Title: Lecture - Statistical Physics, PHYS 602

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From Before: Momentum - Space Field Theory

$$Z = \frac{1}{\sqrt{2}} \prod_k \sqrt{\frac{\beta A^2 4 B_k}{\pi V a^D}} \int D^N \varphi \, e^{-S(\varphi)}$$

where

$$S(\varphi) \simeq \overbrace{\frac{A^2 \beta}{2} \frac{1}{V a^D} \sum_k B_k (1 - \beta B_k) |\varphi_k|^2 - \beta A \frac{H}{a^D} \varphi_0}^{\text{Gaussian Model}}$$

$$\underbrace{+ \frac{\beta^4 A^4}{12} \frac{1}{V^3 a^D} \sum_{k_1 k_2 k_3 k_4} B_{k_1} B_{k_2} B_{k_3} B_{k_4} \varphi_{k_1} \varphi_{k_2}}_{\phi^4 \text{ theory}}$$

$$\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4}$$

- We will write using simpler parameters that can be interpreted physically:

$$B_k = \lambda + 2J\sum_\mu \cos k_\mu a \approx \lambda + 2J\left(D - \frac{k^2a^2}{2}\right)$$

$$\rightarrow \boxed{B_k \approx \frac{1}{\beta_c} - Jk^2a^2}$$

- Where $T_c = (\lambda + 2JD)/k$ is Gaussian T_c

So $\frac{A^2}{a^D}\beta B_k(1-\beta B_k) \approx \frac{A^2}{a^D}\left(t + \beta_c J a^2 k^2\right)$

$$\rightarrow \frac{A^2}{a^D}\beta_c J a^2 k^2 = k^2 \rightarrow \boxed{A = \frac{a^{(D-2)/2}}{\sqrt{\beta_c J}}}$$

- For ϕ^4: $B_{k_i} \approx B_0 \approx \frac{1}{\beta_c}$ ← using Gaussian T_c
- We can thus introduce:

$$r = \frac{A^2 t}{a^D} = \frac{t}{a^2 J \beta_c}$$ ← related to reduce Gaussian T_c

$$h = \frac{\beta A H}{a^D} = \frac{\beta H}{a^{(D+2)/2}\sqrt{\beta_c J}}$$ ← external magnetic field

$$u = \frac{2\beta^4 A^4}{a^D \beta_c^4} = \frac{2\beta^4 a^{D-4}}{\beta_c^6 J^2}$$ ← ϕ^4 prefactor

$$\rightarrow A^2_D \beta_c J a^2 k^2 = k^2 \rightarrow \boxed{A = a^{(D-2)/2}}$$

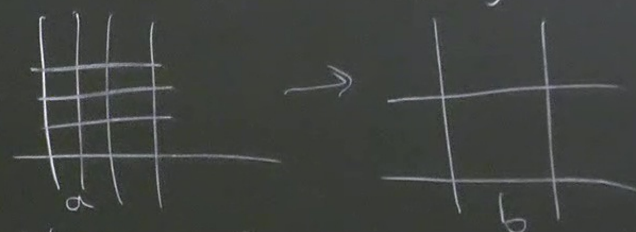
We have:

$$\boxed{Z \simeq \frac{1}{(\pi V)^{N/2}} \int D^N\varphi \, e^{-S(\varphi)}}$$

$$S(\varphi) = \underbrace{\underbrace{\frac{1}{2V} \sum_k (r+k^2)|\varphi_k|^2 - h\varphi_0}_{\text{Gaussian}} + \frac{u}{4!\,V} \sum_{k_1k_2k_3k_4} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4} \, \delta_{k_1+k_2+k_3+k_4,0}}_{\phi^4}$$

RG Procedure

- In position space, we rewrote using a coarser spatial grid, and tried to get back to the original action.



- For lattice spacing a, the maximum momentum is $\Lambda = \frac{\pi}{a}$ in any direction

- For bigger lattice spacing b, max momentum is

$$\frac{\pi}{b} < \frac{\pi}{a}$$

$\Rightarrow$ So what if we integrate out higher $k > \frac{\pi}{b}$

<u>To make this precise:</u> UV IR

1. Split φ_k into "fast" and "slow" modes

$$\varphi_k^+ = \begin{cases} \varphi_k & \text{if } |k_\mu| > \frac{\pi}{b} \\ 0 & \text{otherwise} \end{cases}, \qquad \varphi_k^- = \begin{cases} \varphi_k & \text{if } |k_\mu| < \pi/b \\ 0 & \text{otherwise} \end{cases}$$

Set $b \geq a = 1$ $\quad \pi/b = \Lambda/b$

$\Big(0$ otherwise $)$ $\quad \Big\{ 0$ otherwise

Set $b \geq a = 1 \quad \pi/b = \Lambda/b$

Thus we have:

$$\int^{\Lambda} \mathcal{D}^N\varphi \, e^{-S(\varphi)} = \int^{\Lambda} \mathcal{D}^N\varphi^- \int^{\Lambda} \mathcal{D}\varphi^+ \, e^{-S(\varphi^- + \varphi^+)} \overset{?}{\sim} \int^{\Lambda} \mathcal{D}^N\varphi^- \, e^{-S'(\varphi^-)}$$

Now for <u>Gaussian Model</u> we have $S(\varphi^- + \varphi^+) = S(\varphi^-) + S(\varphi^+)$

because no terms $\varphi_{k_1}\varphi_{k_2}$.

of $\beta A B_k = \dfrac{\beta\, a^{(D-2)/2}}{\beta_c \sqrt{\beta_c J}}$ (works for $a=1$ if small rel. to J)

So we can directly do the integral:

$$) = e^{-S(\varphi^-)} \int^{\Lambda} D\varphi^+ e^{-S(\varphi^+)} \rightarrow \quad S(\varphi^+) = \frac{1}{2V} \sum_{k / |k_\mu| > \frac{\pi}{b}} \left(r + \sum_\mu k_\mu^2\right) |\varphi_k^+|^2$$

$$\varphi^-) = e^{-S(\varphi^-)} \prod_{k/|k_\mu| > \frac{\pi}{b}} \sqrt{\frac{2V\pi}{r+k^2}} \rightarrow \quad S'(\varphi^-) = S(\varphi^-)$$

in terms of $\beta A B_k = \dfrac{\beta a^{(D-2)/2}}{\beta_c \sqrt{\beta_c J}}$ (works for $a=1$ if small rel. to J)

So we can directly do the integral:

$$e^{-S'(\varphi^-)} = e^{-S(\varphi^-)} \int^\Lambda \mathcal{D}\varphi^+ e^{-S(\varphi^+)} \to \quad S(\varphi^+) = \frac{1}{2V} \sum_{k / |k_\mu| > \frac{\pi}{b}} \left(r + \sum_\mu k_\mu^2\right) |\varphi_k^+|^2$$

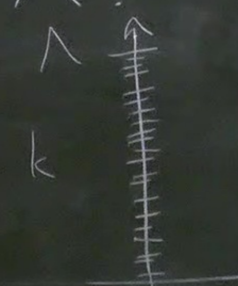
$$\boxed{e^{-S'(\varphi^-)} = e^{-S(\varphi^-)} \prod_{k / |k_\mu| > \frac{\pi}{b}} \sqrt{\frac{2V\pi}{r+k^2}}} \to \quad S'(\varphi^-) = S(\varphi^-)$$

And:

$$Z = \left(\prod_{k \,|\, |k_\mu| > \frac{\Lambda}{b}} \sqrt{\frac{2\pi V}{r+k^2}} \, \frac{1}{(\pi V)^{1/2}} \right) \prod_{k \,|\, |k| < \frac{\Lambda}{b}} \int^{\Lambda/b} D^N \varphi^- \, e^{-S(\varphi^-)}$$

2. <u>Rescaling</u> (renorming) We are now integrating onl
$\rightarrow$ rescale to integrate to the full Λ.

lattice sites $\rightarrow N' = \frac{N}{b^D}$, $V' = \frac{V}{b^D}$ $k'_\mu = b k_\mu$



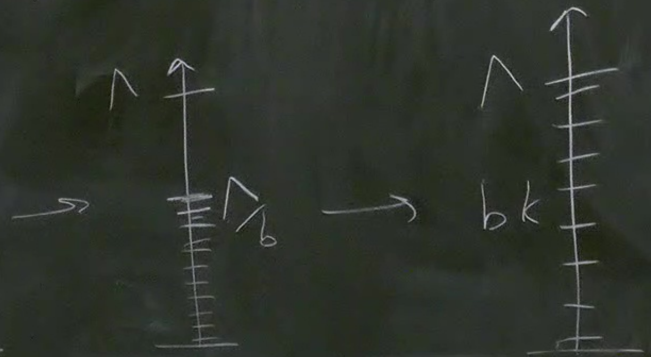
$$= e^{-S(\varphi^-)} \prod_{k \,|\, |k_\mu| > \frac{\pi}{b}} \sqrt{\frac{2V\pi}{r+k^2}} \to S'(\varphi^-) = S(\varphi^-)$$

$$\varphi'_{k'} = z\,\varphi_k$$

we will figure this out

$$d\varphi'_{k'} = z\, d\varphi_k$$

only to Λ/b



stretch them out to Λ

$$\rightarrow \; S(\varphi^-) = \frac{1}{2V} \sum_{k \,|\, |k_\mu| < \frac{\Lambda}{b}} (r+k^2)|\varphi_k|^2 - h\varphi_0$$

$$= \frac{b^{-D}}{2V'} \sum_{k' \,|\, |k'_\mu| < \Lambda} (r + b^{-2}k'^2)\, z^{-2} |\varphi'_{k'}|^2 - h z^{-1}\varphi'_0$$

- To make coefficient of $k'^2 = 1$:

$$b^{-D} \cdot b^{-2} \cdot z^{-2} = 1 \longrightarrow \boxed{z = b^{-\frac{D+2}{2}}}$$

$$\boxed{Z(N,r,h) \simeq \left(\prod_{|k_\mu| > \frac{\Lambda}{b}} \frac{\sqrt{2}}{\sqrt{r+k^2}}\right) b^{Nb^{-D}} Z\left(Nb^{-D}, rb^{2}, hb^{\frac{D+2}{2}}\right)}$$

Fixed point: $r' = RGT(r) = rb^2$

solve: $r = rb^2 \rightarrow r^* = 0$

From here we have: $\rightarrow \boxed{t = 0}$

$$f = -\frac{kT}{N}\log(Z) = -\frac{kT\,b^{-D}}{N'}\log\left(a(r)\,b^{Nb^{-D}} Z\right)$$

$$f(N, r, h) \simeq b^{-D} f\left(Nb^{-D}, rb^2, hb^{\frac{D+2}{2}}\right)$$

$$-kTb^{-D}\log b + \frac{kT}{N}\log\left(\prod_{|k_\mu| > \Lambda/b} \sqrt{\frac{r + |k|^2}{2}}\right)$$

$\longrightarrow$ $\Lambda/b \longrightarrow bk$

Stretch them out to Λ

Thermodynamic limit:

$$f(r,h) \simeq b^{-D} f\left(rb^2, hb^{\frac{D+2}{2}}\right) - kT b^{-D} \log b + \frac{kT}{2} \frac{a^D}{(2\pi)^D} \int_{\Lambda/b<|k_\mu|<\Lambda} d^Dk \ln(r+k^2)$$

+ linear in T term

Critical Exponents

$$\chi = \frac{\partial^2 f}{\partial H^2} \sim \frac{\partial^2 f}{\partial h^2}$$

$$\frac{\partial^2}{\partial h^2} f(r,h) = b^{-D} \frac{\partial^2 f}{\partial h^2}\left(rb^2, hb^{\frac{D+2}{2}}\right) b^{D+2}$$

$$-kT\,b^{-D}\log b + \frac{kT}{N}\log\left(\prod_{|k_\mu|>\frac{1}{b}}\sqrt{\frac{r+k^2}{2}}\right)$$

$$\chi = \frac{\partial^2 f(r,h)}{\partial h^2} = b^2\,\frac{\partial^2 f}{\partial h^2}\left(r b^2, h b^{\frac{D+2}{2}}\right)$$

$h=0$ (criticality)

set $rb^2 = 1$, $b^2 = \frac{1}{r}$

$$= \frac{1}{r}\,\frac{\partial^2 f}{\partial h^2}(1, 0)$$

$$\chi \sim \frac{1}{r} \qquad \boxed{\gamma = 1}$$

$b^2 = \frac{1}{r}$

How does ξ go?

$$k' = bk \,, \quad x' = \frac{x}{b}$$

$$\xi(r,h) = b\,\xi\big(\underbrace{b^2 r}_{1},\, b^{\frac{D+2}{2}} h\big)$$

$$\xi(r,0) = \frac{1}{\sqrt{r}}\,\xi(1,0) \qquad \xi \sim r^{-1/2} \rightarrow \boxed{\nu = \frac{1}{2}}$$

$$-kTb^{-D}\log b + \frac{kT}{N}\log\left(\prod_{|k_\mu|>1/b}\sqrt{\frac{r+k^2}{2}}\right)$$

$h=0$ (criticality …

set $rb^2=1$, b …

$$\chi = \frac{\partial^2 f(r,h)}{\partial h^2} = b^2\,\frac{\partial^2 f}{\partial h^2}\left(rb^2, hb^{\frac{D+2}{2}}\right)$$

$$= \frac{1}{r}\,\frac{\partial^2 f}{\partial h^2}(1, 0)$$

$$\chi \sim \frac{1}{r} \qquad \boxed{\gamma = 1}$$

Scaling Fields and Fixed Points

- More systematic understanding of RG

Assume these parameters:

$$K = (K_1, K_2, \ldots)$$

Ex: r, h, u

After momentum space RGT:

$$Z(N, K) \propto Z(N', K') \propto \int^{\Lambda} D^{N'}\varphi' \, e^{-S(\varphi', K')}$$

$$K' = RGT_b(K), \quad N' = N b^{-D}, \quad \varphi'_{k'} = z\,\varphi_{k'/b}$$

- Fixed Points given by

$$K^* = RGT_b(K^*)$$

$$\frac{\partial K'_i}{\partial K_j}$$

- Linearizing near fixed points:

$$K'_i \approx K^*_i + \sum_j M_{ij}(b)\,(K_j - K^*_j)$$

- Fixed Points given by

$$K^* = RGT_b(K^*)$$

- Linearizing near fixed points:

$$K_i' \simeq K_i^* + \sum_j M_{ij}(b)\left(K_j - K_j^*\right)$$

$$\frac{\partial K_i'}{\partial K_j}$$

$$\underbrace{(K_i' - K_i^*)}_{\delta K'} \simeq \sum_j \underbrace{M_{ij}(b)}_{\frac{\partial K_i'}{\partial K_j}} \underbrace{(K_j - K_j^*)}_{\delta K_j}$$

$$\delta K' = M(b)\,\delta K$$

$$\delta K' \quad \frac{\partial K_i}{\partial K_j} \quad \delta K_j$$

$$\delta K' = M(b)\,\delta K \longrightarrow M(b)\,\vec{v}_a = \lambda_a(b)\,\vec{v}_a$$

RGT: <u>Semigroup</u> $M(b_1)M(b_2) = M(b_1 b_2)$

$$\lambda_a(b_1)\lambda_a(b_2) = \lambda_a(b_1 b_2)$$

This works for $\lambda_a(b) = b^{y_a}$

$$b_1^{y_a}\, b_2^{y_a} = (b_1 b_2)^{y_a}$$

$\delta K = \sum_a t_a \vec{v}_a$ ← eigenvectors $\qquad \delta K' = \sum_a t'_a \vec{v}_a$

t_a & t'_a are <u>scaling fields</u>

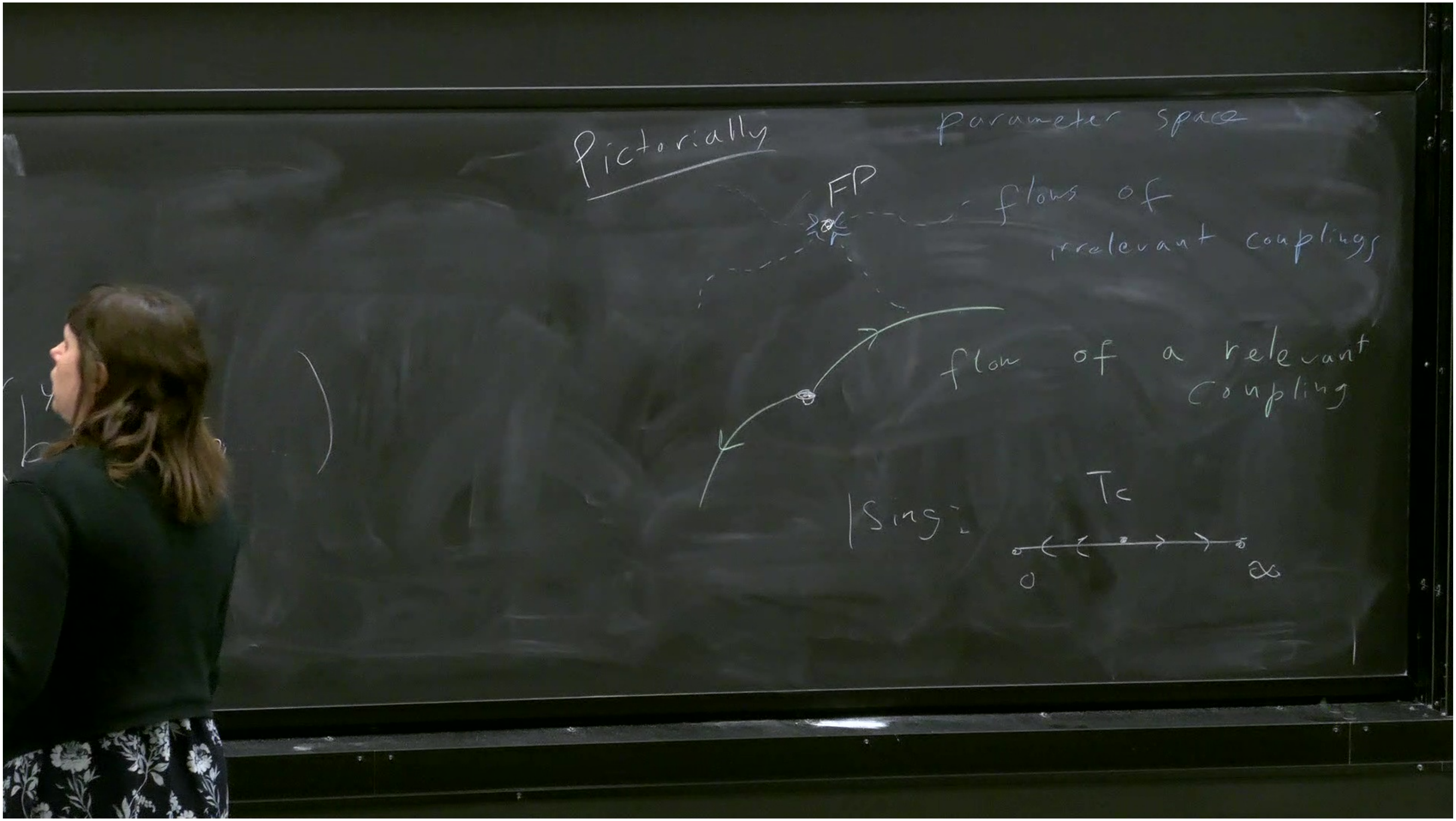
– We have $t'_a = b^{y_a} t_a \qquad RGT_b(t_1, t_2, \dots) = (b^{y_1} t_1, b^{y_2} t_2, \dots)$

A scaling field t_a is

<u>relevant</u> if $y_a > 0$ $\quad t'_a$ increases as b increases

<u>irrelevant</u> if $y_a < 0$

<u>marginal</u> if $y_a = 0$ → we need to go beyond linear to understand


Pictorially
parameter space
FP
flows of
irrelevant couplings
flow of a relevant
coupling
Ising:
Tc
0
∞

if $y_a < 0$

if $y_a = 0$ $\longrightarrow$ we need to go beyond linear to understand

$\frac{\partial^2 f}{\partial h^2}\left(r b^2, h b^{\frac{D+2}{2}}\right)$

$\frac{\partial^2 f}{\partial h^2}(1, 0)$

$\boxed{r = 1}$

$h = 0$ (criticality)

set $r b^2 = 1$, $b^2 = \frac{1}{r}$

Situation:

$y_1 > 0 > y_2 > y_3 \dots$

t_1 can be interpretated as reduced temp.

$\xi \sim \frac{1}{|t_1|^{1/y_1}} \rightarrow \boxed{\nu = \frac{1}{y_1}}$ $\quad g(x) \sim |x|^{2-D-\eta}$

How d

$k' = b$

$\xi(r, h) =$

$\xi(r, 0)$