Title: Lecture - Statistical Physics, PHYS 602

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Collection/Series: Statistical Physics (Core), PHYS 602, October 7 - November 6, 2024

Subject: Condensed Matter, Other

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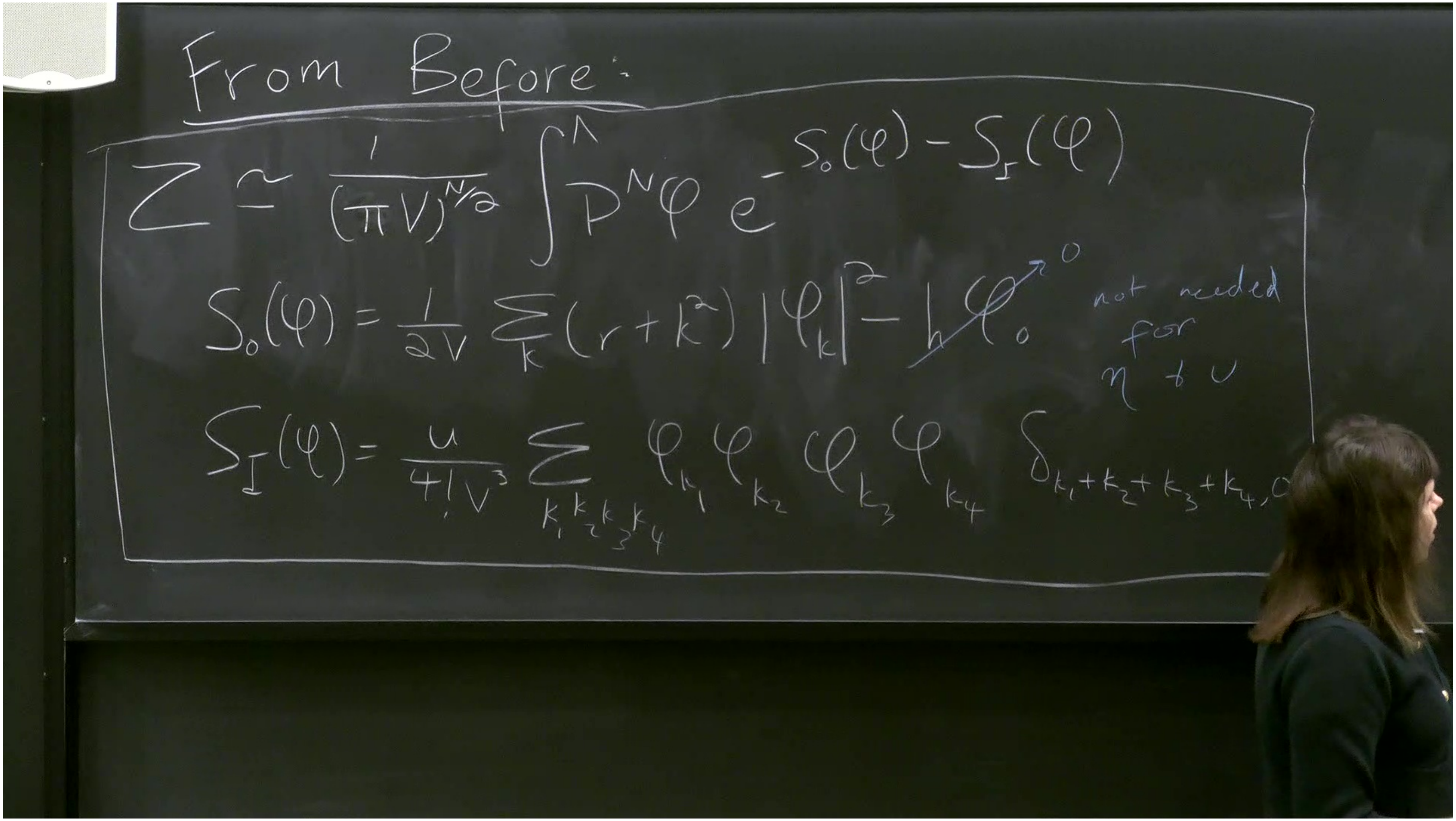
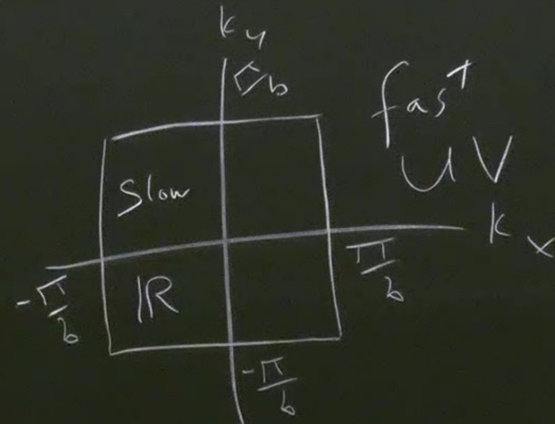

From Before:
$Z \simeq \frac{1}{(\pi V)^{N/2}} \int^{\Lambda} \mathcal{D}^N \varphi \, e^{-S_0(\varphi) - S_I(\varphi)}$
$S_0(\varphi) = \frac{1}{2V} \sum_k (r + k^2) |\varphi_k|^2 - h\varphi_0$
not needed for η + ν
$S_I(\varphi) = \frac{u}{4! V^3} \sum_{k_1 k_2 k_3 k_4} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{k_4} \delta_{k_1+k_2+k_3+k_4, 0}$

"fast" and "slow" modes

$$\varphi_k^+ = \begin{cases} \varphi_k & \text{if } |k_\mu| > \frac{\pi}{b} \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_k^- = \begin{cases} \varphi_k & \text{if } |k_\mu| < \frac{\pi}{b} \\ 0 & \text{otherwise} \end{cases}$$



We want: $\int^{\Lambda} \mathcal{D}^N \varphi^- \, e^{-S'(\varphi^-)}$

$e^{-S'(\varphi^-)} \propto$

so focus on:

$$e^{-S'(\varphi^-)} \sim e^{-S_0(\varphi^-)} \int^{\Lambda} \mathcal{D}^N \varphi^+ \, e^{-S_0(\varphi^+)} \, e^{-S_I(\varphi^- + \varphi^+)}$$

$$\propto e^{-S_0(\varphi^-)} \frac{\int^{\Lambda} \mathcal{D}^N \varphi^+ \, e^{-S_0(\varphi^+)} \, e^{-S_I(\varphi^- + \varphi^+)}}{\int^{\Lambda} \mathcal{D}^N \varphi^+ \, e^{-S_0(\varphi^+)}}$$

$\leftarrow$ fine for computing correlations

$$e^{-S'(\varphi^-)} \propto e^{-S_0(\varphi^-)} \left\langle e^{-S_I(\varphi^- + \varphi^+)} \right\rangle_+$$

↑ avg with Gaussian $e^{-S_0(\varphi^+)}$

<u>cumulant expansion</u>

Expand $\ln\langle e^{-S_I}\rangle_+$:

$$\ln\langle e^{-S_I}\rangle_+ \approx -\langle S_I\rangle_+ + \frac{1}{2}\left(\langle S_I^2\rangle_+ - \langle S_I\rangle_+^2\right)$$

$\left(\frac{u}{v^3} \text{ small}\right)$

for computing correlations

So we'll get terms like: $\langle \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{k_4} \rangle_+$ even φ

$$= \langle \varphi_{k_1} \varphi_{k_2} \rangle_+ \langle \varphi_{k_3} \varphi_{k_4} \rangle_+ + \langle \varphi_{k_1} \varphi_{k_3} \rangle_+ \langle \varphi_{k_2} \varphi_{k_4} \rangle_+ + \langle \varphi_{k_1} \varphi_{k_4} \rangle_+ \langle \varphi_{k_2} \varphi_{k_3} \rangle_+$$

and $\langle \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \rangle_+ = 0$ odd

Tong notes pg. 79-80

Trick: use identity $\langle e^{B_a \varphi_a} \rangle_+ = e^{\frac{1}{2} B_a \langle \varphi_a \varphi_b \rangle_+ B_b}$

...e for
Computing Correlations

n φ

Interaction piece:

$$S_I(\varphi^- + \varphi^+) = \frac{u}{4!\,V^3} \sum_{k_1 k_2 k_3 k_4} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}$$

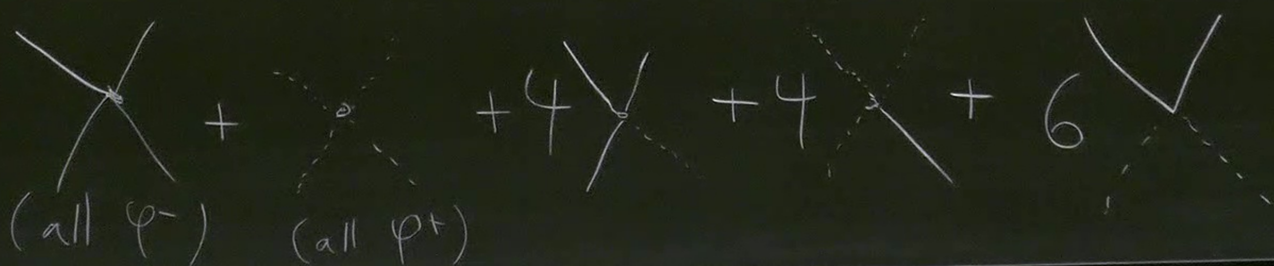
$\langle \varphi_{k_2}\varphi_{k_4}\rangle_+$

$\varphi_{k_3}\rangle_+$

Combinations like $\varphi^+\varphi^+\varphi^-\varphi^-$, $\varphi^+\varphi^-\varphi^+\varphi^-$, ...

—— $= \varphi^-$ .... $= \varphi^+$

The 2^4 terms:

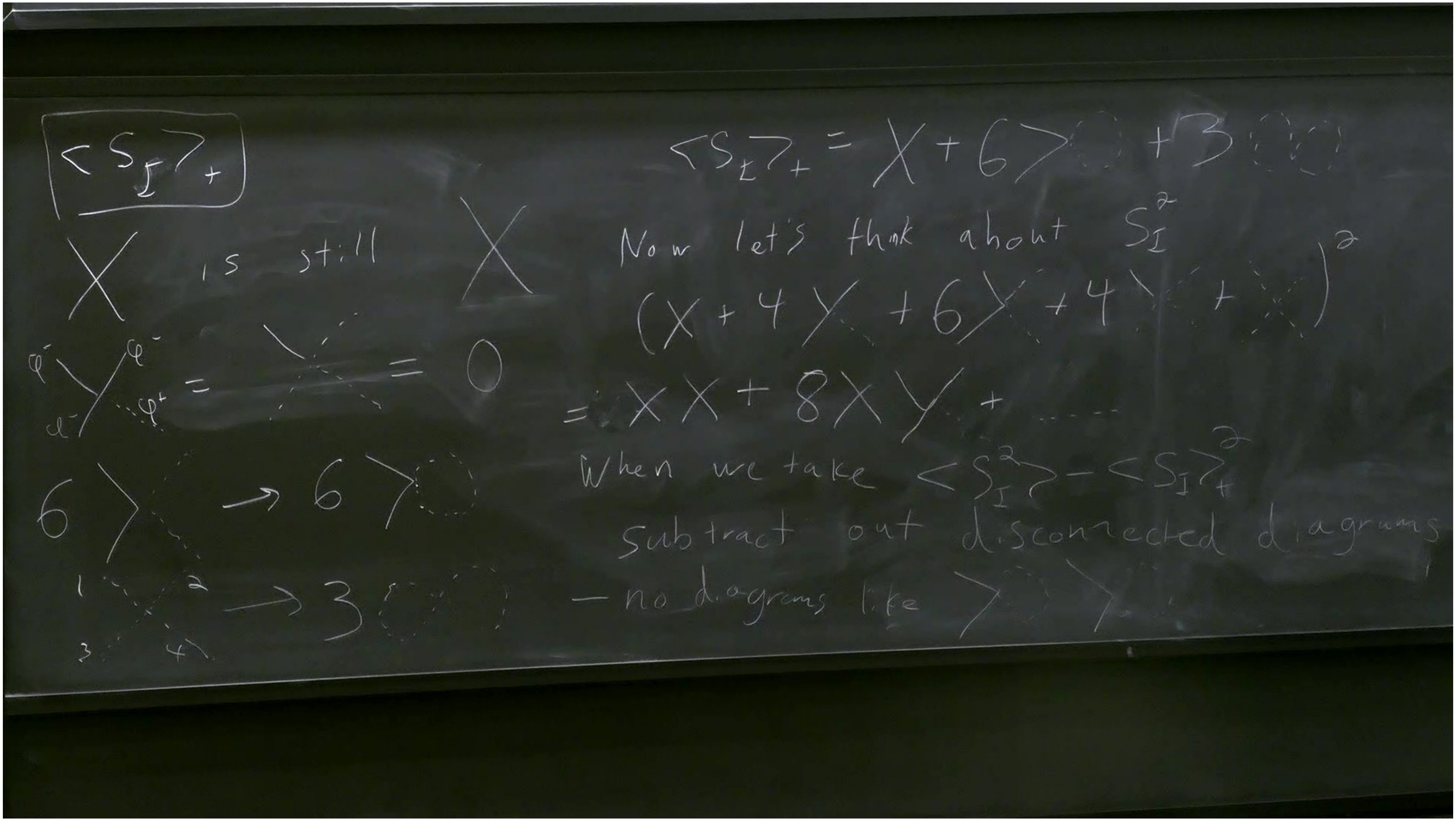


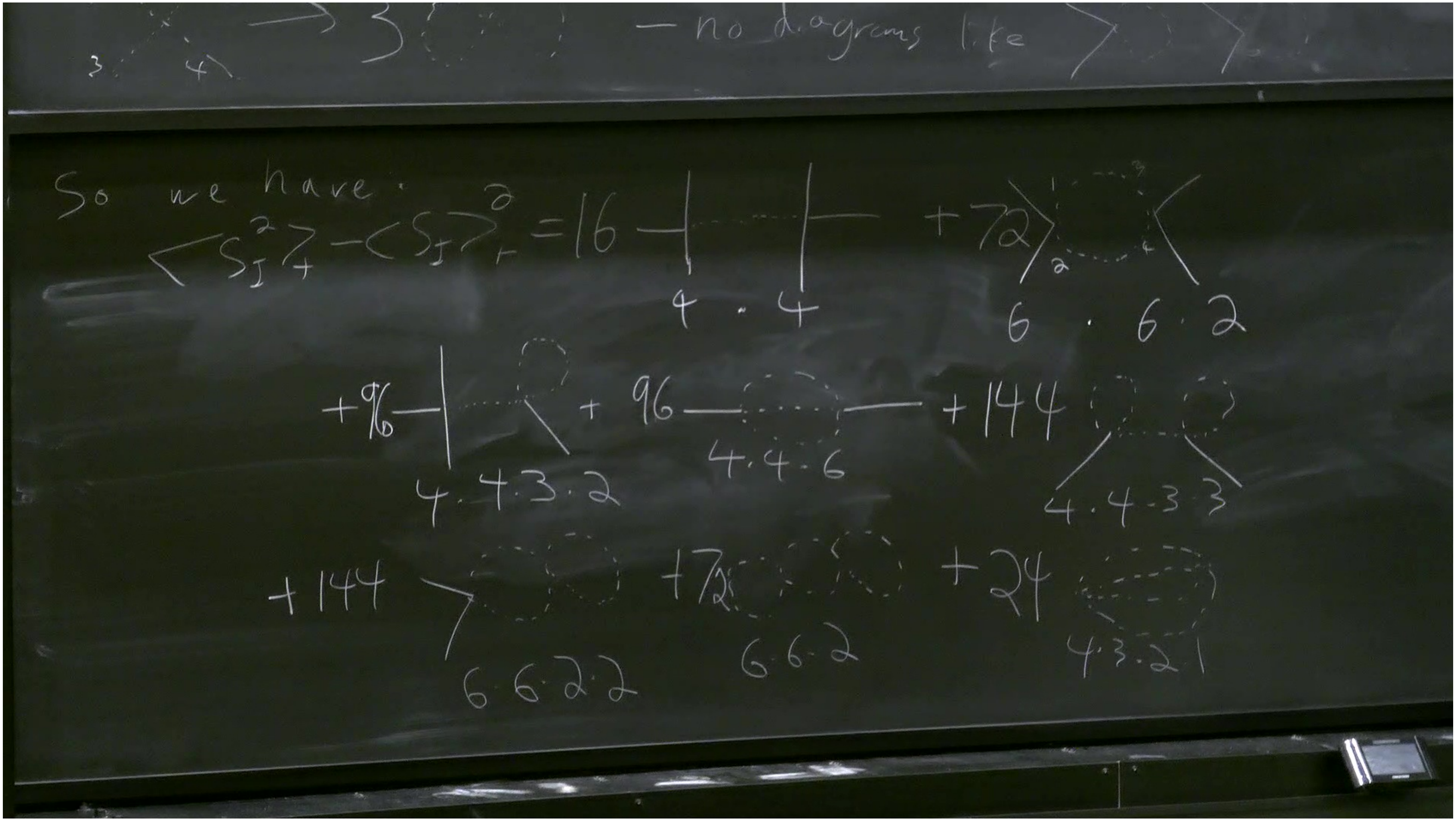
Rules:

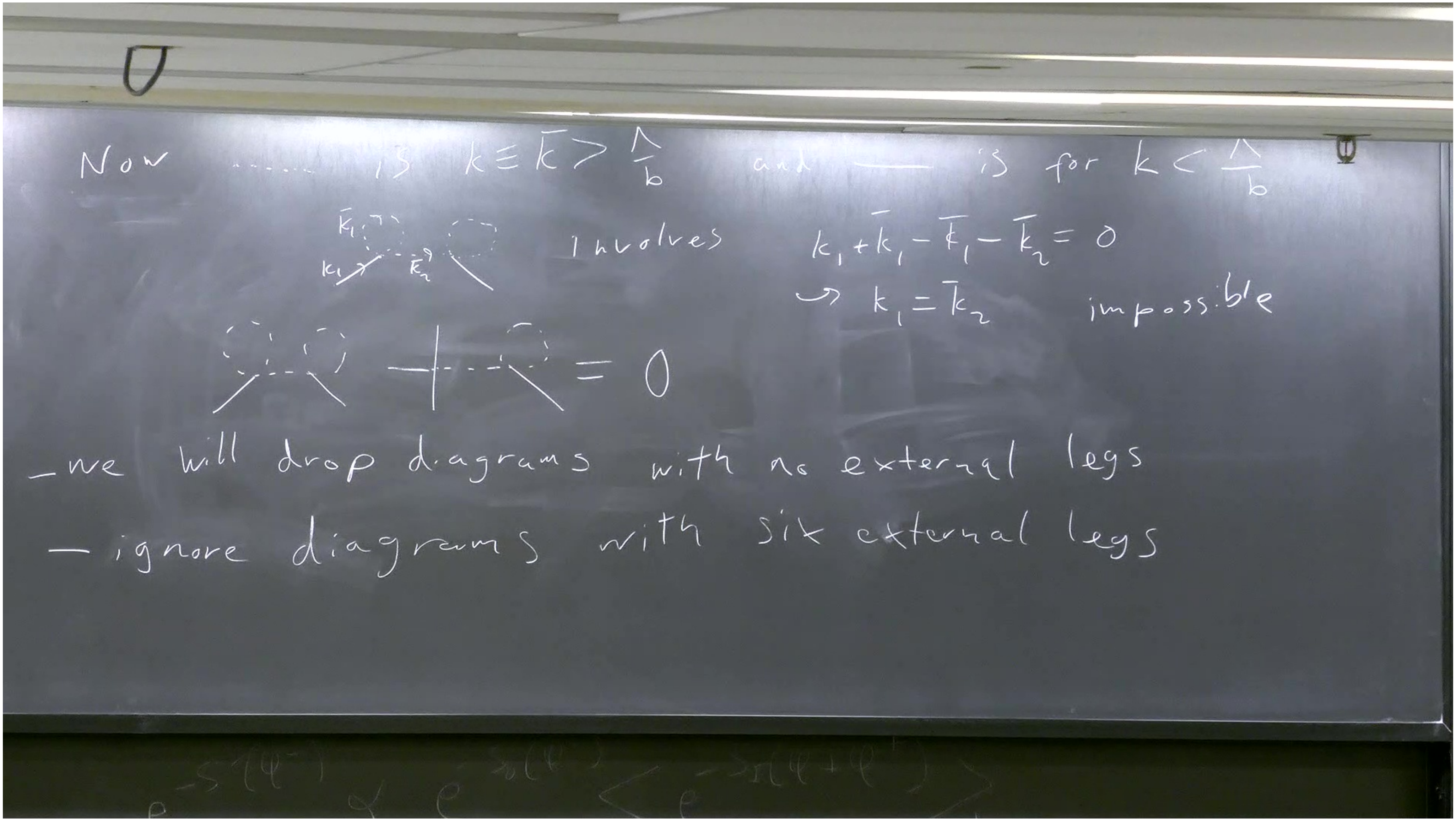
- each solid external line carries a φ_k^-
- each vertex involves a $\frac{u}{4!\,V^3}\,\delta_{k_1+k_2+k_3+k_4,\,0}$
- all dashed must be internal and are φ_k^+
- each internal line denotes $\langle \varphi_k^+ \varphi_q^+ \rangle_+$ which can be computed as

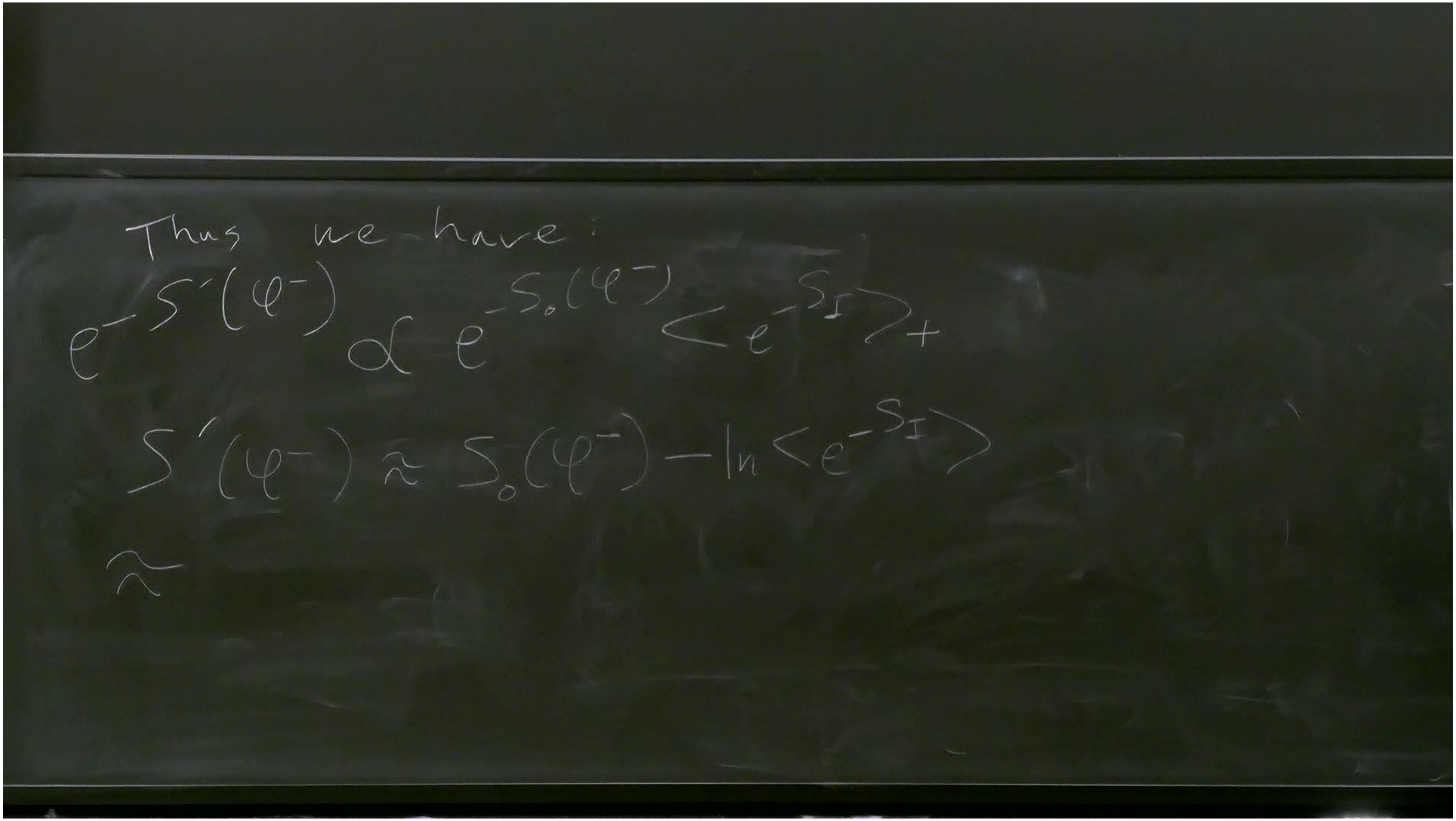
$$\langle \varphi_k^+ \varphi_q^+ \rangle_+ = \begin{cases} V\,\dfrac{\delta_{k+q,0}}{r+k^2}, & \text{if } k=-q \\ 0, & \text{otherwise} \end{cases}$$

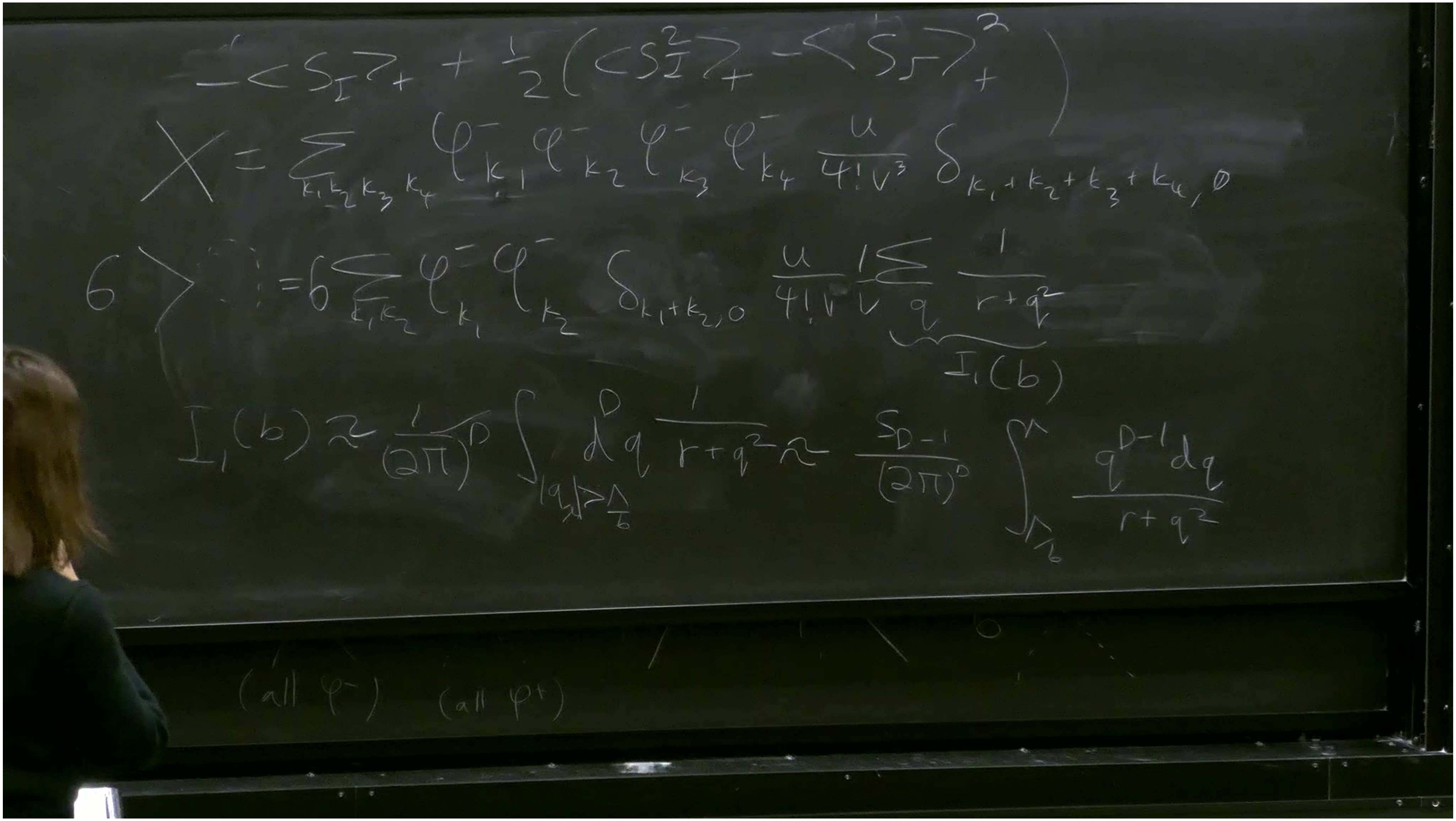
- all momenta are summed over

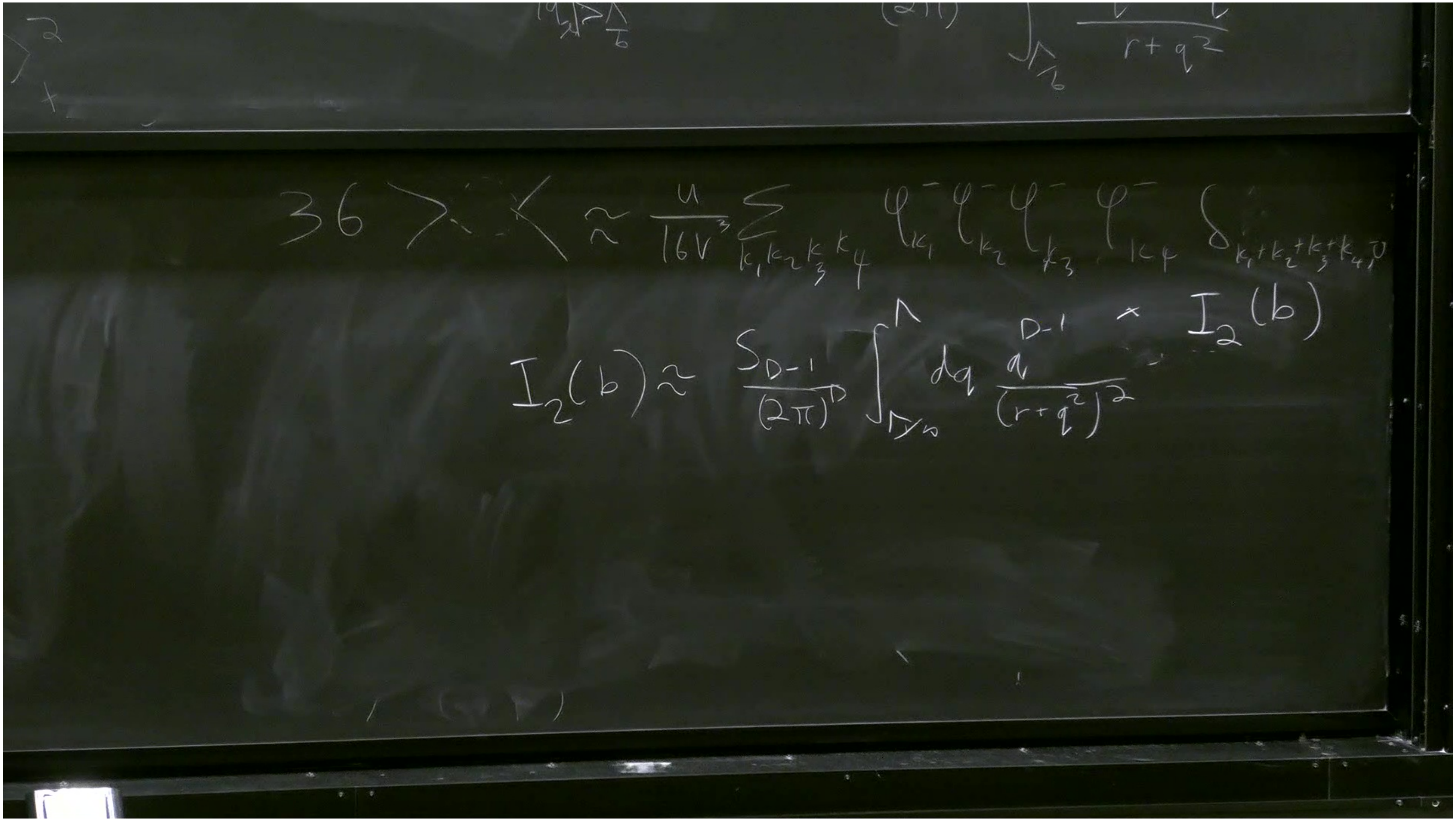

⟨S_I⟩+
is still
⟨S_I⟩+ = X + 6 + 3
Now let's think about S_I^2
When we take ⟨S_I^2⟩ − ⟨S_I⟩+^2
subtract out disconnected diagrams
— no diagrams like


— no diagrams like
So we have:
<S_I^2>_+ − <S_I>_+^2 = 16
4 · 4
+ 72
6 · 6 · 2
+ 96
4 · 4 · 3 · 2
+ 96
4 · 4 · 6
+ 144
4 · 4 · 3 · 3
+ 144
6 · 6 · 2 · 2
+ 72
6 · 6 · 2
+ 24
4 · 3 · 2 · 1


Now .... is k ≡ k̄ > Λ/b and —— is for k < Λ/b
involves k₁ + k̄₁ − k̄₁ − k̄₂ = 0
⇝ k₁ = k̄₂ impossible
= 0
– we will drop diagrams with no external legs
– ignore diagrams with six external legs


Thus we have:
$e^{-S'(\varphi^-)} \propto e^{-S_0(\varphi^-)} \langle e^{-S_I} \rangle_+$
$S'(\varphi^-) \approx S_0(\varphi^-) - \ln \langle e^{-S_I} \rangle$
$\approx$


$-\langle S_I\rangle_+ + \frac{1}{2}\left(\langle S_I^2\rangle_+ - \langle S_I\rangle_+^2\right)$
$X = \sum_{k_1 k_2 k_3 k_4} \varphi^-_{k_1}\varphi^-_{k_2}\varphi^-_{k_3}\varphi^-_{k_4}\frac{u}{4!V^3}\delta_{k_1+k_2+k_3+k_4,0}$
$6 \; = 6\sum_{k_1 k_2}\varphi^-_{k_1}\varphi^-_{k_2}\delta_{k_1+k_2,0}\frac{u}{4!V}\underbrace{\frac{1}{V}\sum_q \frac{1}{r+q^2}}_{I_1(b)}$
$I_1(b) \simeq \frac{1}{(2\pi)^D}\int_{|q_\lambda|>\Lambda/b} d^Dq\,\frac{1}{r+q^2} \simeq \frac{S_{D-1}}{(2\pi)^D}\int_{\Lambda/b}^{\Lambda}\frac{q^{D-1}dq}{r+q^2}$
(all φ^-)
(all φ^+)

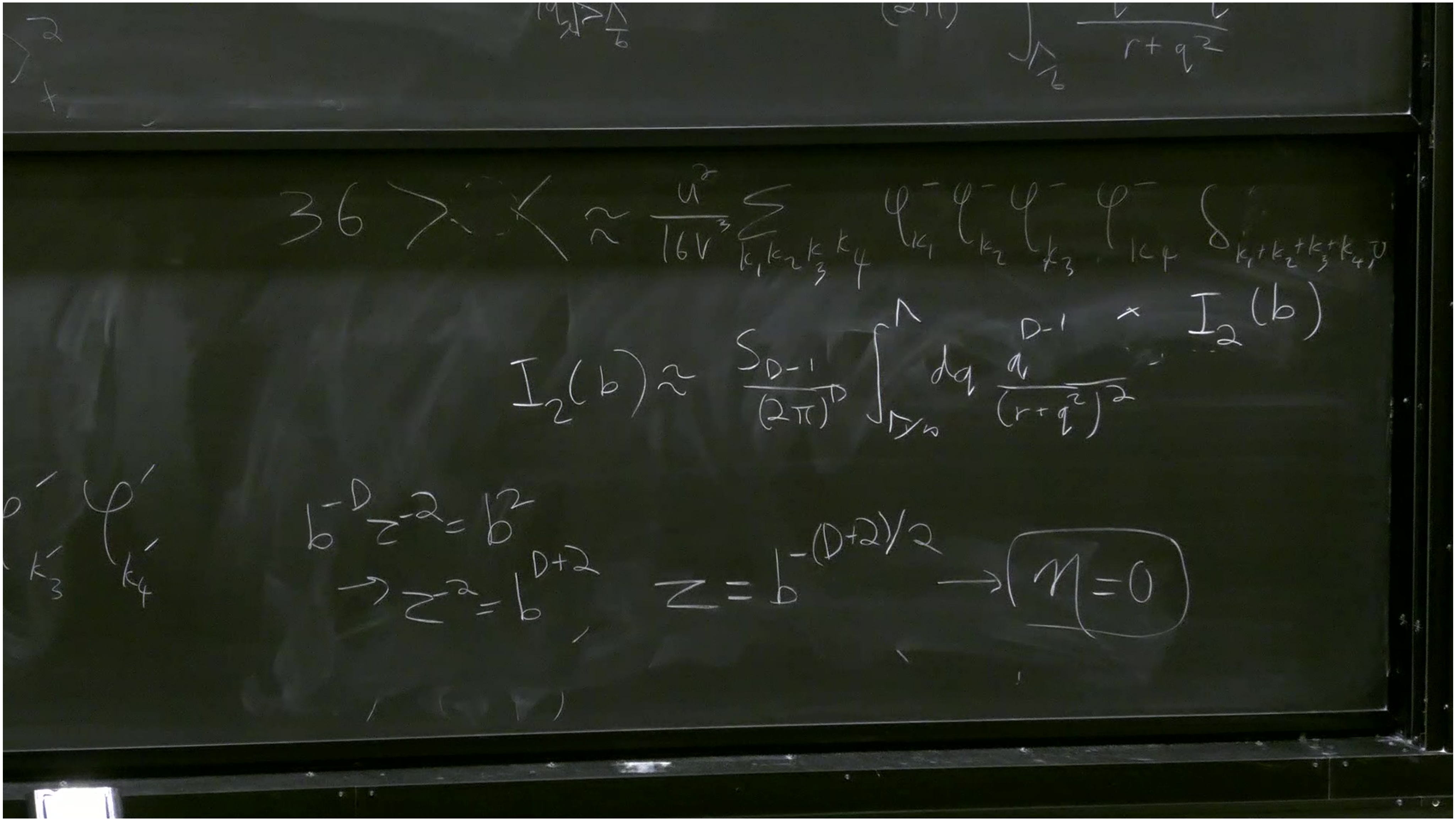

36 >< ≈ u/(16V^3) Σ_{k1k2k3k4} φ^-_{k1} φ^-_{k2} φ^-_{k3} φ^-_{k4} δ_{k1+k2+k3+k4,0} × I_2(b)
I_2(b) ≈ S_{D-1}/(2π)^D ∫_{Λ/b}^{Λ} dq q^{D-1}/(r+q^2)^2

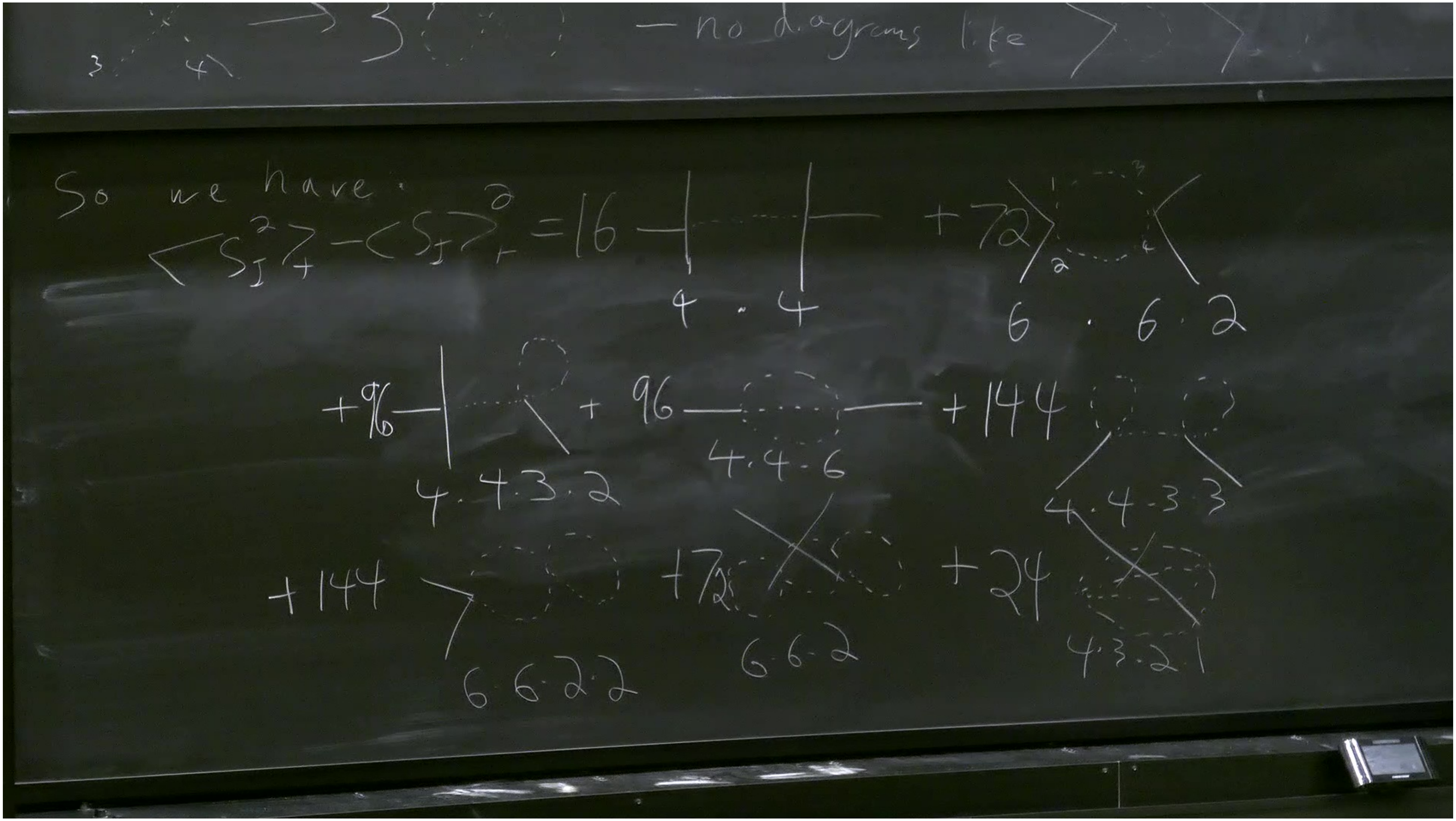
$\langle S_I \rangle \qquad - \langle S_I^2 \rangle_+ + \langle S_I \rangle_+^2$

After rescaling: $N' = \frac{N}{b^D}$, $k' = bk$, $\varphi'_{k'} = z\varphi_k$

$$S'(\varphi^-) = \frac{b^{-D}}{2V'} \sum_{|k'|<\Lambda} \left(r + b^{-2}k^2 + \frac{u I_1(b)}{2} \right) z^{-2} |\varphi'_{k'}|^2$$

$$+ \frac{b^{-3D}}{4! V'^3} \sum_{|k_\mu|<\frac{\Lambda}{b}} \left(u - \frac{3}{2} u^2 I_2(b) \right) z^{-4} \varphi'_{k_1'} \varphi'_{k_2'} \varphi_{k'} \ldots$$


36 ≈ u^2/(16 r^3) Σ_{k1 k2 k3 k4} φ^-_{k1} φ^-_{k2} φ^-_{k3} φ^-_{k4} δ_{k1+k2+k3+k4, 0} × I_2(b)
I_2(b) ≈ S_{D-1}/(2π)^D ∫_{Λ/b}^{Λ} dq q^{D-1}/(r+q^2)^2
b^{-D} z^{-2} = b^2
→ z^{-2} = b^{D+2}
z = b^{-(D+2)/2} → η = 0


— no diagrams like
So we have:
⟨S_I^2⟩_+ − ⟨S_I⟩_+^2 = 16
+ 72
4 · 4
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4 · 4 · 3 · 3
+ 144
+ 72
+ 24
6 · 6 · 2 · 2
6 · 6 · 2
4 · 3 · 2 · 1

RGT

$$\begin{cases} r' = b^2\left(r + \dfrac{u I_1(b)}{2}\right) \\ u' = b^{4-D}\left(u - \dfrac{3}{2} u^2 I_2(b)\right) \end{cases}$$

$$\frac{d}{db}$$

What is the β-function?

Leibniz Rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

β - function:

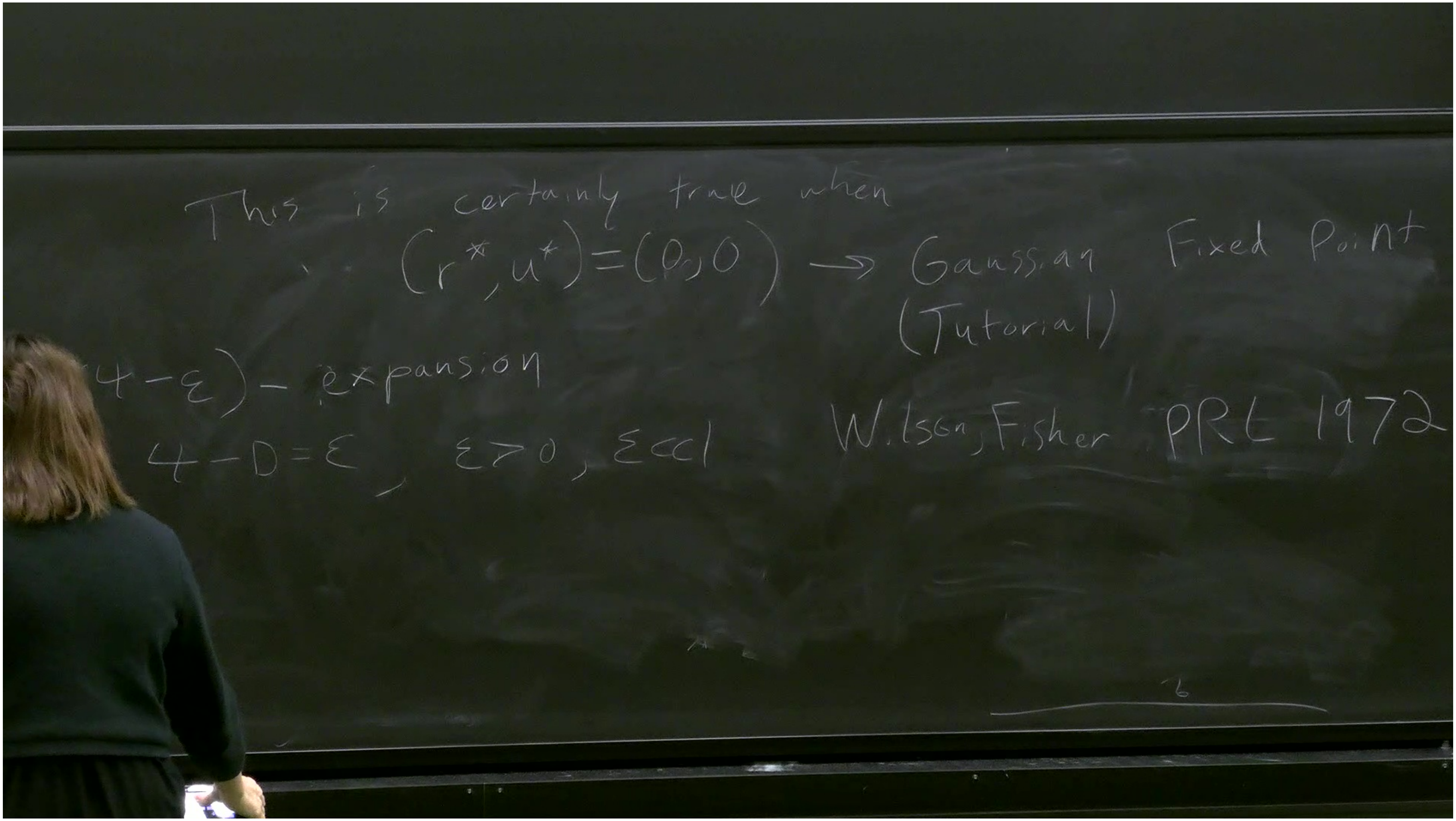
$$\frac{dr'}{db} = 2b\left(r + \frac{u}{2}\frac{K_D}{r+\Lambda^2 b^{-2}}\right)$$

$$\frac{du'}{db} = (4-D)\,b^{3-D}\left(u - \frac{3u^2}{2}\frac{K_D}{(r+\Lambda^2 b^{-2})^2}\right)$$

$$K_D = \frac{S_{D-1}\Lambda^D b^{-1-D}}{(2\pi)^D}$$

Fixed Points $b=1$:

$$
\left.
\begin{cases}
\left.\dfrac{dr'}{db}\right|_{b=1} = 2\left(r + \dfrac{u}{2}\dfrac{K_D}{r+\Lambda^2}\right) \\[2ex]
\left.\dfrac{du'}{db}\right|_{b=1} = (4-D)\left(u - \dfrac{3u^2}{2}\dfrac{K_D}{(r+\Lambda^2)^2}\right)
\end{cases}
\right\} = 0
$$


This is certainly true when
$(r^*, u^*) = (0, 0) \rightarrow$ Gaussian Fixed Point
(Tutorial)
$(4-\varepsilon)$ - expansion
$4 - D = \varepsilon$, $\varepsilon > 0$, $\varepsilon \ll 1$
Wilson, Fisher PRL 1972

For (2)

$$\varepsilon - \frac{3u}{2}\frac{k_D}{(r+\Lambda^2)^2} = 0 \rightarrow u = \frac{2}{3}\varepsilon\frac{(r+\Lambda^2)^2}{k_D}$$

For (1)

$$2r + \frac{\varepsilon}{3}(r+\Lambda^2) = 0$$

$$\rightarrow \boxed{r^* = -\frac{\varepsilon\Lambda^2}{6}}$$

back to (2)

$$\boxed{u = \frac{2}{3}\varepsilon\frac{\left(-\frac{\varepsilon\Lambda^2}{6}+\Lambda^2\right)}{k_D}}$$

For (2)

$$\varepsilon - \frac{3u}{2}\frac{k_D}{(r+\Lambda^2)^2} = 0 \rightarrow u = \frac{2}{3}\varepsilon\frac{(r+\Lambda^2)^2}{k_D}$$

For (1)

$$2r + \frac{\varepsilon}{3}(r+\Lambda^2) = 0$$

$$\rightarrow r^* = -\frac{\varepsilon\Lambda^2}{6}$$

back to (2)

$$u = \frac{2}{3}\varepsilon\frac{\left(-\frac{\varepsilon\Lambda^2}{6}+\Lambda^2\right)^2}{k_D}$$

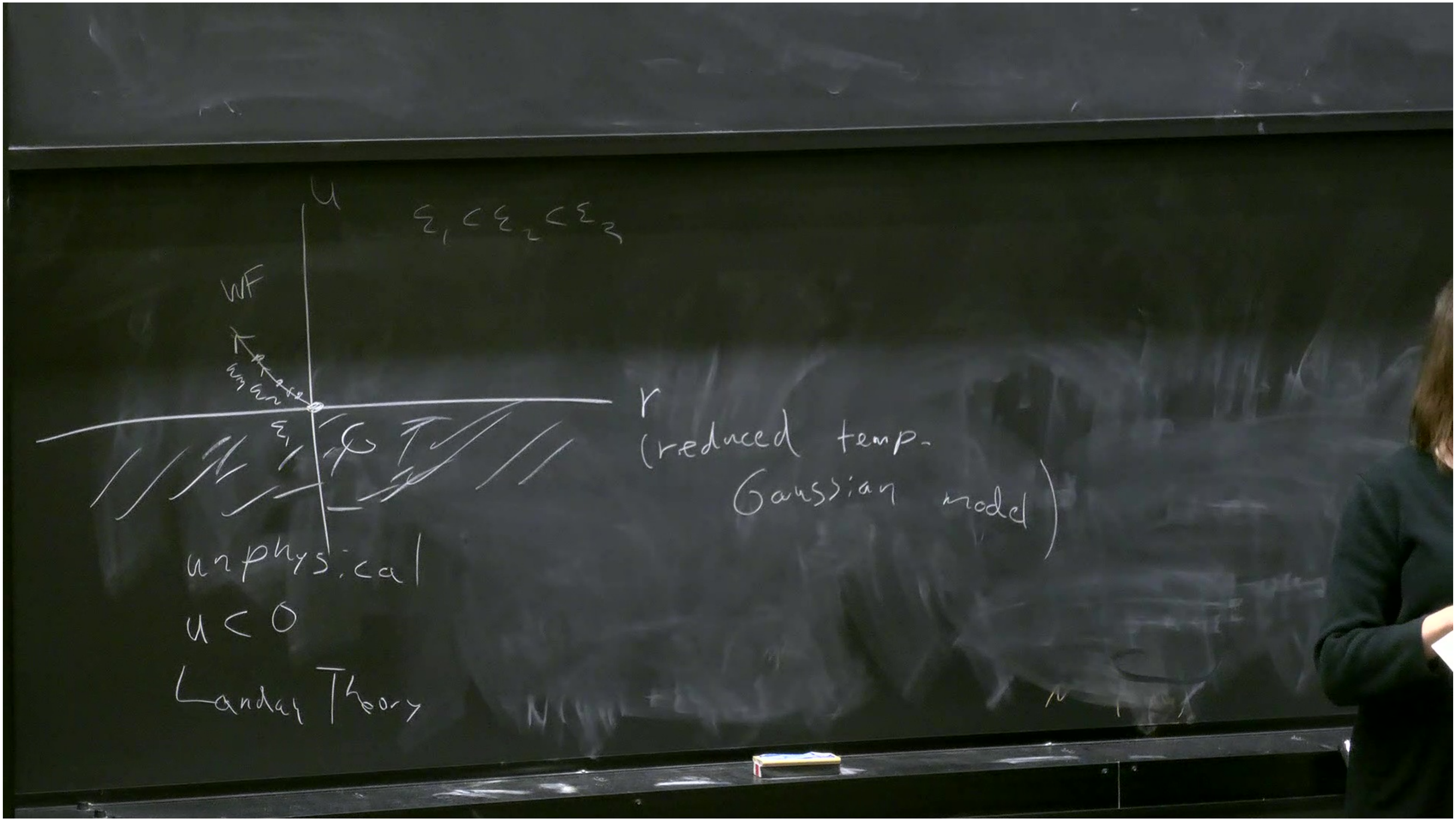
$K_{4-\varepsilon} \approx K_4 + O(\varepsilon)$ ← Pathria Appendix C

We have

$$(r^*, u^*) = \left(-\frac{\varepsilon}{6}\Lambda^2, \ \frac{2}{3}\frac{\Lambda^4}{K_4}\varepsilon\right)$$

Wilson-Fisher Fixed Point

↑ close to $D=4$


u
$\epsilon_1 < \epsilon_2 < \epsilon_3$
WF
r
(reduced temp.
Gaussian model)
unphysical
u < 0
Landau Theory