Title: Lecture - Statistical Physics, PHYS 602

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Subject: Condensed Matter, Other

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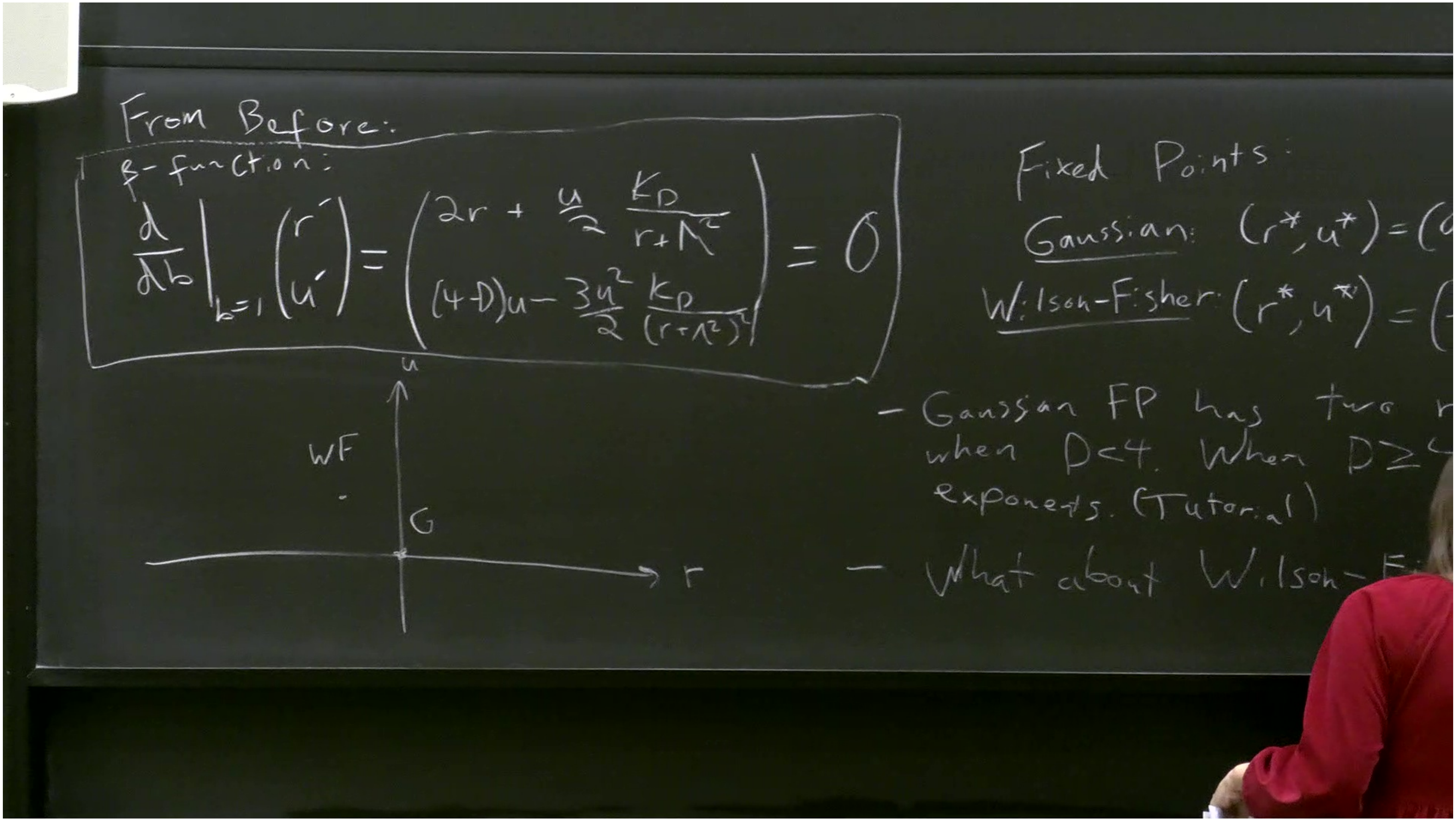


From Before:

β-function:

$$\frac{d}{db}\Big|_{b=1}\begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} 2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2} \\ (4-D)u - \frac{3u^2}{2}\frac{K_D}{(r+\Lambda^2)^2} \end{pmatrix} = 0$$

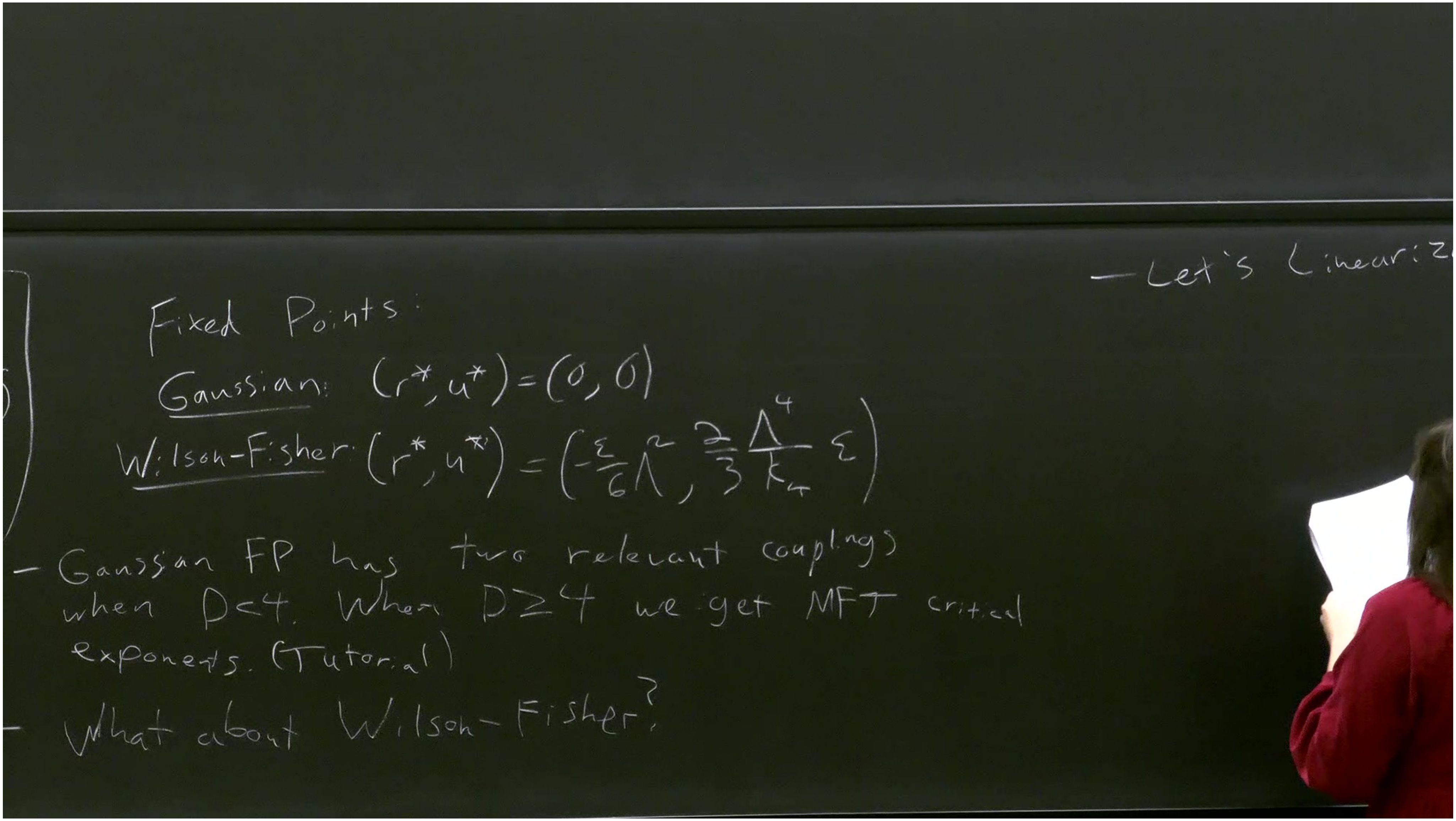
Fixed Points:

Gaussian: $(r^*, u^*) = ($

Wilson-Fisher: $(r^*, u^*) = ($

- Gaussian FP has two r
when $D<4$. When $D \geq$
exponents. (Tutorial)

- What about Wilson-F



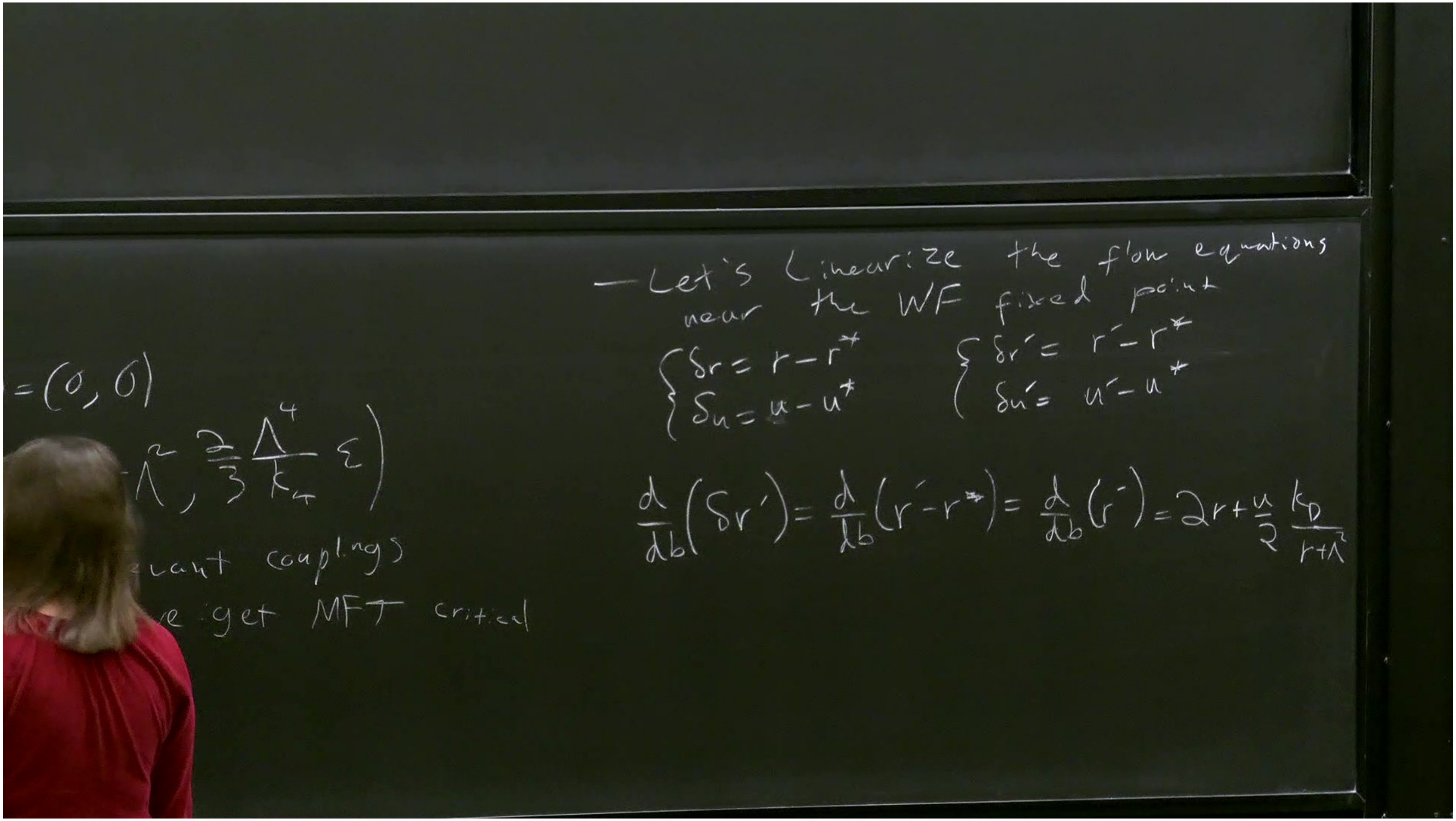
— Let's Lineariz

Fixed Points:

Gaussian: $(r^*, u^*) = (0, 0)$

Wilson-Fisher: $(r^*, u^*) = \left(-\frac{\varepsilon}{6}\Lambda^2, \frac{2}{3}\frac{\Lambda^4}{K_4}\varepsilon\right)$

- Gaussian FP has two relevant couplings when $D<4$. When $D \geq 4$ we get MFT critical exponents. (Tutorial)
- What about Wilson-Fisher?



$= (0, 0)$

$\Lambda^2, \frac{2}{3}\frac{\Lambda^4}{K_4}\varepsilon \Big)$

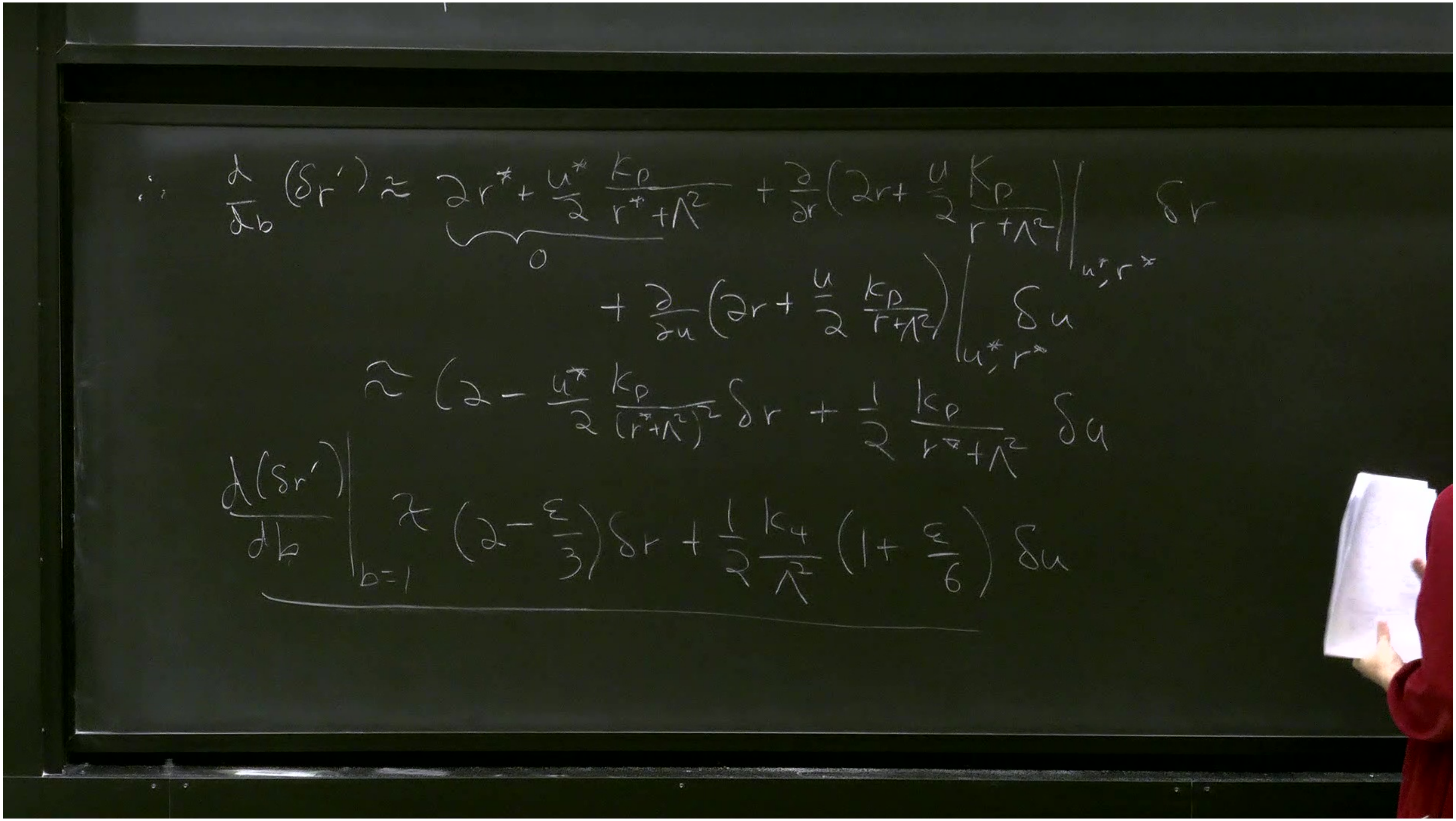
evant couplings

e get MFT critical

— Let's linearize the flow equations near the WF fixed point

$$\begin{cases} \delta r = r - r^* \\ \delta u = u - u^* \end{cases} \qquad \begin{cases} \delta r' = r' - r^* \\ \delta u' = u' - u^* \end{cases}$$

$$\frac{d}{db}\left(\delta r'\right) = \frac{d}{db}\left(r' - r^*\right) = \frac{d}{db}\left(r'\right) = 2r + \frac{u}{2}\frac{K_D}{r+\Lambda^2}$$

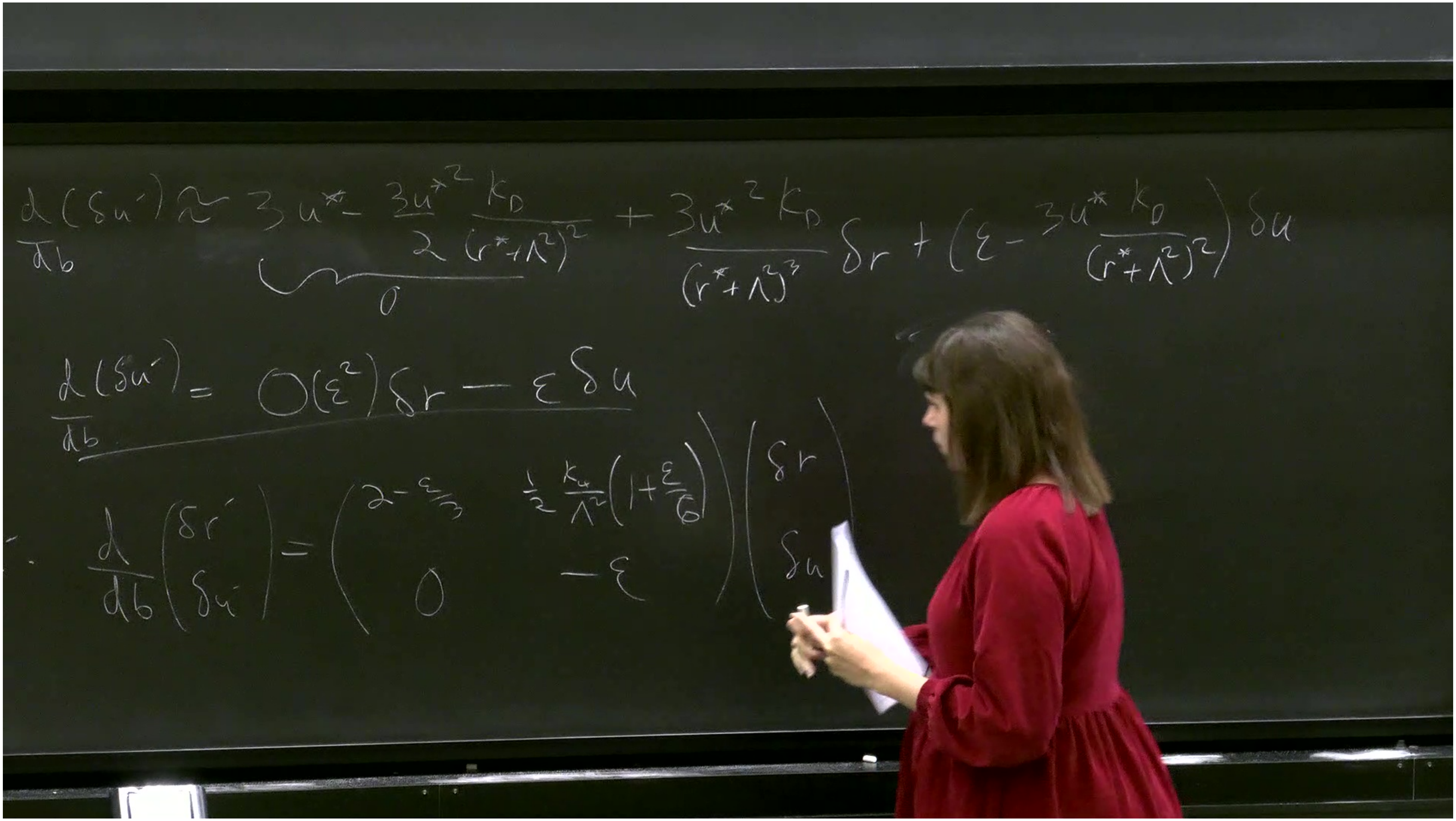


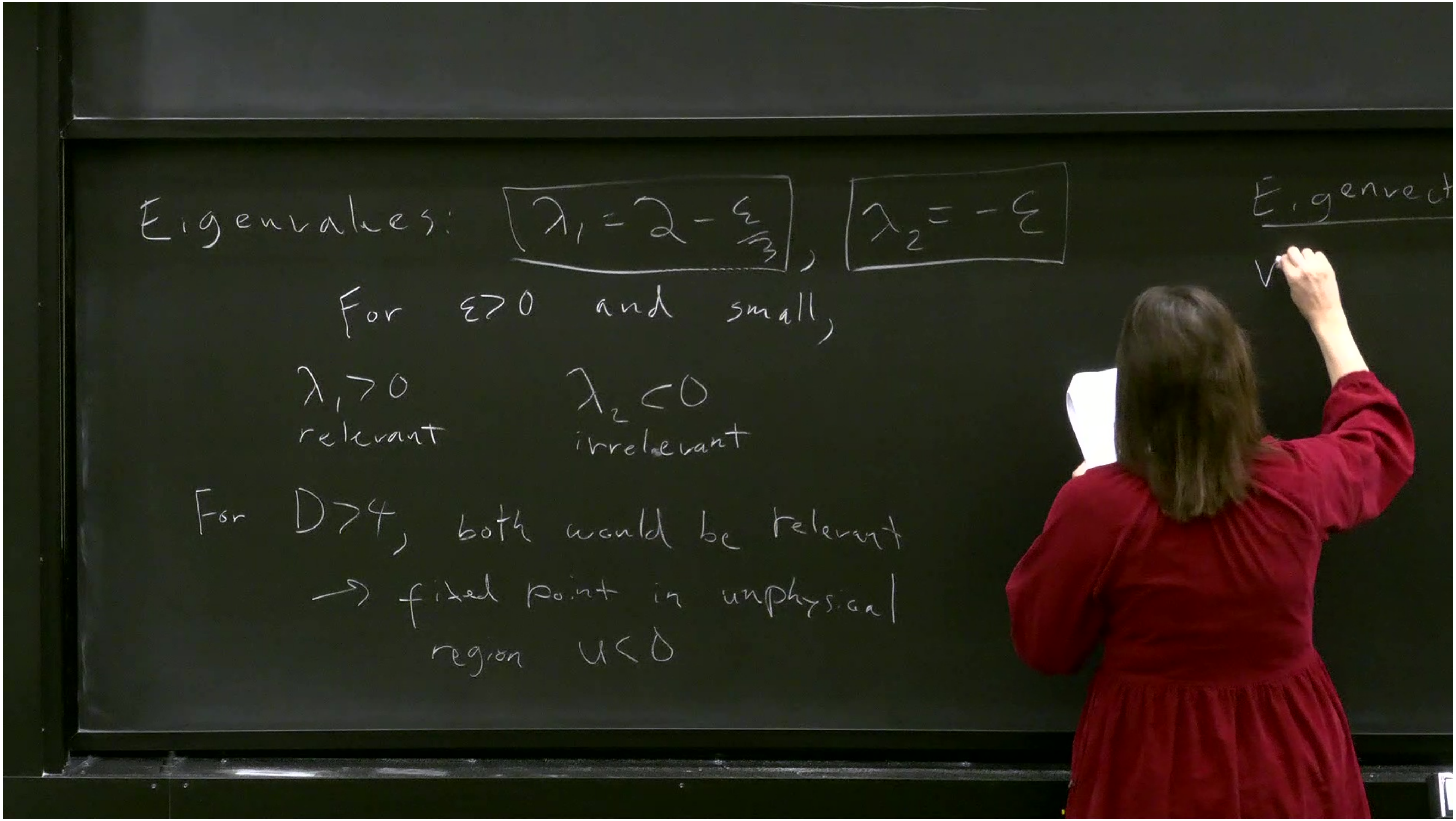
$$\therefore \frac{d}{db}(\delta r') \approx \underbrace{2r^* + \frac{u^*}{2}\frac{K_p}{r^*+\Lambda^2}}_{0} + \frac{\partial}{\partial r}\left(2r + \frac{u}{2}\frac{K_p}{r+\Lambda^2}\right)\Bigg|_{u^*, r^*} \delta r$$

$$+ \frac{\partial}{\partial u}\left(2r + \frac{u}{2}\frac{K_p}{r+\Lambda^2}\right)\Bigg|_{u^*, r^*} \delta u$$

$$\approx \Big(2 - \frac{u^*}{2}\frac{K_p}{(r^*+\Lambda^2)^2}\delta r + \frac{1}{2}\frac{K_p}{r^*+\Lambda^2}\delta u$$

$$\underline{\frac{d(\delta r')}{db}\Bigg|_{b=1} \approx \left(2 - \frac{\varepsilon}{3}\right)\delta r + \frac{1}{2}\frac{K_4}{\Lambda^2}\left(1 + \frac{\varepsilon}{6}\right)\delta u}$$





Eigenvalues: $\boxed{\lambda_1 = 2 - \frac{\varepsilon}{3}}$, $\boxed{\lambda_2 = -\varepsilon}$

For $\varepsilon > 0$ and small,

$\lambda_1 > 0$ relevant

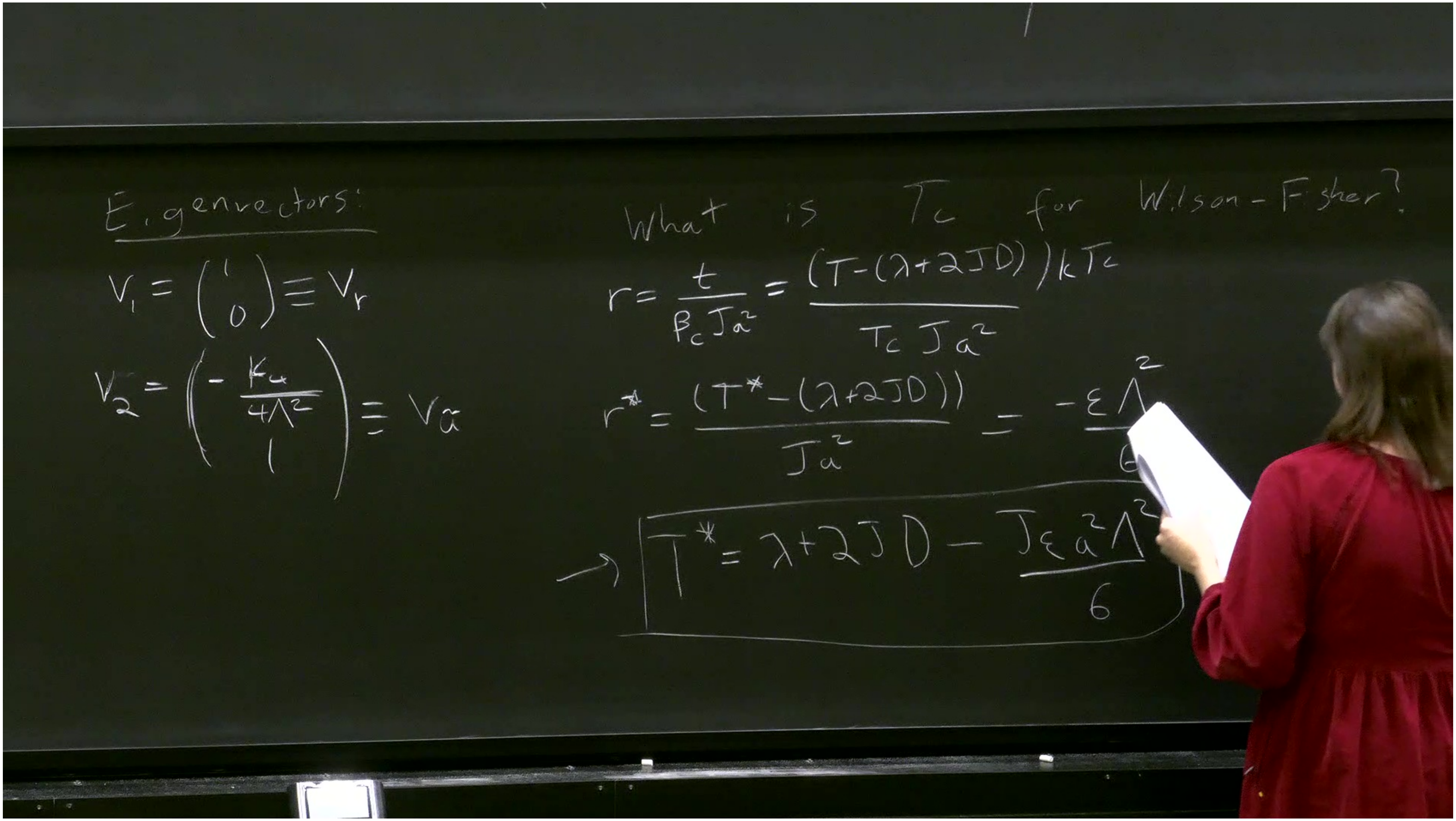
$\lambda_2 < 0$ irrelevant

For $D > 4$, both would be relevant

→ fixed point in unphysical region $u < 0$

Eigenvect

V



Eigenvectors:

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv V_r$$

$$V_2 = \begin{pmatrix} -\frac{K_u}{4\Lambda^2} \\ 1 \end{pmatrix} \equiv V_{\tilde{a}}$$

What is T_c for Wilson-Fisher?

$$r = \frac{t}{\beta_c J a^2} = \frac{(T-(\lambda+2JD))kT_c}{T_c J a^2}$$

$$r^* = \frac{(T^*-(\lambda+2JD))}{Ja^2} = -\frac{\varepsilon\Lambda^2}{6}$$

$$\rightarrow \boxed{T^* = \lambda + 2JD - \frac{J\varepsilon a^2\Lambda^2}{6}}$$

ctors:

$) \equiv V_r$

$\Lambda^2) \equiv V_{\tilde{a}}$

What is T_c for Wilson-Fisher?

$$r = \frac{t}{\beta_c J a^2} = \frac{(T-(\lambda+2JD))kT_c}{T_c J a^2}$$

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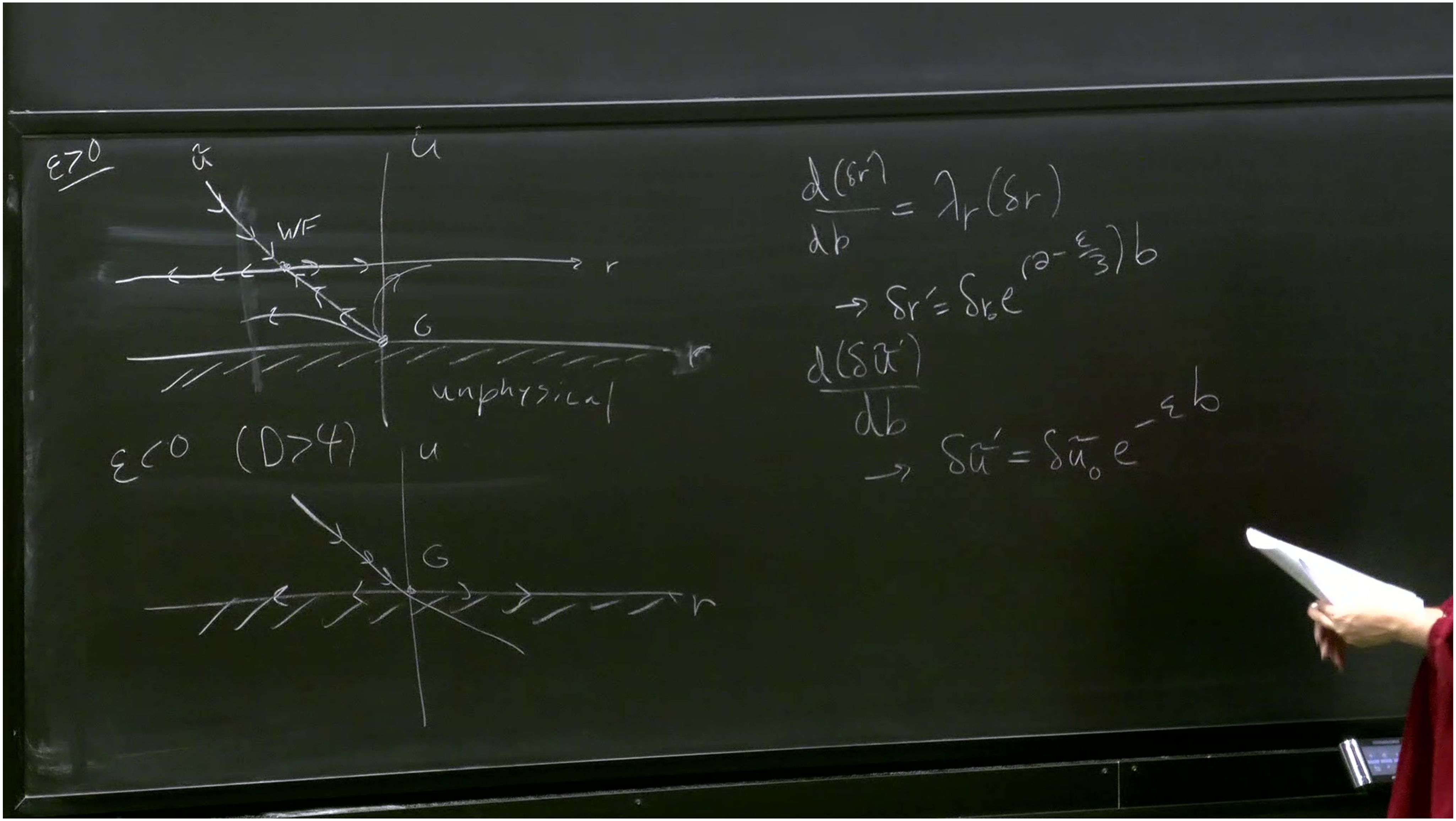
lower than Gaussian prediction.



$$\frac{d(\delta r)}{db} = \lambda_r(\delta r)$$

$$\rightarrow \delta r' = \delta r_0 e^{(2-\frac{\varepsilon}{3})b}$$

$$\frac{d(\delta \tilde{u}')}{db}$$

$$\rightarrow \delta \tilde{u}' = \delta \tilde{u}_0 e^{-\varepsilon b}$$

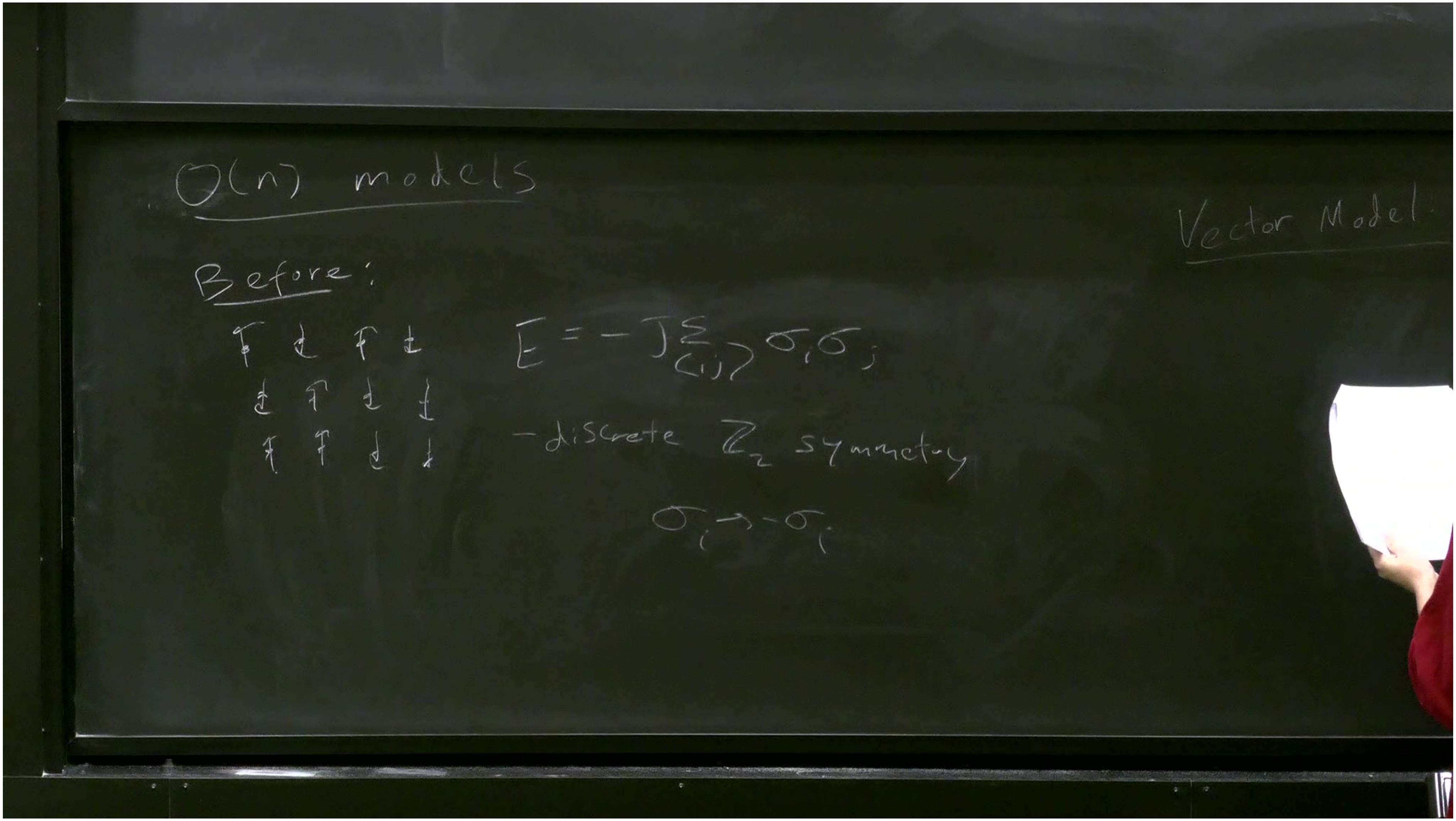
Critical Exponents:

$\boxed{\eta=0}$ $\quad \xi \sim \frac{1}{t_r^{1/\lambda_r}} \quad \nu = \frac{1}{2-\varepsilon/3} \approx \frac{1}{2}+\frac{\varepsilon}{12}$

$\alpha \approx \frac{\varepsilon}{6}$, $\beta \approx \frac{1}{2}+\frac{\varepsilon}{6}$, $\gamma \approx 1+\frac{\varepsilon}{6}$, $\delta = 3\varepsilon$

Something unjustified; what if $\varepsilon=1$? $(D=3)$

	α	β	γ	δ	η	ν
num.	0.11	0.33	1.24	4.79	0.04	0.62
$\varepsilon=1$	0.17	0.33	1.17	4	0	0.58



O(n) models

Before:

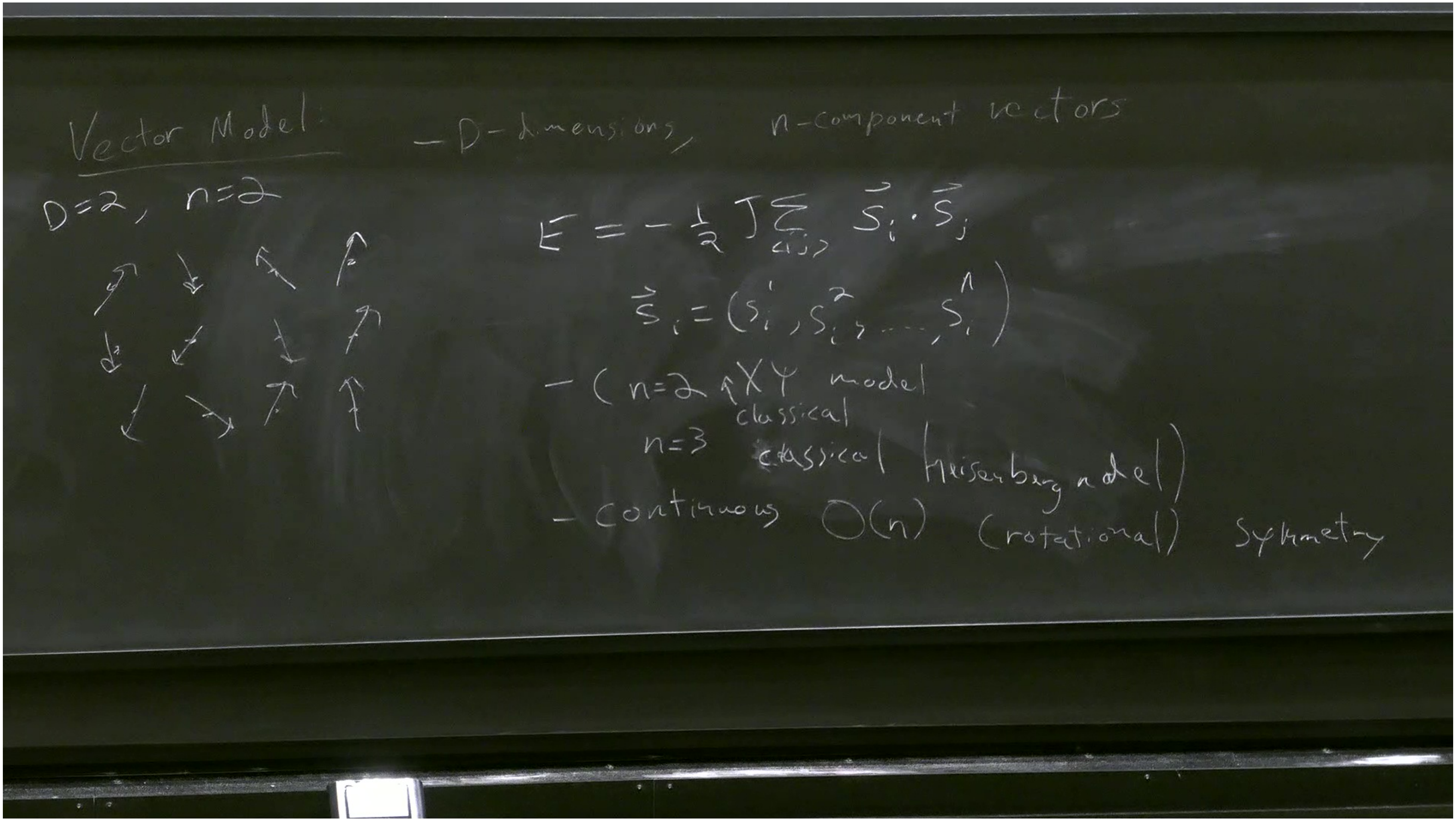
↑ ↓ ↑ ↓
↓ ↑ ↓ ↓
↑ ↑ ↓ ↓

$$E = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

- discrete $\mathbb{Z}_2$ symmetry

$$\sigma_i \to -\sigma_i$$

Vector Model:



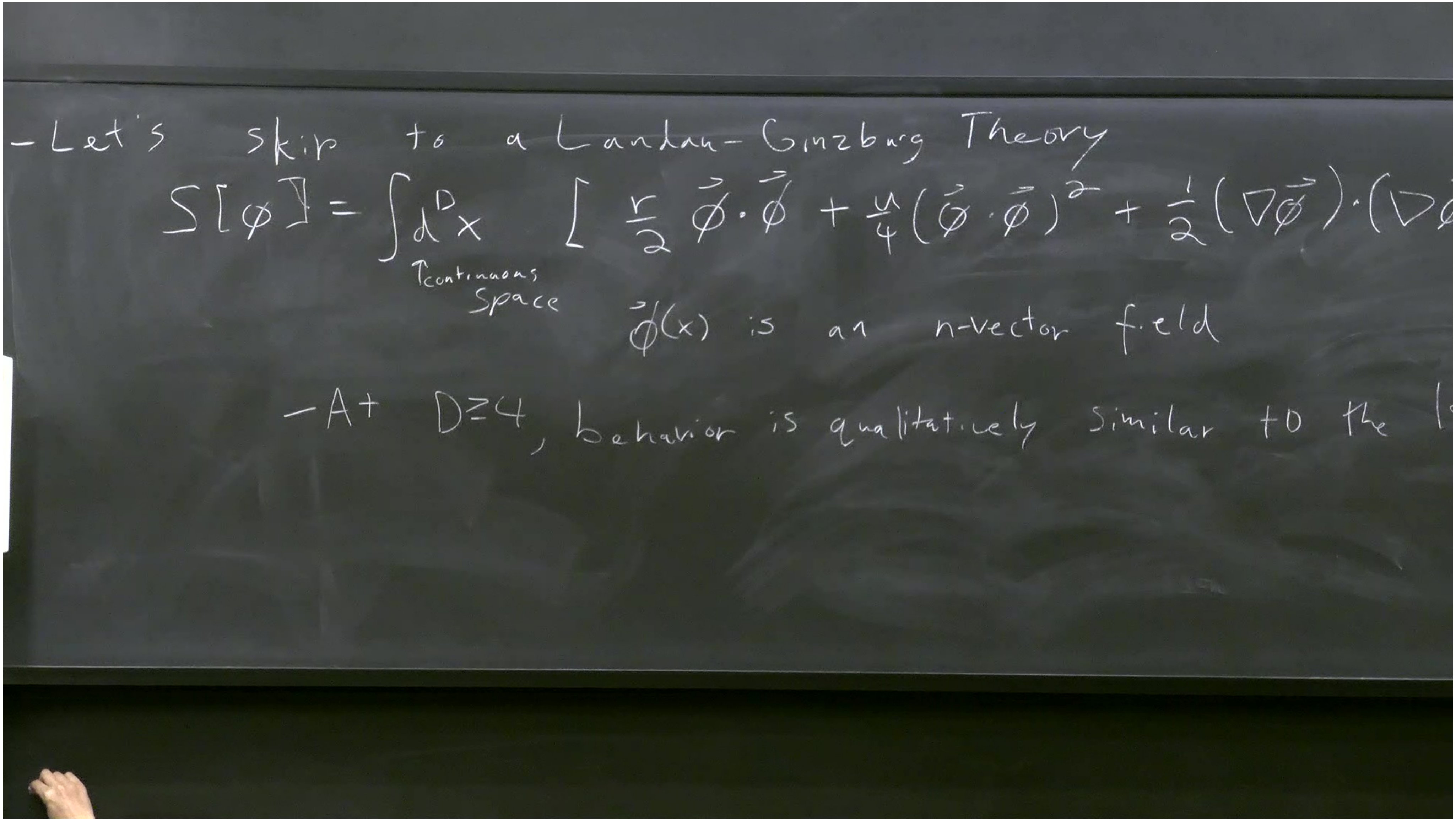
Vector Model: - D-dimensions, n-component vectors

D=2, n=2

$$E = -\frac{1}{2} J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$\vec{S}_i = (S_i^1, S_i^2, \ldots, S_i^n)$$

- (n=2) XY model
  classical
- n=3 classical (Heisenberg model)
- continuous O(n) (rotational) symmetry



- Let's skip to a Landau-Ginzburg Theory

$$S[\phi] = \int \underset{\uparrow \text{continuous space}}{d^D x} \Big[\frac{r}{2}\vec{\phi}\cdot\vec{\phi} + \frac{u}{4}(\vec{\phi}\cdot\vec{\phi})^2 + \frac{1}{2}(\nabla\vec{\phi})\cdot(\nabla$$

$\vec{\phi}(x)$ is an n-vector field

- At $D \geq 4$, behavior is qualitatively similar to the classical

$n=3$ classical (Heisenberg model)

- continuous $O(n)$ (rotational) sym

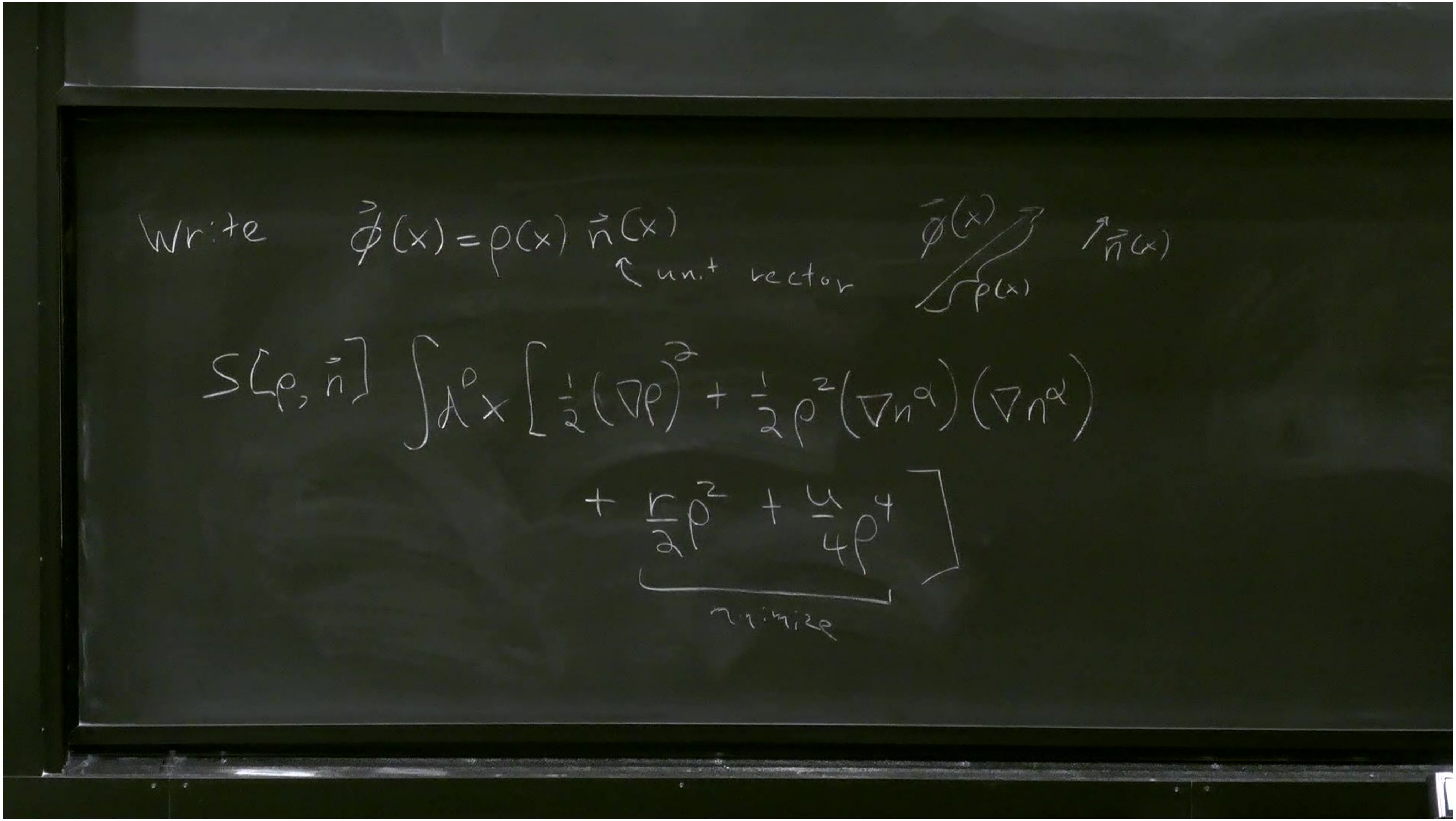
- Ginzburg Theory

$$\cdot\vec{\phi} + \frac{u}{4}(\vec{\phi}\cdot\vec{\phi})^2 + \frac{1}{2}(\nabla\vec{\phi})\cdot(\nabla\vec{\phi})]$$

$$\sum_{\mu=1}^{D} \frac{\partial\vec{\phi}}{\partial x^\mu}\cdot\frac{\partial\vec{\phi}}{\partial x^\mu}$$

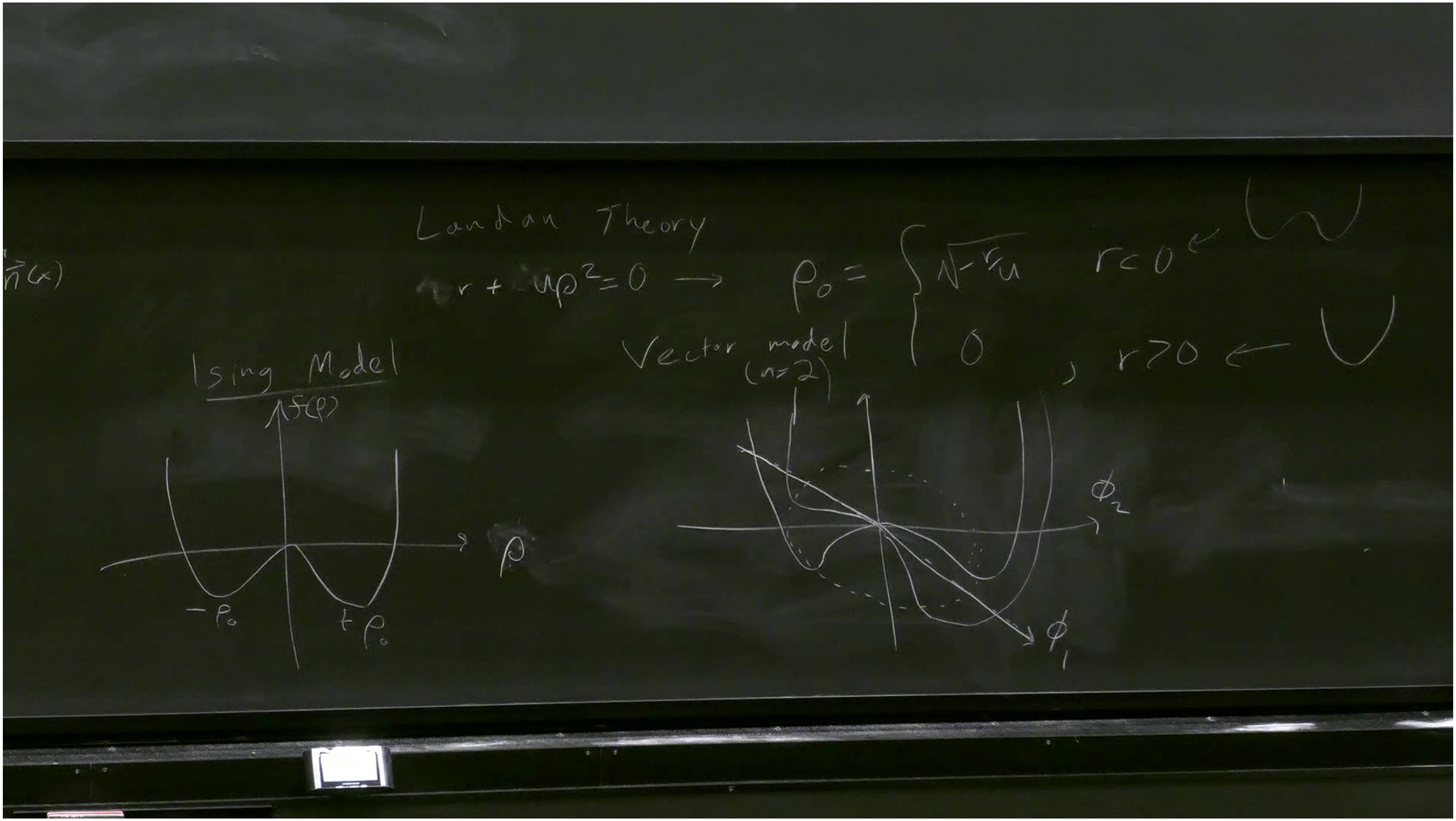
is an n-vector field

is qualitatively similar to the Ising



Write $\vec{\phi}(x) = \rho(x)\, \underset{\uparrow \text{ unit vector}}{\vec{n}(x)}$

$$S[\rho, \vec{n}] \int d^D x \left[\frac{1}{2}(\nabla\rho)^2 + \frac{1}{2}\rho^2 (\nabla n^\alpha)(\nabla n^\alpha) + \underbrace{\frac{r}{2}\rho^2 + \frac{u}{4}\rho^4}_{\text{minimize}} \right]$$



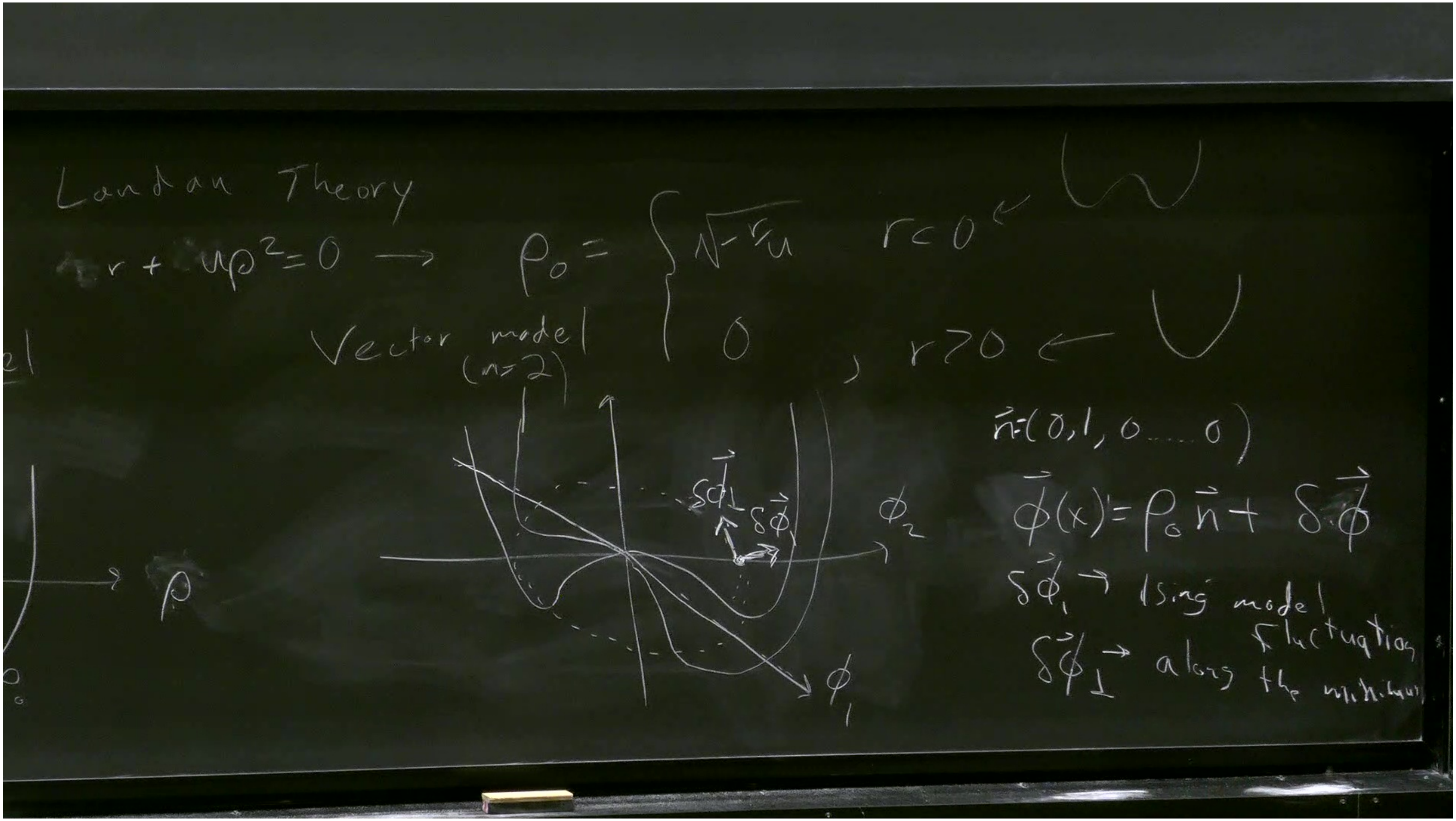
Landau Theory

$\vec{n}(x)$

$$r + u\rho^2 = 0 \longrightarrow \rho_0 = \begin{cases} \sqrt{-r/u} & r < 0 \leftarrow \\ 0, & r > 0 \leftarrow \end{cases}$$

Ising Model

Vector model (n=2)



Landau Theory

$$r + u\rho^2 = 0 \longrightarrow \rho_0 = \begin{cases} \sqrt{-r/u} & r<0 \\ 0 & r>0 \end{cases}$$

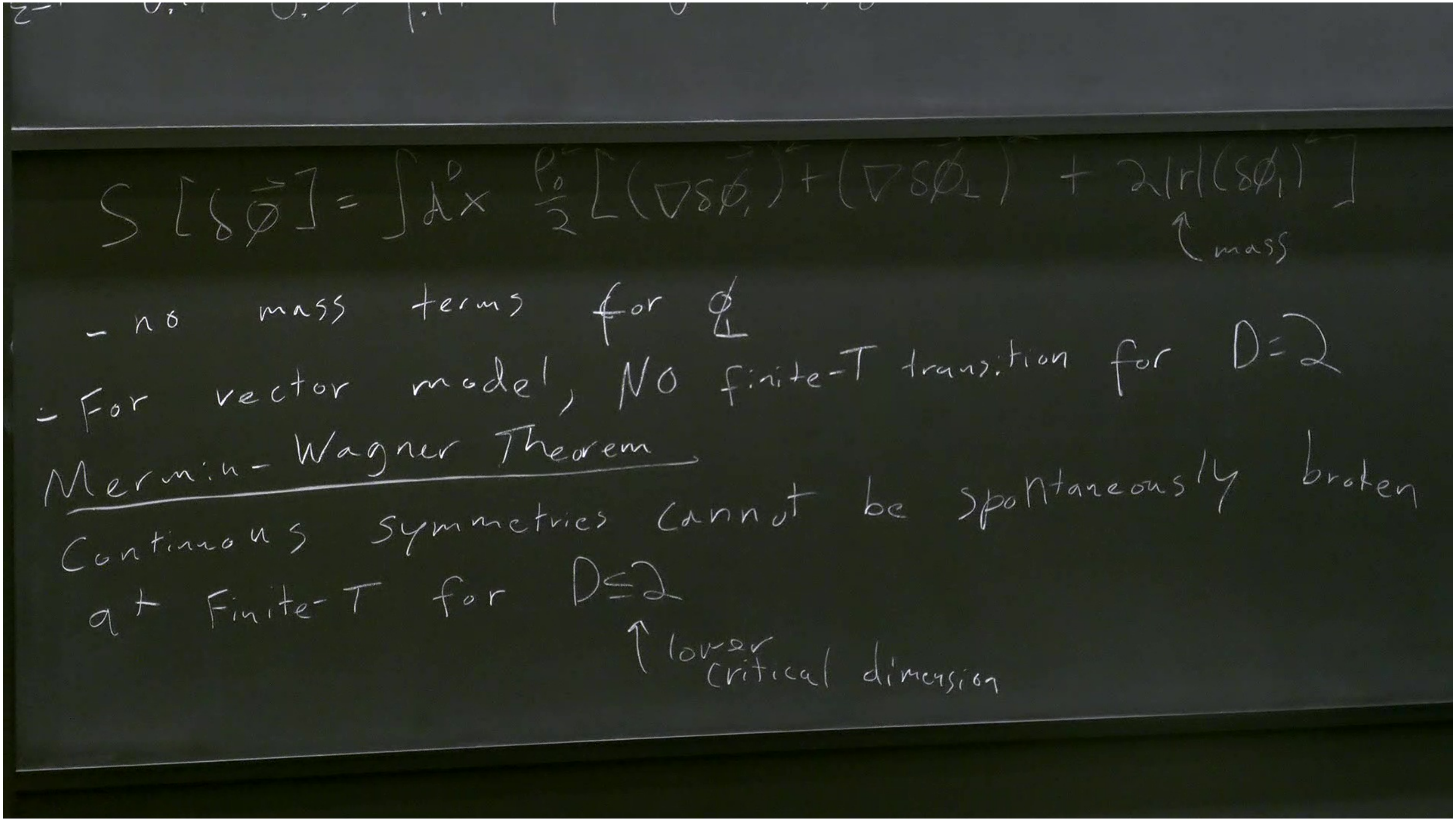
Vector model
(n=2)

$\vec{n} = (0, 1, 0 \ldots\ldots 0)$

$\vec{\phi}(x) = \rho_0 \vec{n} + \delta\vec{\phi}$

$\delta\vec{\phi}_1 \rightarrow$ Ising model

$\delta\vec{\phi}_\perp \rightarrow$ fluctuation along the minimum



$$S[\delta\vec{\phi}] = \int d^Dx \, \frac{\rho_0}{2}\left[(\nabla\delta\vec{\phi}_1)^2 + (\nabla\delta\vec{\phi}_\perp)^2 + 2|r|(\delta\phi_1)^2\right]$$

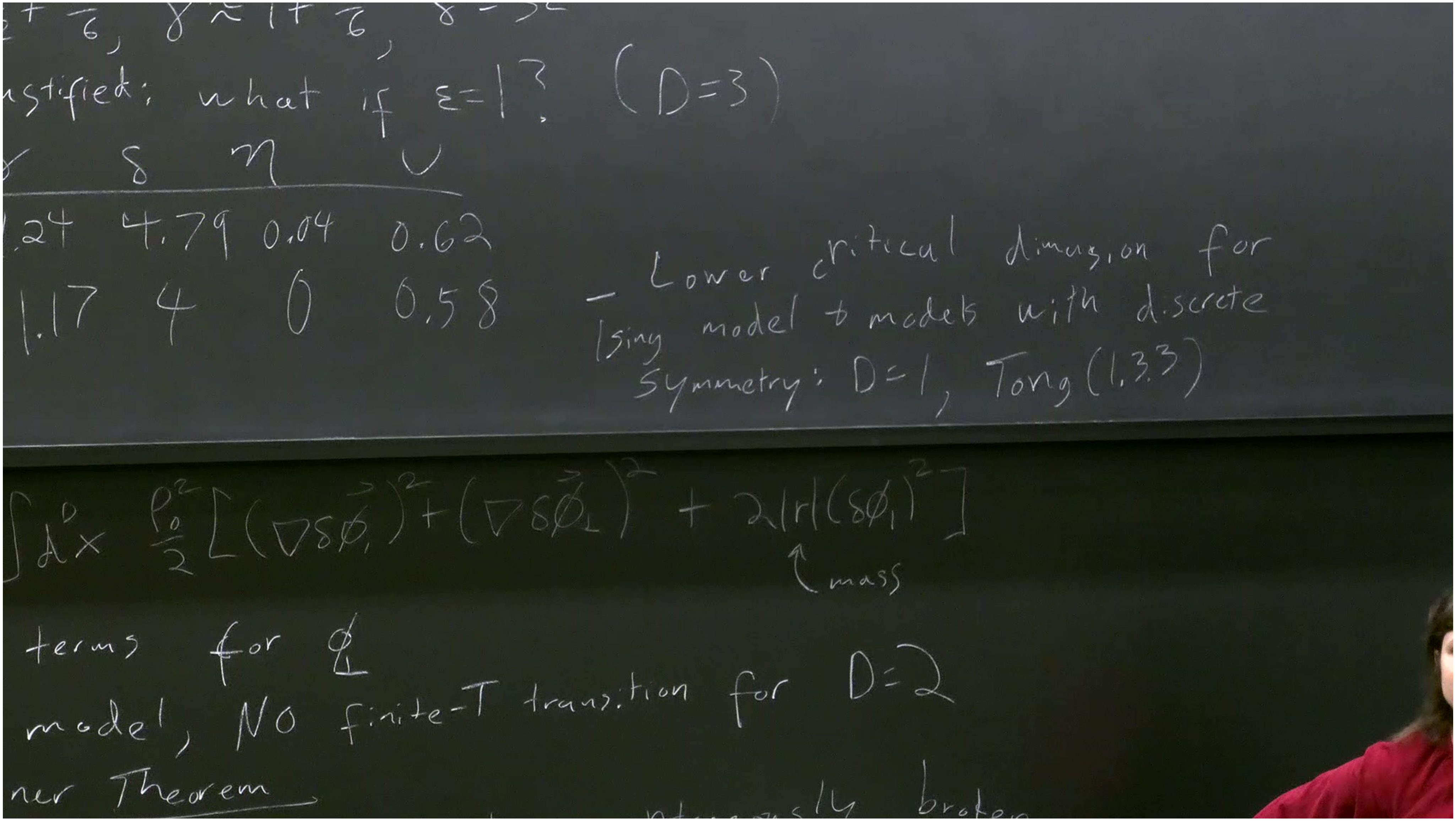
mass

- no mass terms for $\phi_\perp$
- For vector model, NO finite-T transition for D=2

Mermin-Wagner Theorem

Continuous symmetries cannot be spontaneously broken at Finite-T for $D \leq 2$

lower critical dimension



$+\ \frac{\ }{6},\quad \gamma \approx 1 + \frac{\ }{6},\quad \delta = 3$

stified: what if $\varepsilon = 1$? $(D=3)$

γ	δ	η	ν
.24	4.79	0.04	0.62
1.17	4	0	0.58

– Lower critical dimension for Ising model + models with discrete symmetry: $D=1$, Tong (1.3.3)

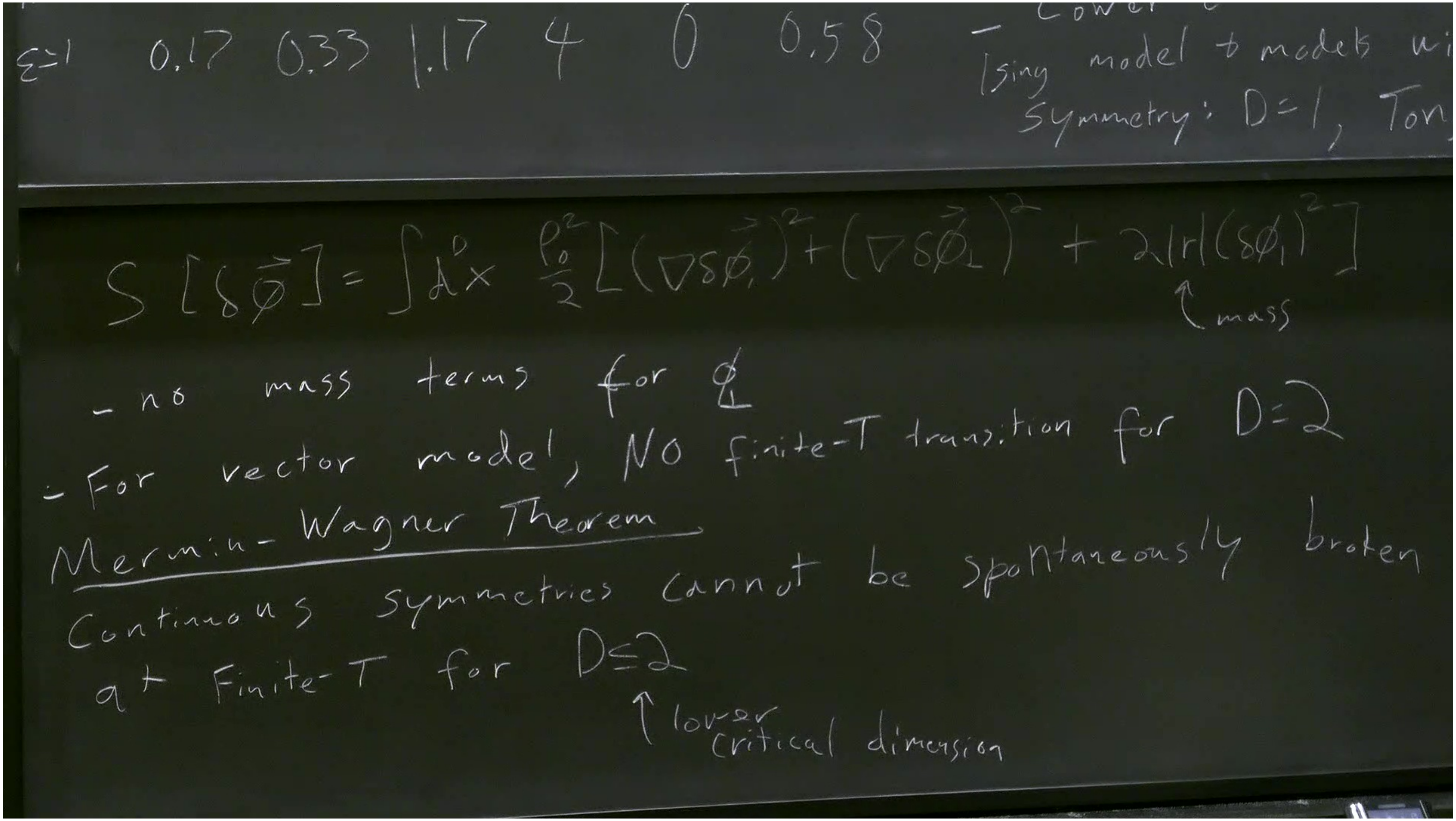
$$\int d^D x\ \frac{\rho_0^2}{2}\left[(\nabla\delta\vec{\phi}_1)^2 + (\nabla\delta\vec{\phi}_\perp)^2 + \underset{\text{mass}}{2|r|}(\delta\phi_1)^2\right]$$

terms for $\phi_\perp$

model, NO finite-T transition for $D=2$

ner Theorem

ntaneously broken

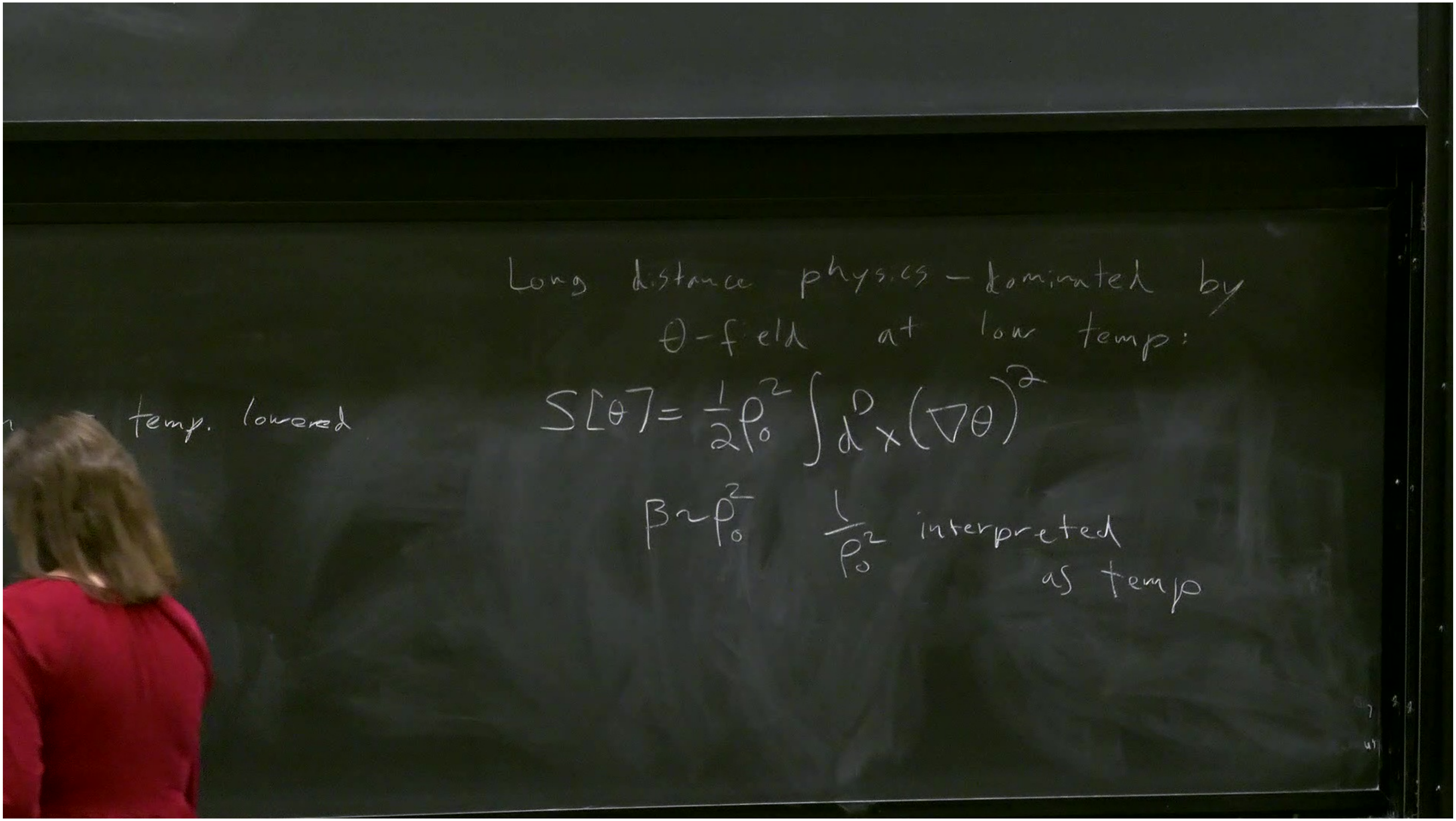


Kosterlitz-Thouless Transition   Tong 4.4

$D=2$, $n=2$ (classical XY model)

- XY does exhibit another type of phase transition as
- No order parameter
- ($n=2$) Write using complex field $\psi(x)=\rho(x)e^{i\theta(x)}$

$$S[\psi]=\int d^Dx\left[\frac{1}{2}|\nabla\psi|^2+\frac{r}{2}|\psi|^2+\frac{u}{4}|\psi|^4\right.$$



temp. lowered

Long distance physics – dominated by θ-field at low temp:

$$S[\theta] = \frac{1}{2}\rho_0^2 \int d^D x \, (\nabla\theta)^2$$

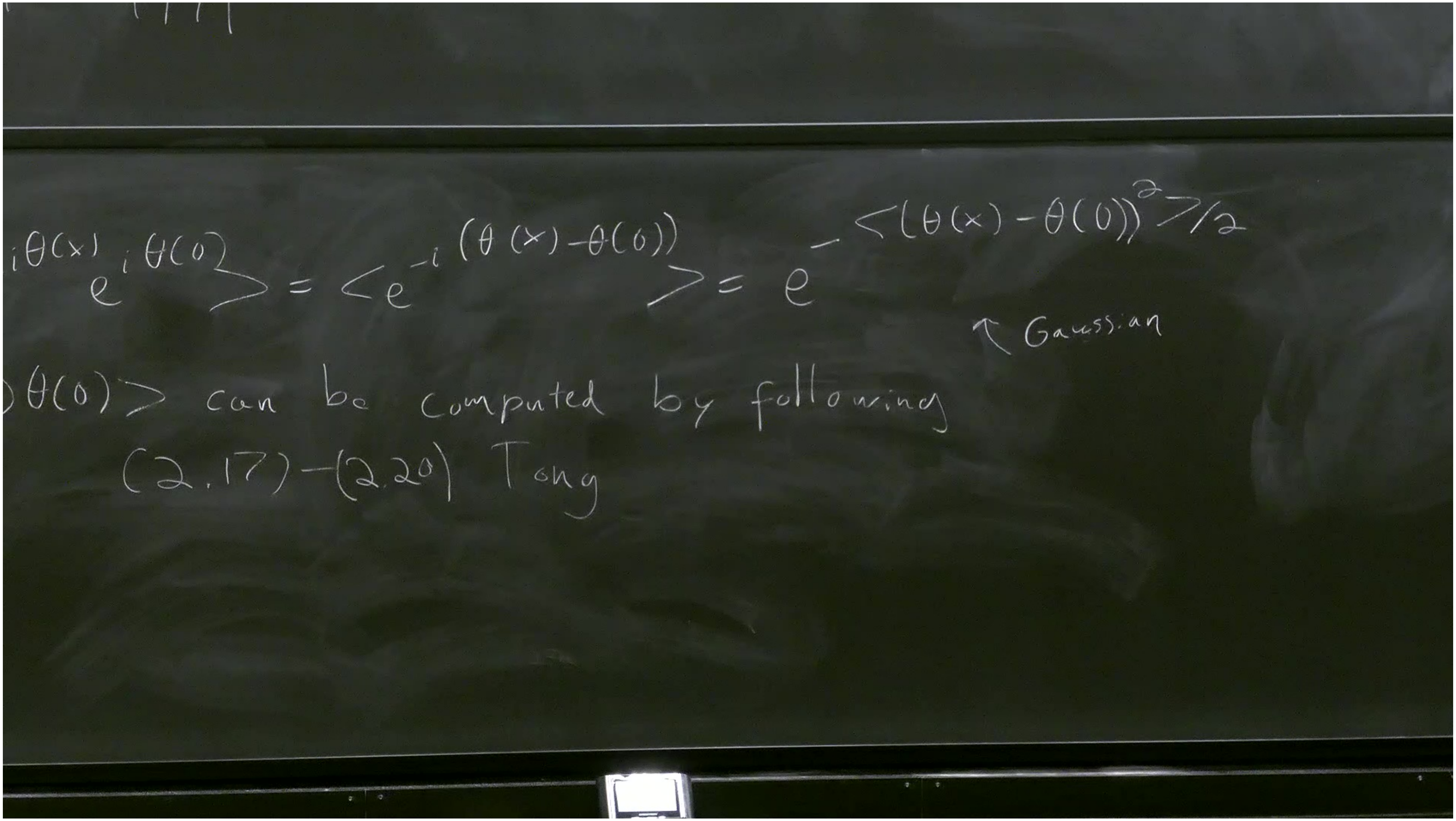
$\beta \sim \rho_0^2$ $\frac{1}{\rho_0^2}$ interpreted as temp

Correlation function:

We want $\langle \psi^\dagger(x)\psi(0)\rangle \sim \langle e^{-i\theta(x)} e^{i\theta(0)}\rangle = \langle e^{-i(\theta(x)-\theta(0}$

Now $\langle \theta(x)\theta(0)\rangle$ can be computed

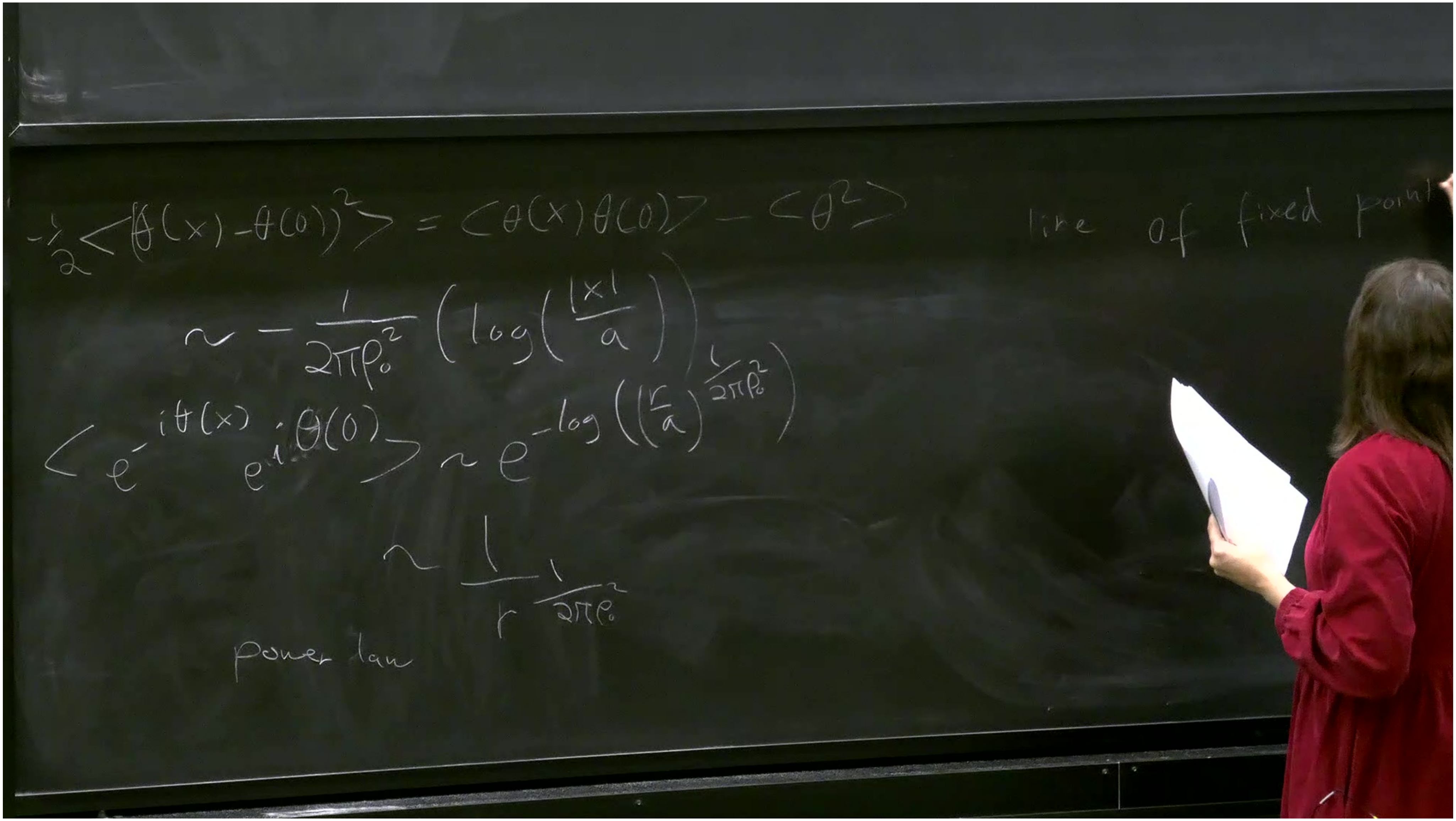
(2.17)–(2.20) Tong



$$^{i\theta(x)} e^{i\theta(0)}\rangle = \langle e^{-i(\theta(x)-\theta(0))}\rangle = e^{-\langle(\theta(x)-\theta(0))^2\rangle/2}$$

↑ Gaussian

$)\theta(0)\rangle$ can be computed by following (2.17)–(2.20) Tong



$$-\frac{1}{2}\langle(\theta(x)-\theta(0))^2\rangle = \langle\theta(x)\theta(0)\rangle - \langle\theta^2\rangle$$

line of fixed point

$$\sim -\frac{1}{2\pi\rho_0^2}\left(\log\left(\frac{|x|}{a}\right)\right)$$

$$\langle e^{-i\theta(x)}\, e^{i\theta(0)}\rangle \sim e^{-\log\left(\left(\frac{r}{a}\right)^{\frac{1}{2\pi\rho_0^2}}\right)}$$

$$\sim \frac{1}{r^{\frac{1}{2\pi\rho_0^2}}}$$

power law

↑ lower critical dimension

$$-\frac{1}{2}\langle(\theta(x)-\theta(0))^2\rangle = \langle\theta(x)\theta(0)\rangle - \langle\theta^2\rangle$$

$$\sim -\frac{1}{2\pi\rho_0^2}\left(\log\left(\frac{|x|}{a}\right)\right)$$

$$\langle e^{-i\theta(x)}\, e^{i\theta(0)}\rangle \sim e^{-\log\left(\left(\frac{r}{a}\right)^{\frac{1}{2\pi\rho_0^2}}\right)}$$

$$\sim \frac{1}{r^{\frac{1}{2\pi\rho_0^2}}}$$

power law

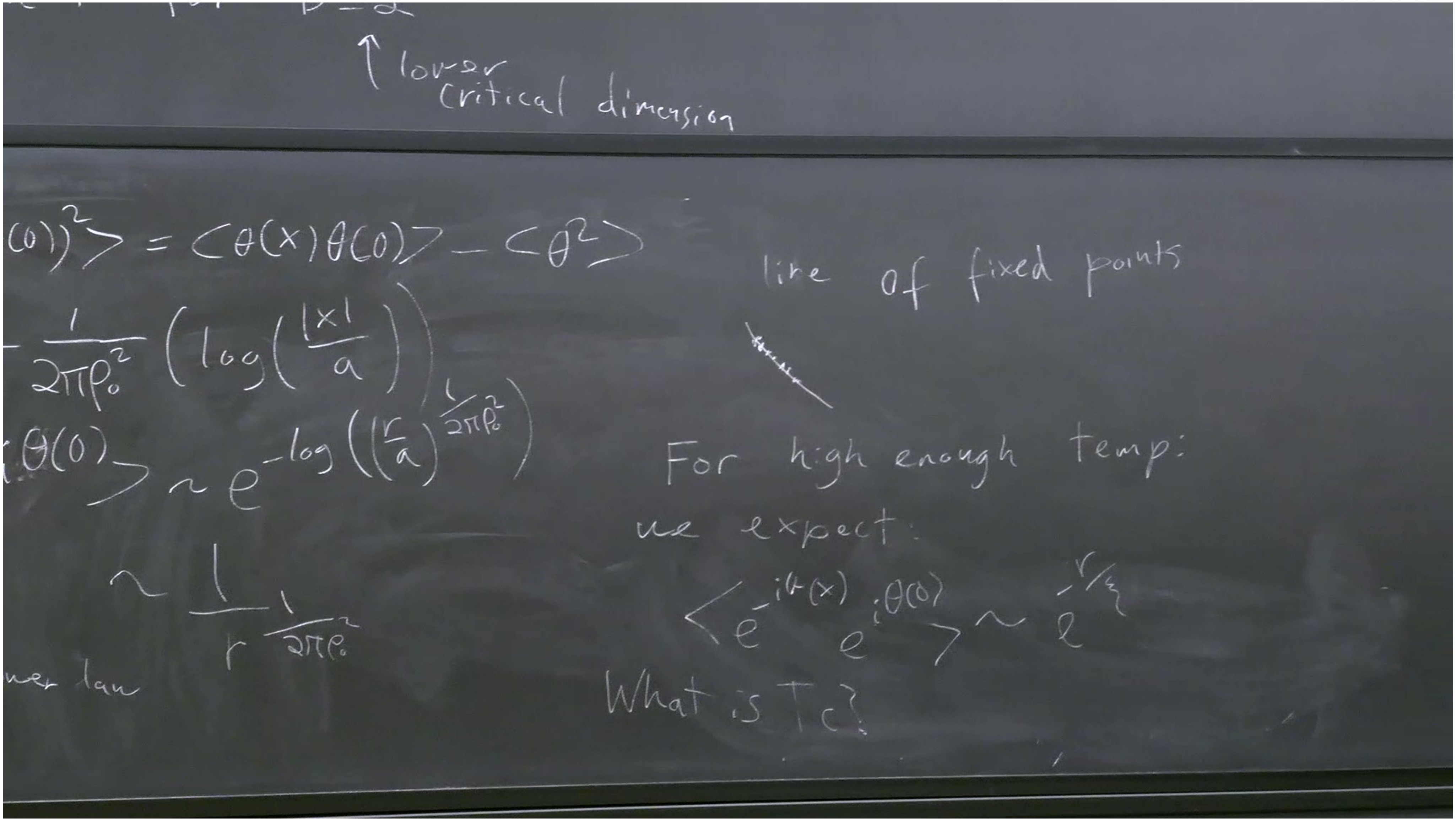
line of fixed poi

For high enough tem

we expect:

$$\langle e^{-i\theta(x)}\, e^{i\theta(0)}\rangle \sim e^{-r/\xi}$$

What is Tc?



↑ lower critical dimension

$$(0))^2\rangle = \langle\theta(x)\theta(0)\rangle - \langle\theta^2\rangle$$

$$\frac{1}{2\pi\rho_0^2}\left(\log\left(\frac{|x|}{a}\right)\right)$$

$$\theta(0)\rangle \sim e^{-\log\left(\left(\frac{r}{a}\right)^{\frac{1}{2\pi\rho_0^2}}\right)}$$

$$\sim \frac{1}{r^{\frac{1}{2\pi\rho_0^2}}}$$

wer law

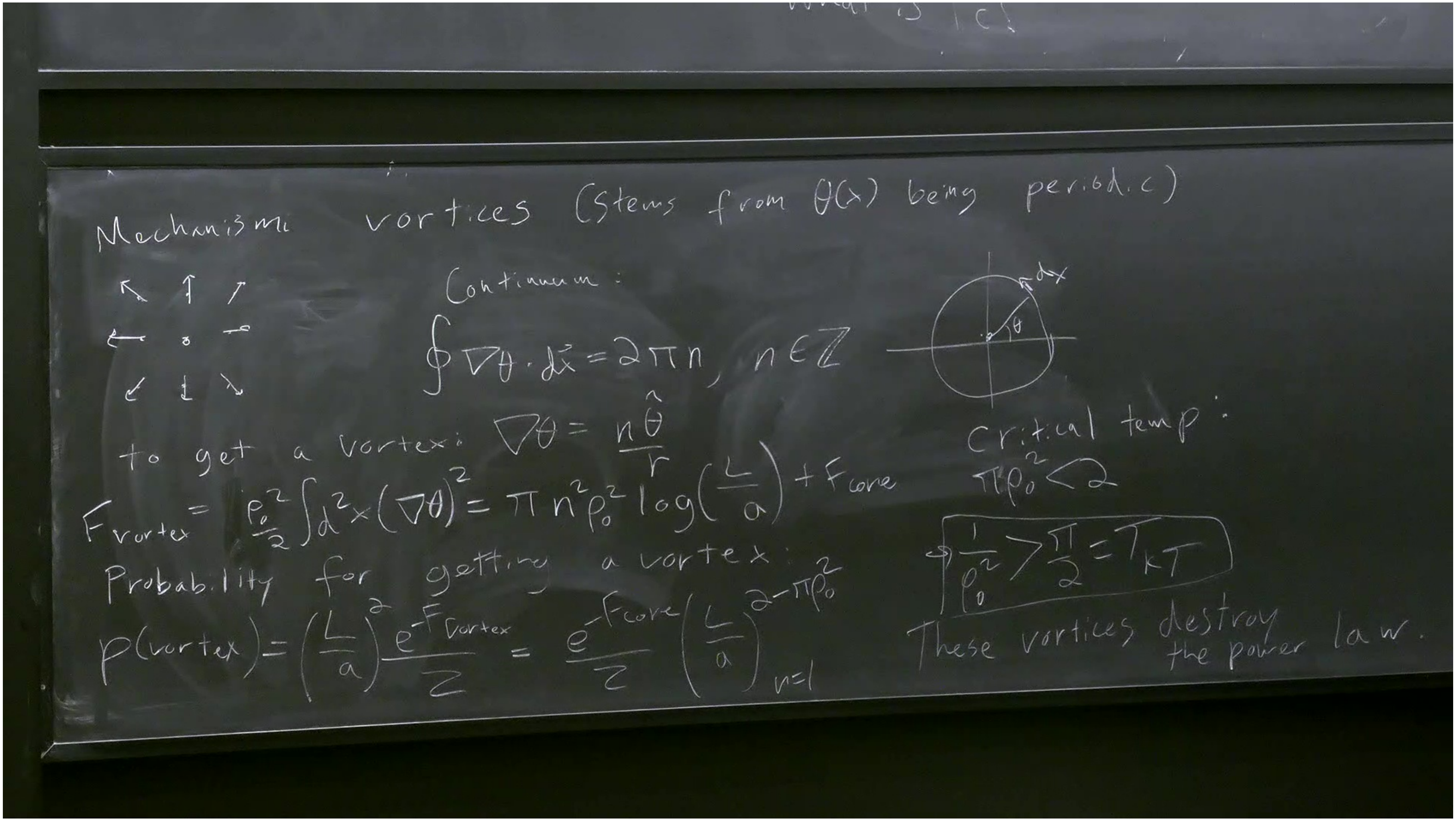
line of fixed points

For high enough temp:

we expect:

$$\langle e^{-i\theta(x)} e^{i\theta(0)}\rangle \sim e^{-r/\xi}$$

What is T_c?



Mechanism: vortices (Stems from $\theta(x)$ being periodic)

Continuum:

$$\oint \nabla\theta \cdot d\vec{x} = 2\pi n, \quad n \in \mathbb{Z}$$

to get a vortex: $\nabla\theta = \dfrac{n\hat{\theta}}{r}$

$$F_{vortex} = \frac{\rho_0^2}{2}\int d^2x\,(\nabla\theta)^2 = \pi n^2 \rho_0^2 \log\left(\frac{L}{a}\right) + F_{core}$$

Probability for getting a vortex:

$$p(\text{vortex}) = \left(\frac{L}{a}\right)^2 \frac{e^{-F_{vortex}}}{Z} = \frac{e^{-F_{core}}}{Z}\left(\frac{L}{a}\right)^{2-\pi\rho_0^2}_{n=1}$$

Critical temp:

$$\pi\rho_0^2 < 2$$

$$\Rightarrow \boxed{\frac{1}{\rho_0^2} > \frac{\pi}{2} = T_{KT}}$$

These vortices destroy the power law.