_L
swap ⇒ $e^{2\pi i/3}$
$(\hat{\beta}_R)_{\frac{2a-1}{2a}} \sim \hat{\sigma}_a^\dagger\,\hat{\mu}_{a\mp 1/2}$
Naive guess: $\hat{\beta}_L \longrightarrow \psi(z)$
$\hat{\beta}_R \longrightarrow \bar{\psi}(\bar{z})$


Matching to CFT fields
Operator zoo: T(z), T̄(z̄), W, W̄, S(z,z̄), E(z,z̄), Ψ(z), Ψ̄(z̄),...
Warm-up
S(z,z̄)
most relevant primary; Δ = 2/15
breaks ℤ3 symmetry
OPE gives "s³ = 1"
Essentially any lattice op which breaks ℤ3 (e.g. σ̂) becomes S.
Bonus: σ̂³ = 1, and ⟨σ̂†σ̂⟩ ~ 1/(r^{2/15})²

Matching to CFT fields: what about ψ?

Naive guess: $\psi \sim \hat{\beta} = \hat{\sigma}^{\dagger} \hat{\mu}$.

more relevant

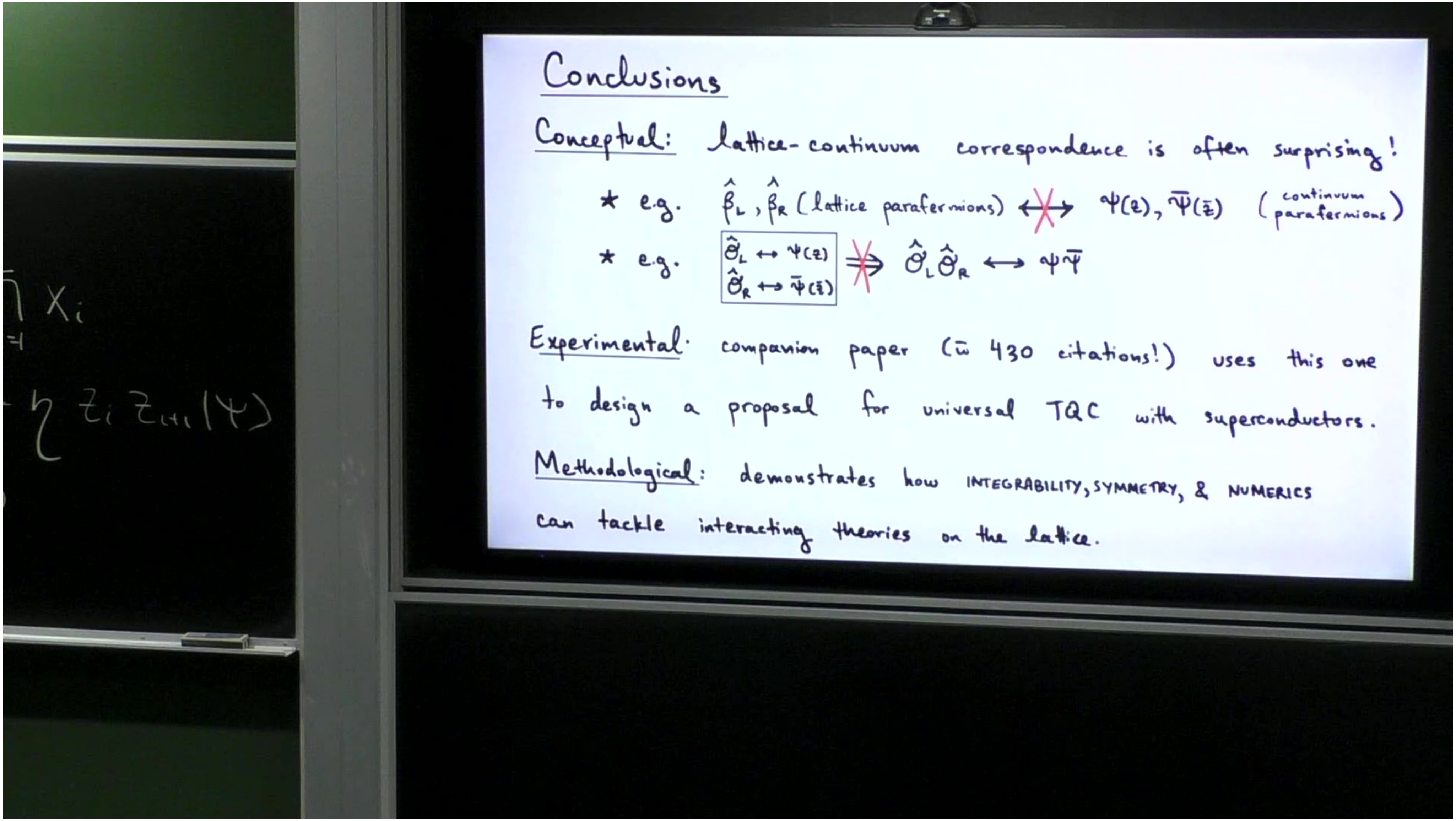
eq.(25)

Why it doesn't work: $\boxed{S^{\dagger}(z,\bar{z})\,\mu(0,0) \overset{\text{OPE}}{\sim} \#\,\psi(0) + \#\,\Phi_{\sigma\bar{\epsilon}}}$

Indeed, syms fully constrain
$$\begin{cases} (\hat{\beta}_R)_a \sim \#\,\bar{\psi} + \#\,(-1)^a\,\Phi_{\epsilon\bar{\sigma}} \\ (\hat{\beta}_L)_a \sim \#\,\psi + \#\,(-1)^a\,\Phi_{\sigma\bar{\epsilon}} \end{cases}$$

Fun fact: $\psi\bar{\psi} \overset{?}{\sim} (\hat{\beta}+\hat{\beta})(\hat{\beta}+\hat{\beta}) \neq \hat{B}$.

wrong.


Conclusions
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★ e.g. $\hat{\beta}_L, \hat{\beta}_R$ (lattice parafermions) $\not\leftrightarrow$ $\psi(z), \bar{\psi}(\bar{z})$ (continuum parafermions)
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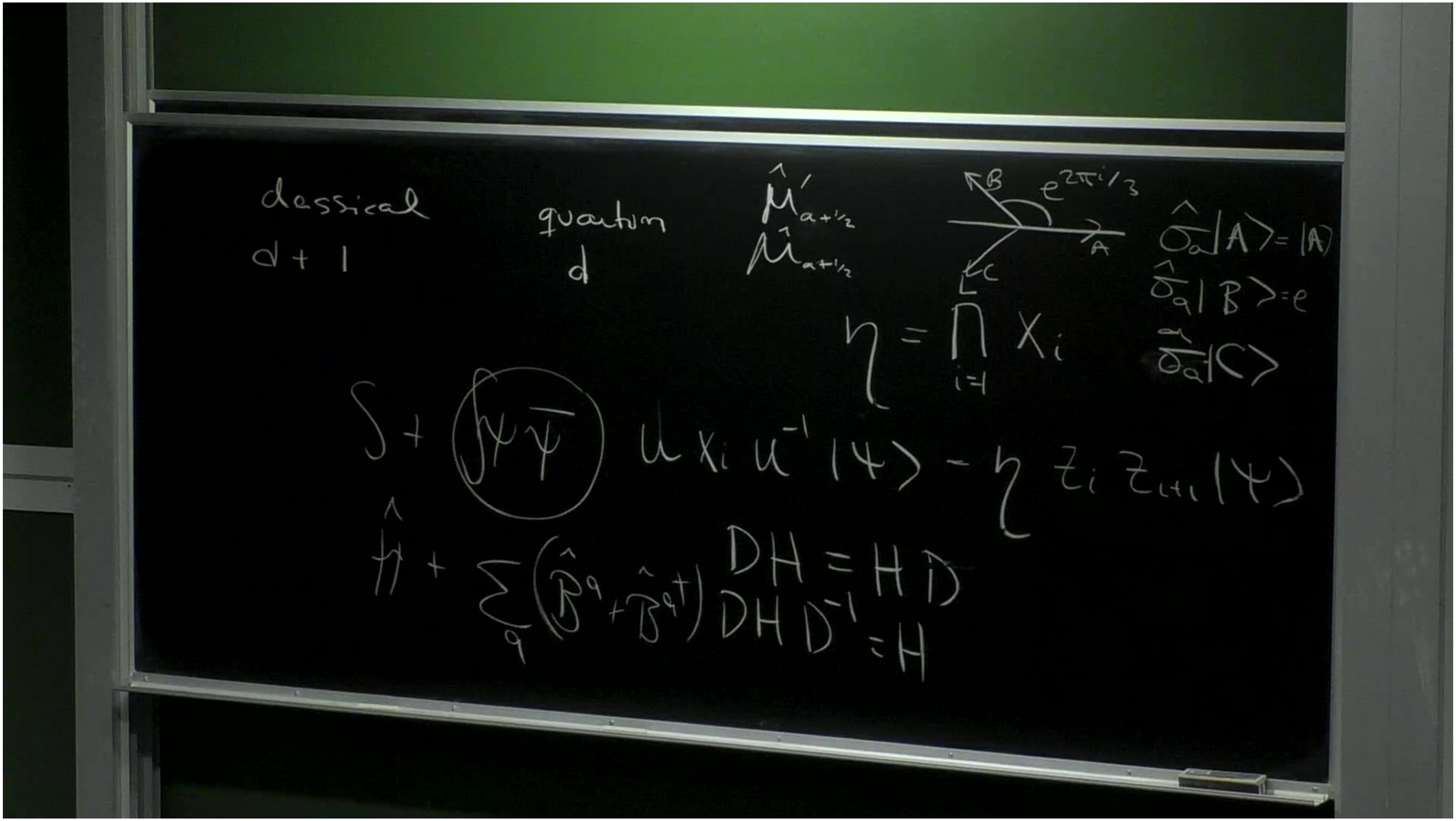
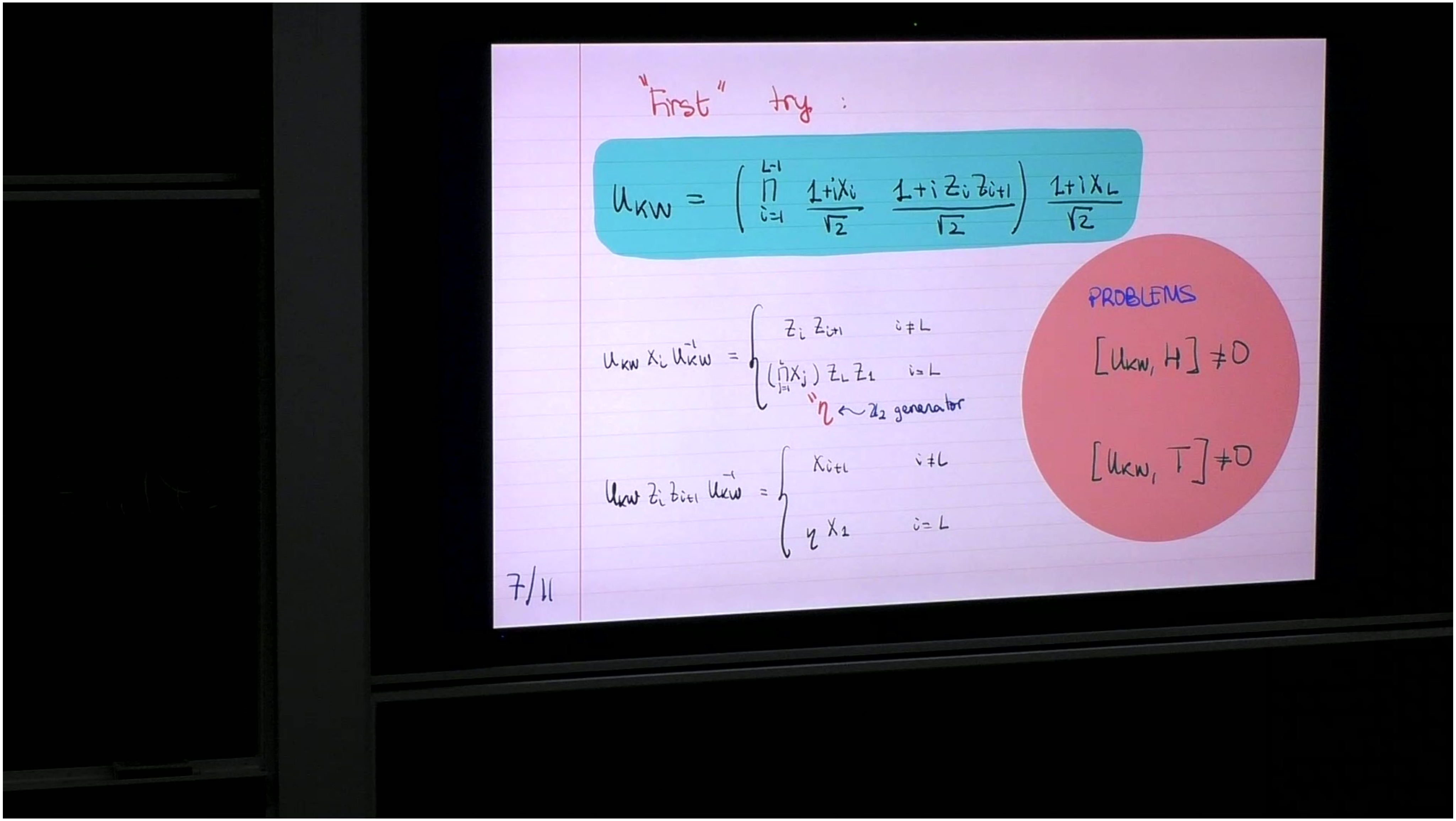

classical
d + 1
quantum
d
$\hat{\mu}'_{a+1/2}$
$\hat{\mu}_{a+1/2}$
$e^{2\pi i/3}$
B
A
C
$\hat{\sigma}_a |A\rangle = |A\rangle$
$\hat{\sigma}_a |B\rangle = e$
$\hat{\sigma}_a |C\rangle$
$\eta = \prod_{i=1} X_i$
$S + \int \psi \bar{\psi}$
$U X_i U^{-1} |\psi\rangle - \eta Z_i Z_{i+1} |\psi\rangle$
$\hat{H} + \sum_q (\hat{B}^q + \hat{B}^{q\dagger})$
$DH = HD$
$DHD^{-1} = H$


"First" try :
$U_{KW} = \left(\prod_{i=1}^{L-1} \frac{1+iX_i}{\sqrt{2}} \frac{1+i Z_i Z_{i+1}}{\sqrt{2}} \right) \frac{1+iX_L}{\sqrt{2}}$
$U_{KW} X_i U_{KW}^{-1} = \begin{cases} Z_i Z_{i+1} & i \neq L \\ (\prod_{j=1}^{L} X_j) Z_L Z_1 & i = L \end{cases}$
"η" ← $\mathbb{Z}_2$ generator
$U_{KW} Z_i Z_{i+1} U_{KW}^{-1} = \begin{cases} X_{i+1} & i \neq L \\ \eta X_1 & i = L \end{cases}$
PROBLEMS
$[U_{KW}, H] \neq 0$
$[U_{KW}, T] \neq 0$
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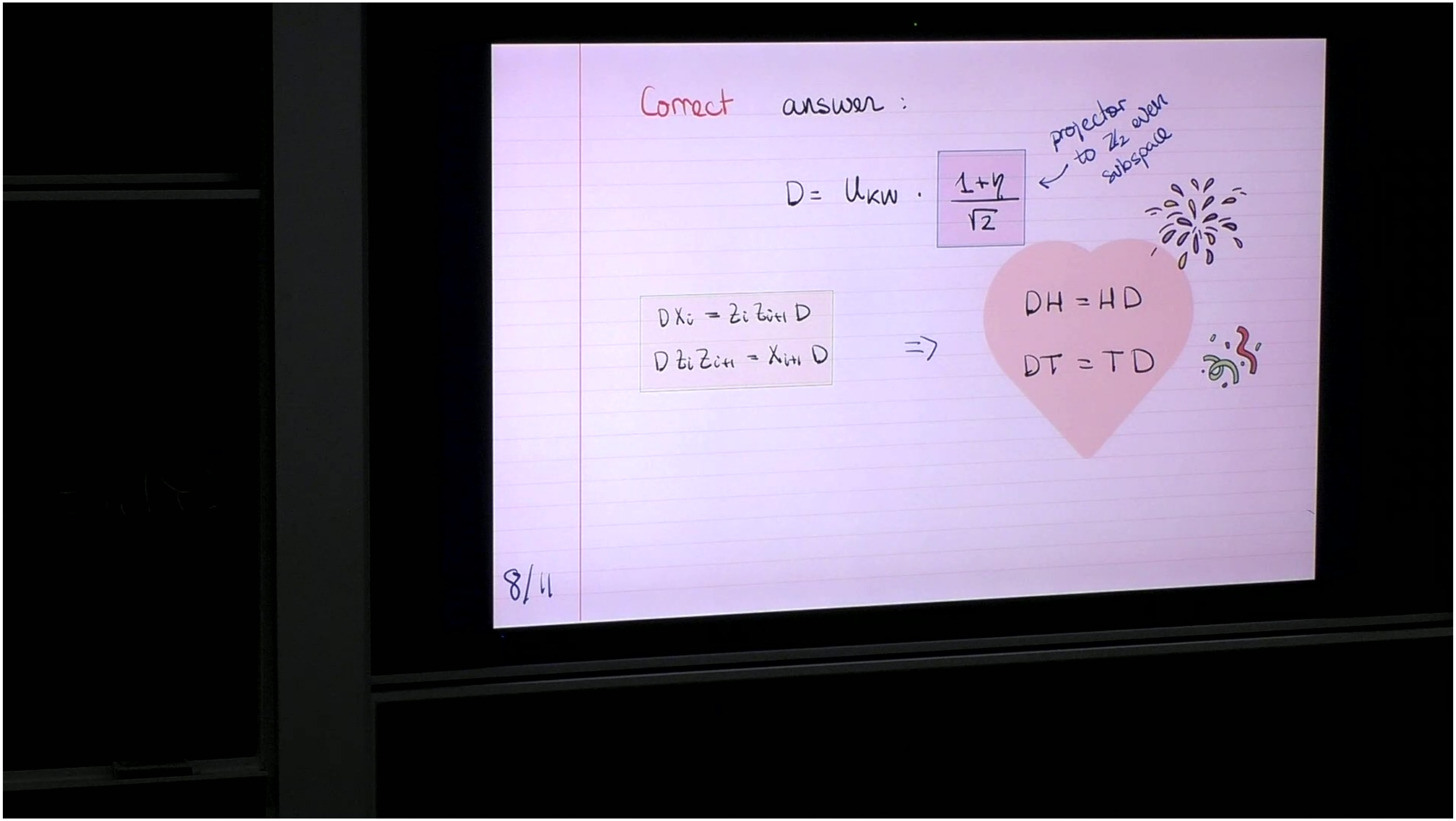
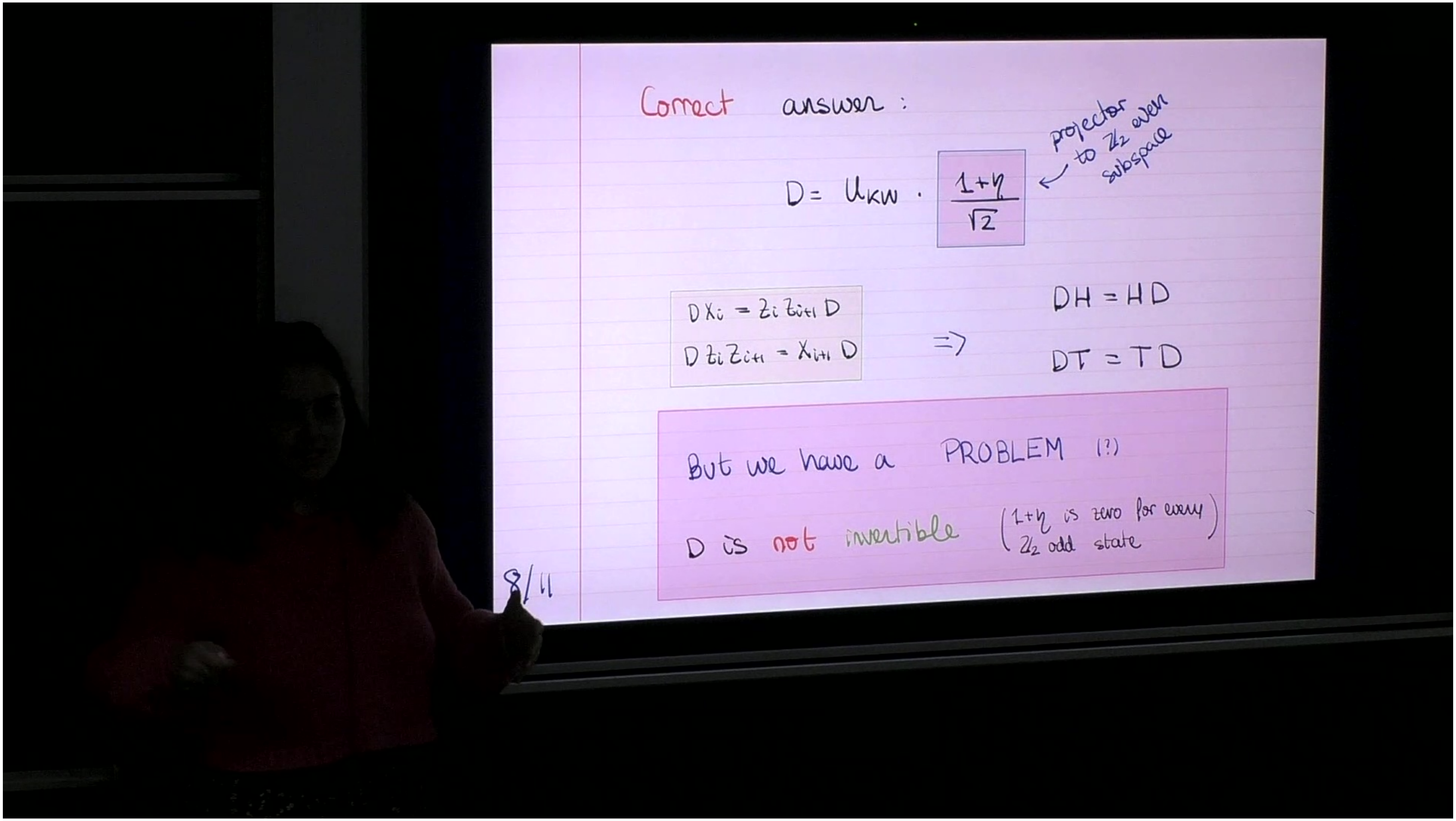


Correct answer:
$D = U_{KW} \cdot \boxed{\frac{1+\eta}{\sqrt{2}}}$
projector to $\mathbb{Z}_2$ even subspace
$D X_i = Z_i Z_{i+1} D$
$D Z_i Z_{i+1} = X_{i+1} D$
$\Rightarrow$
$DH = HD$
$DT = TD$
8/11


Correct answer :
D = U_KW · (1+η)/√2
projector to ℤ2 even subspace
D X_i = Z_i Z_i+1 D
D Z_i Z_i+1 = X_i+1 D
=>
DH = HD
DT = TD
But we have a PROBLEM (?)
D is not invertible (1+η is zero for every ℤ2 odd state)
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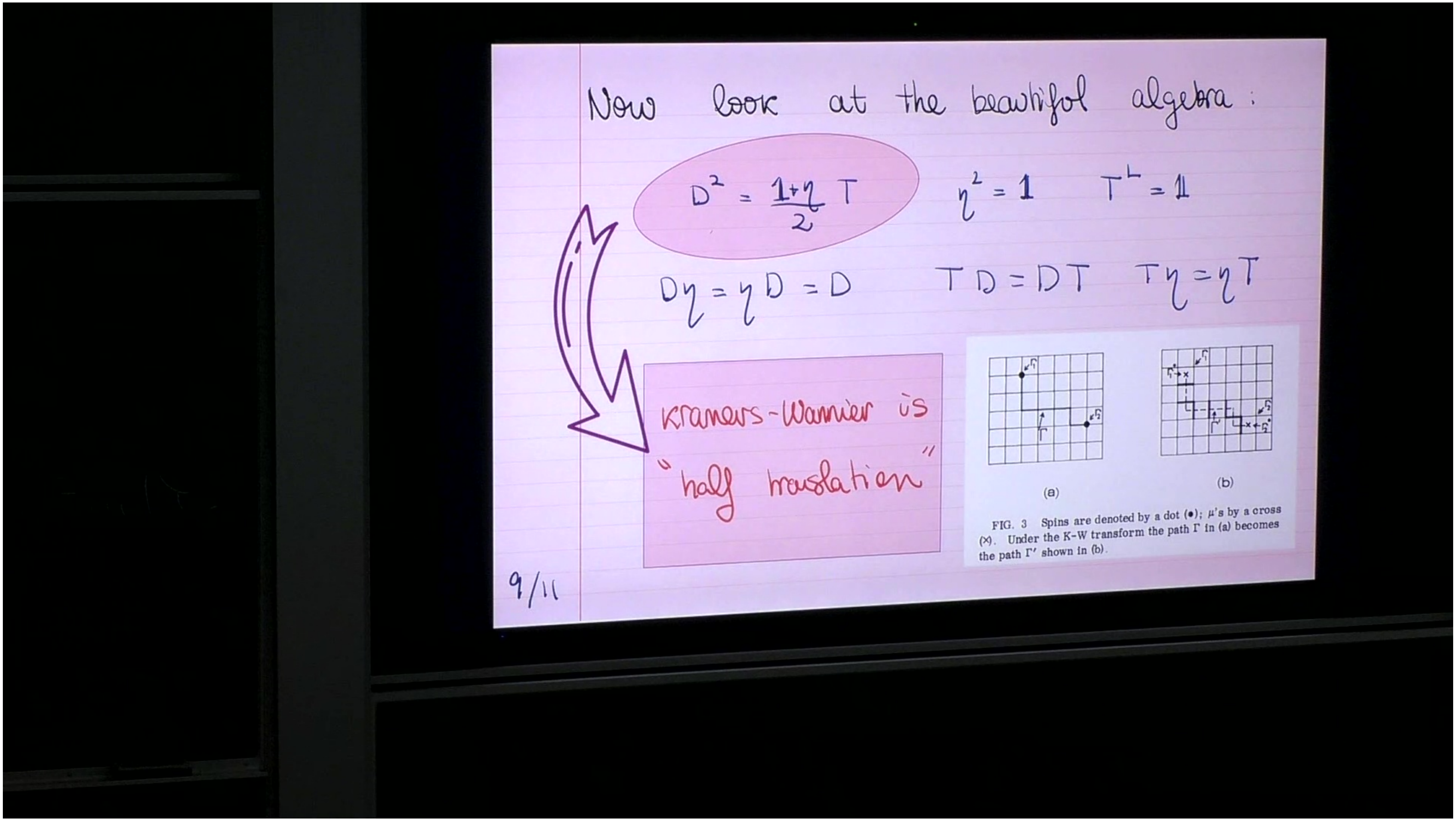
Now look at the beautiful algebra :
$D^2 = \frac{1+\eta}{2} T$
$\eta^2 = 1$
$T^L = 1$
$D\eta = \eta D = D$
$TD = DT$
$T\eta = \eta T$
Kramers-Wannier is
"half translation"
(a)
(b)
FIG. 3 Spins are denoted by a dot (•); μ's by a cross (×). Under the K-W transform the path Γ in (a) becomes the path Γ′ shown in (b).
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D obeys the following:

1) acts on the Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{L} \mathcal{H}_i$

2) Commutes with the Hamiltonian $[D, H] = 0$

3) Flows to the non-invertible D of the Ising CFT when $L \to \infty$

Then, maybe we need to enlarge what our notion of symmetry ...

(Wigner was wrong?)

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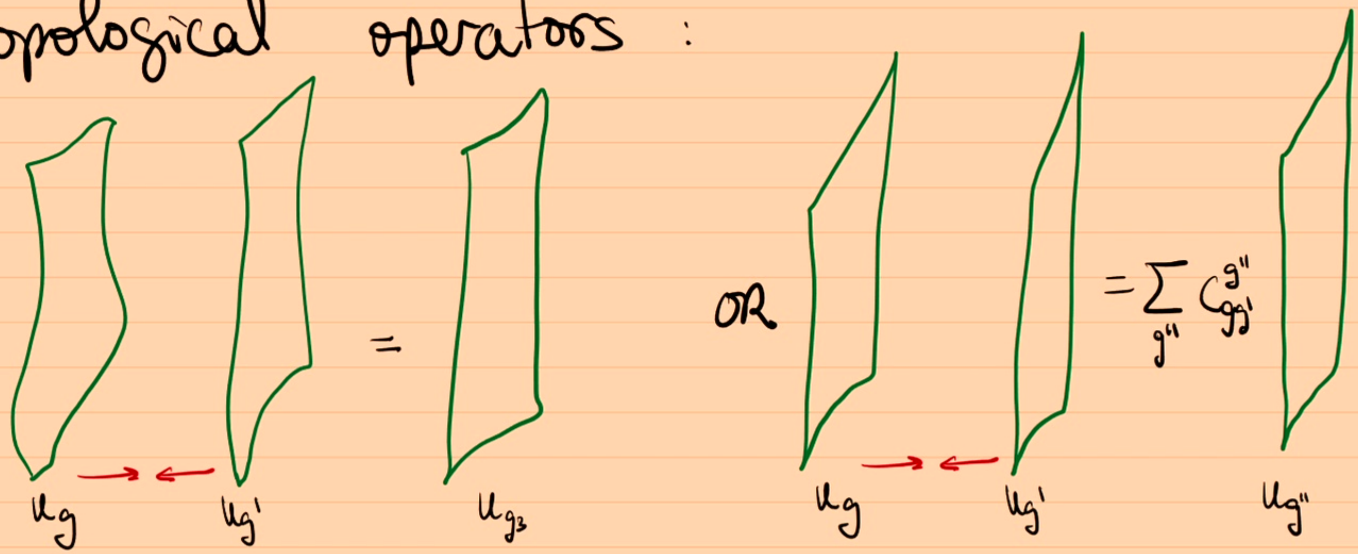
From $[U,H]=0$ in quantum mechanics to

topological operators :



invertible — non invertible

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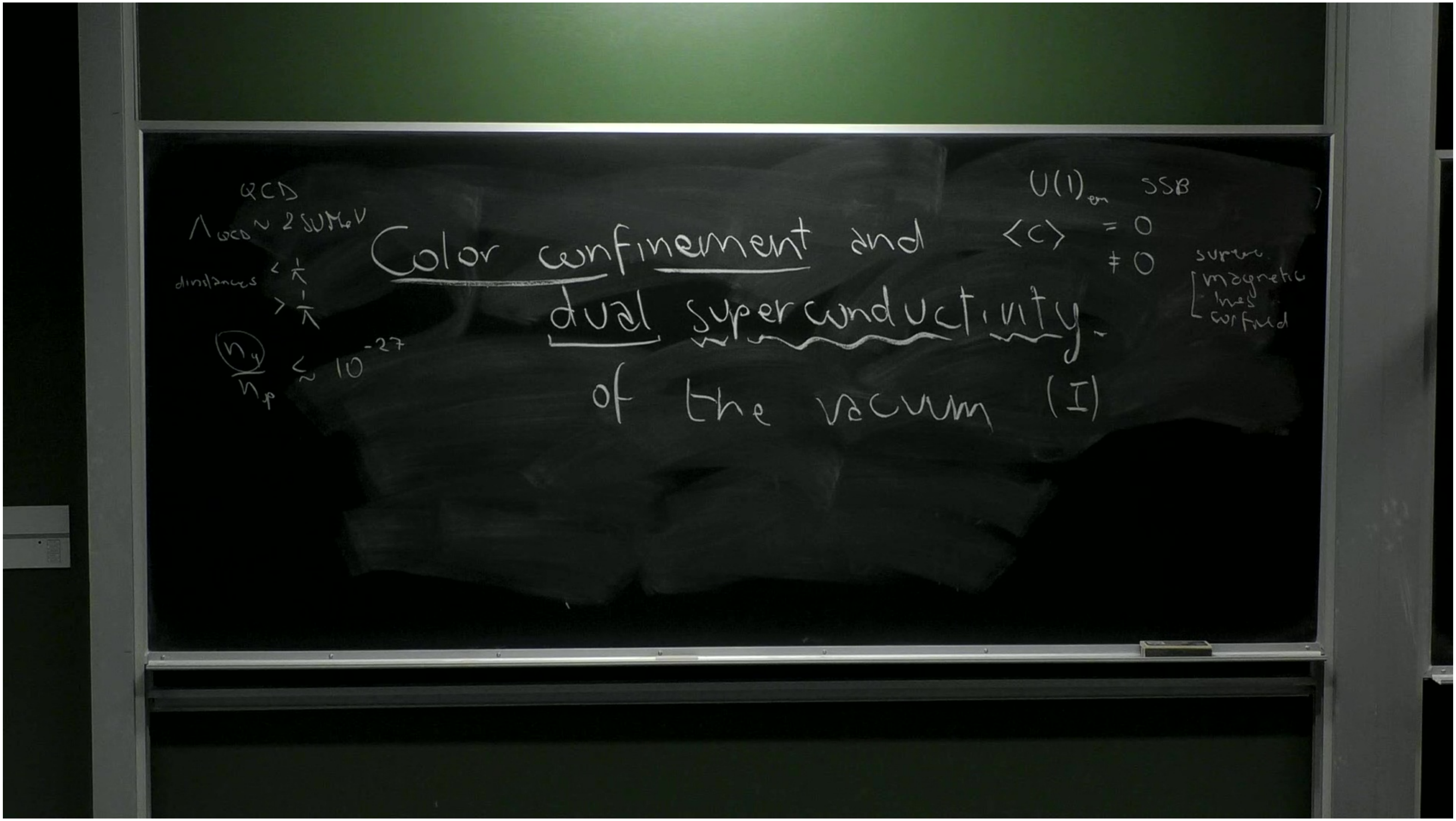

QCD
Color confinement and
dual superconductivity
of the vacuum (I)
U(1)em SSB
<C> = 0
≠ 0
magnetic lines confined

symmetries:

~~Color SU(2)~~

$\cancel{\mathbb{Z}_2}$

? Dual symmetries

- [illegible]
- $J_\mu \to Q$
- $\langle \mu \rangle$

3+1 Gauge Theory:

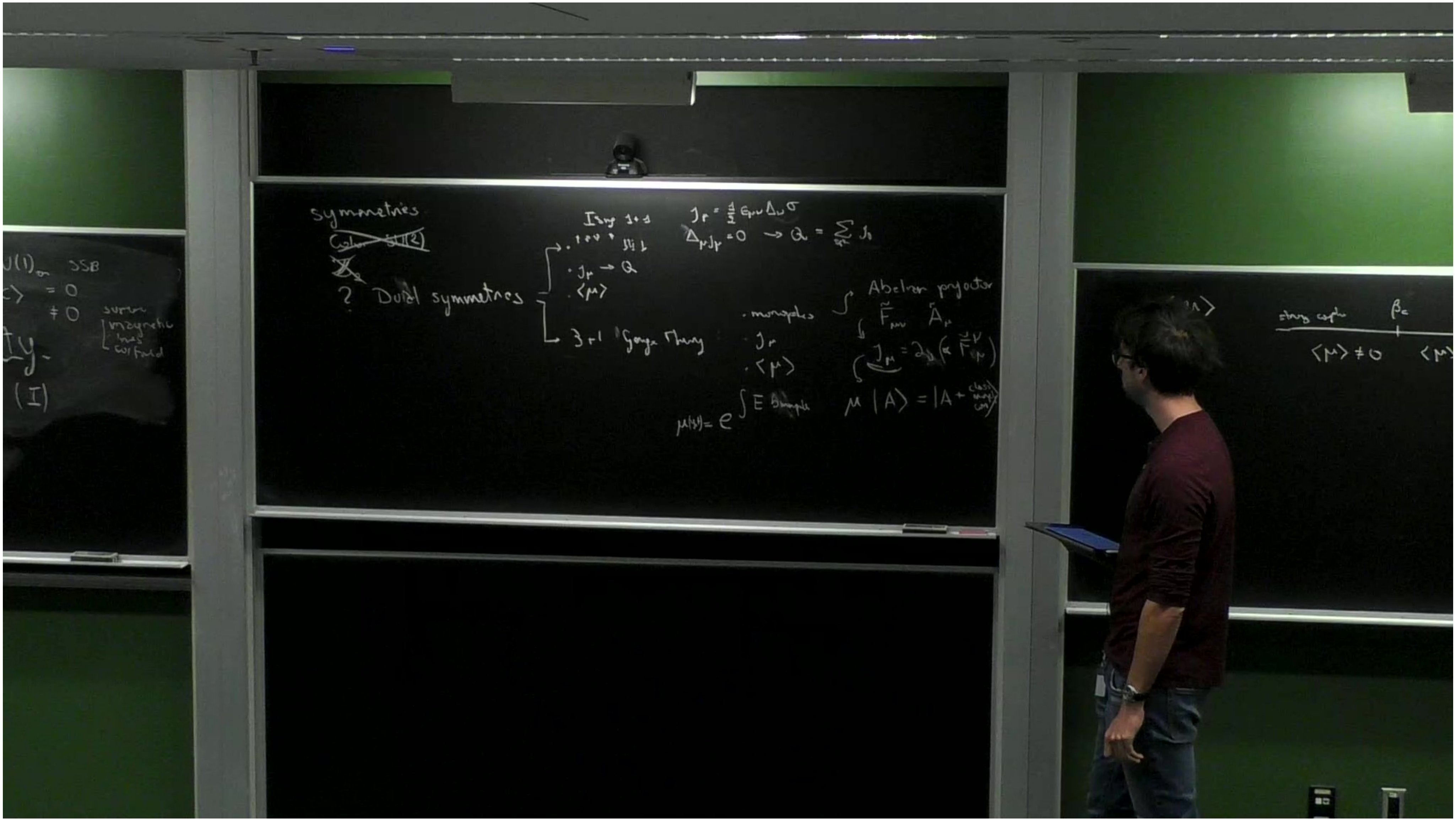
- monopoles
- J_μ
- $\langle \mu \rangle$

Abelian projector

$\tilde{F}_{\mu\nu} \quad \tilde{A}_\mu$

$$J_\mu = \partial_\nu \left(\propto \tilde{F}^{\nu}{}_{\mu} \right)$$

$$\mu \,|A\rangle = \left|A + \begin{smallmatrix}\text{class}\\ \text{monopole}\\ \text{conf}\end{smallmatrix}\right\rangle$$


symmetries
Dual symmetries
monopoles
Abelian projector
Gauge Theory