**Title:** Lecture - Statistical Physics, PHYS 602

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From Before:

$$Z = \sum_{\{\sigma\}} e^{\frac{\beta}{2}\sigma^T J \sigma + \beta \tilde{H}^T \sigma}$$

$\downarrow$ Gaussian Integral $\int d^N x \, e^{-\frac{1}{2}x^T M x + x^T w} = \sqrt{\det(2\pi M^{-1})}\; e^{\frac{1}{2}w^T M^{-1} w}$

$$Z = \sqrt{\frac{\beta^N \det B}{(2\pi)^N}} \int d^N y \; e^{-\frac{\beta}{2} y^T B y} \sum_{\{\sigma\}} e^{\beta (\underbrace{B}_{J+\lambda I} y + \tilde{H})^T \sigma}$$

- Hubbard Stratonovich took us to sum of non-interacting spin pieces
- Everything is still exact

From Ising Model to Ising Field Theory (cont.):

## From Ising Model to Ising Field Theory (cont):

– We introduce a change of variables for $y$

Our Real space field:

$$\phi = \frac{1}{A}\left(y + B^{-1}\tilde{H}\right)$$

constant

$$By + \tilde{H} = BA\phi$$

$$Z = \left( \quad \right) \int d^N\phi \, e^{-\frac{\beta}{2}}$$

$$\sum_{\{\sigma\}} e^{\beta A \sigma^T B \phi}$$

$$e^{-\frac{\beta}{2}\left(A^2\phi^T B\phi + \tilde{H}^T B^{-1}\tilde{H} - 2A\tilde{H}^T\phi\right)} \sum_{\{\sigma\}} e^{\beta A \sigma^T B\phi}$$

$$\sigma^T B\phi$$

$$= \sum_{\{\sigma\}} e^{\beta A \sum_{ij} \sigma_i B_{ij}\phi_j}$$

$$= \sum_{\{\sigma\}} \prod_i e^{\beta A\left(\sigma_i \sum_j B_{ij}\phi_j\right)}$$

$$= \prod_i \sum_{\sigma_i = \pm 1} e^{\beta A\left(\sigma_i (B\phi)_i\right)} = \prod_i 2\cosh\left(\beta A (B\phi)_i\right) = 2^N e^{\sum_i \ln\cosh\left(\beta A (B\phi)_i\right)}$$

Approximations:

For small $x$ $\quad \cosh x \approx 1 + \frac{x^2}{2} + \frac{x^4}{4!}$ , $\quad \ln(1+x) \approx x - x^2/2$

$$\ln(\cosh x) \approx \ln\left(1 + x^2/2 + x^4/4!\right) \approx \frac{x^2}{2} - \frac{x^4}{12}$$

$$S(\phi) \approx \underbrace{\underbrace{\frac{\beta A^2}{2} \phi^T B(\mathbb{1} - \beta B)\phi - \beta A \tilde{H}^T \phi}_{\text{gaussian model}} + \frac{\beta^4 A^4}{12} \sum_i (B\phi)_i^4}_{\phi^4 \text{ theory}}$$

We have the Real-Space Ising Field Theory At critical

$$Z = \sqrt{\frac{\beta^N A^{2N} 2^N \det B}{\pi^N}}\; e^{-\beta/2\, \tilde{H}^T B^{-1} \tilde{H}} \int d^N\phi\; e^{-S(\phi)}$$

$\tilde{H} \to 0$

$\to e^{-\beta/2\, \tilde{H}^T B^{-1} \tilde{H}} \to 1$

$$S(\phi) = \frac{\beta}{2}\tilde{A}\,\phi^T B \phi - \beta A \tilde{H}^T \phi - \sum_i \ln\left(\cosh\left(\beta A (B\phi)_i\right)\right)$$

Exact

# The Gaussian Model: Non-RG Solution

Gaussian Integral

$$\Big\downarrow \quad M = \beta A^2 B(\mathbb{1} - \beta B), \qquad w = \beta A \tilde{H}^T$$

$$Z = \sqrt{\det\left(\frac{2\beta A^2 B}{\pi}\right)} \sqrt{\det\left(\frac{2\pi}{\beta A^2} B^{-1}(\mathbb{1}+\beta B)^{-1}\right)}\; e^{\frac{1}{2}\beta \tilde{H}^T B^{-1}(\mathbb{1}-\beta B)^{-1}\tilde{H}}$$

<u>But</u> $\mathbb{1} - \beta B$ must be positive definite

– eigenvalues of $B$ (from momentum space) are $B_k = \lambda + 2J\sum_\mu \cos k_\mu a$

are $B_k = \lambda + 2J\sum_\mu \cos k_\mu a$

$\rightarrow \quad B_k \leq B_0$

$1 - \beta B_k \geq 1 - \beta B_0 \rightarrow \beta B_0 < 1$

dimensions

$$\frac{B_0}{k} < T \rightarrow T > \frac{B_0}{k} = \frac{\lambda + 2JD}{k} = T_c$$

Critical Exponents: $T > T_c$

$$f = -\frac{1}{N} kT \ln Z = -k_B T \ln 2 - \frac{\beta_c H^2}{2t} + \frac{k_B T}{2N}\sum_k \ln(1 - \beta B_k)$$

$$\chi = -\frac{\partial^2 f}{\partial H^2} = \frac{\beta_c}{t} \to \boxed{\gamma = 1}$$

Can also get $\alpha$ from $C$

$$\boxed{\alpha = \begin{cases} 2-\frac{D}{2} & \text{if } D<4 \\ 0 & \text{if } D\geq 4 \end{cases}}$$

Can get $\eta$

$$g(x_i - x_j) = \langle \phi_i \phi_j \rangle - \overset{\to 0}{\cancel{\langle \phi_i \rangle \langle \phi_j \rangle}}$$

$$\to \boxed{\eta = 0}$$

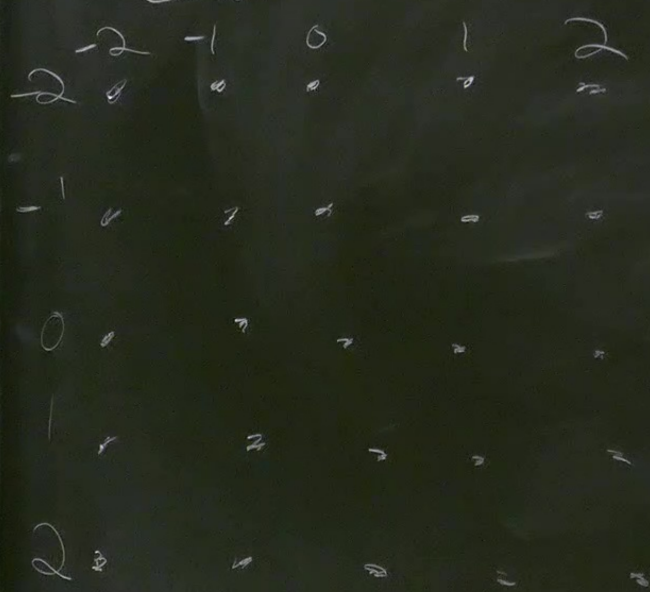
minimize

$$S(\phi) \approx S(\varphi) - \frac{1}{2}\sum_{ij} (\phi - \varphi)_i (\phi - \varphi)_j \frac{\partial^2 S}{\partial \phi_i \partial \phi_j}(\varphi)$$

for all $T$

# Momentum Space:

Lattice:




Assume $N = (2n+1)^D$

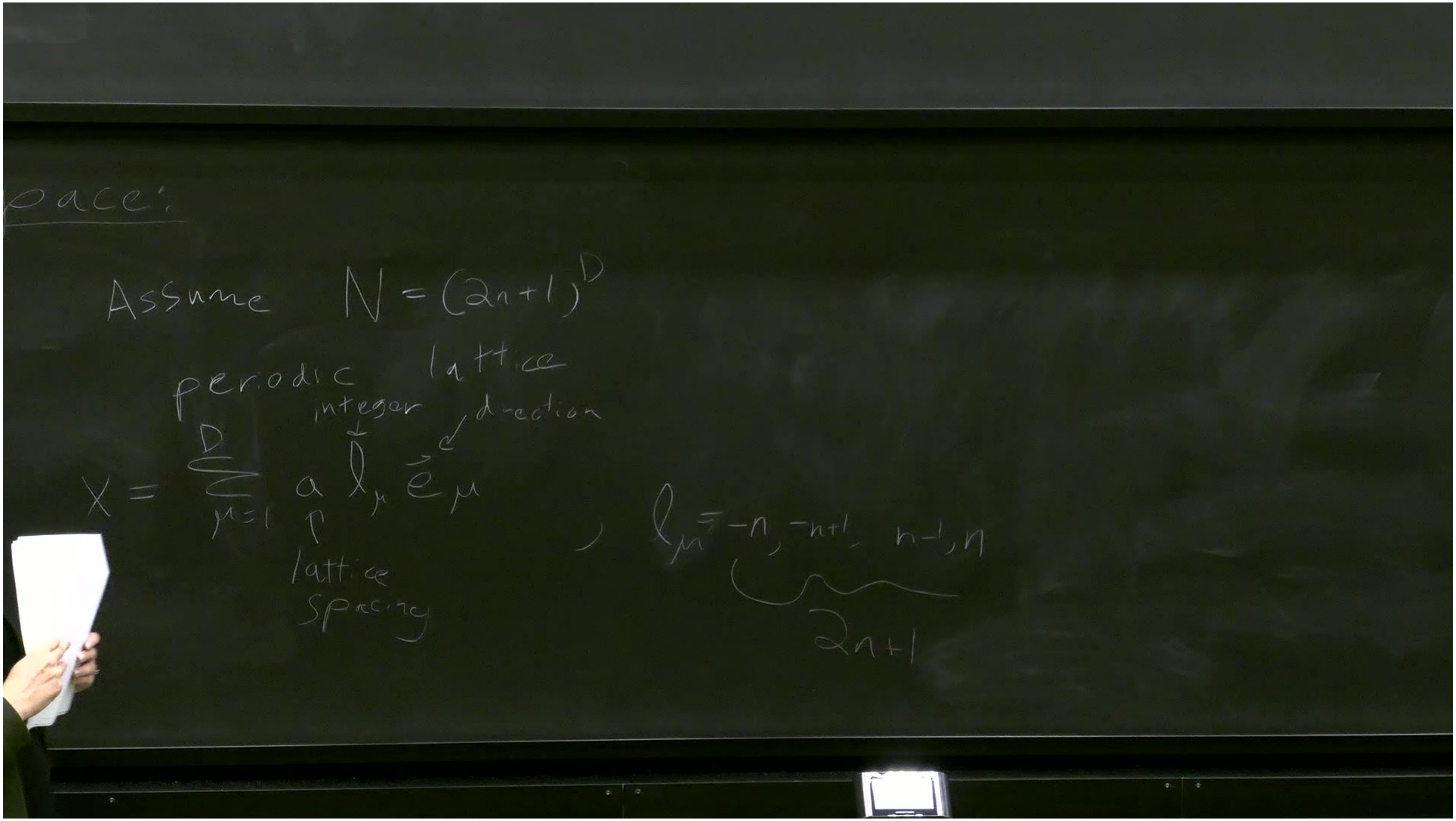
periodic lattice

integer

direction

$$x = \sum_{\mu=1}^{D} a \, l_\mu \, \vec{e}_\mu \quad , \quad l_\mu =$$

lattice spacing


pace:
Assume $N = (2n+1)^D$
periodic lattice
integer
direction
$X = \sum_{\mu=1}^{D} a \, l_\mu \, \vec{e}_\mu$
lattice spacing
$l_\mu = -n, -n+1, \ldots, n-1, n$
$2n+1$

periodicity: $x = x + (2n+1) a \vec{e}_\mu$

Lattice Fourier transform, $f(x_i)$

$$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i) e^{-ik \cdot x_i}$$

$k$ lives in $k$-space

$$\Lambda = \frac{\pi}{a}$$

$$k = \sum_{\mu=1}^{D} k_\mu \vec{e}_\mu$$

$n-1, n$

$$k_\mu = \frac{2m\Lambda}{(2n+1)} \qquad m = -n, -n+1, \ldots n-1, n$$

$n+1$

$\phi^4$ theory

periodicity: $x = x + (2n+1)\, a\, \vec{e}_\mu$

Lattice Fourier transform; $f(x_i)$

$$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i)\, e^{-i k \cdot x_i}$$

$k$ lives in $k$-space

$$k = \sum_{\mu=1}^{D} k_\mu \vec{e}_\mu \qquad \Lambda = \frac{\pi}{a}$$

$$k_\mu = \frac{2m\Lambda}{(2n+1)} \qquad m = -n, -n+1, \ldots n-1, n$$

$$l_\mu = \underbrace{-n, -n+1, \ldots n-1, n}_{2n+1}$$

This comes from periodic boundary conditions:

$$\sum_{i=1}^{N} f(x_i) e^{-ik\cdot x_i} = \sum_{i=1}^{N} f(x_i) e^{-ik\cdot(x_i + (2n+1) a \vec{e}_\mu)}$$

We know that $e^{-i2\pi m} = 1$

So we need! $ik_\mu (2n+1)_a = i2\pi m$

$$\hookrightarrow \quad k_\mu = \frac{2m\pi}{(2n+1)a}$$

$(2n+1)$

Inverse?

$$f(x_i) = \frac{1}{N}\sum_k \tilde{f}(k)\, e^{ik\cdot x_i}$$

$$\frac{1}{N}\sum_k \tilde{f}(k)\, e^{ik\cdot x_j} = \frac{1}{N}\sum_{k,i} f(x_i)\, e^{-ik\cdot(x_i - x_j)}$$

$$= \frac{1}{N}\sum_i f(x_i) \underbrace{\sum_k e^{-ik\cdot(x_i - x_j)}}_{N\,\delta_{x_i, x_j}}$$

$$= f(x_j)$$

**Note:** assuming $f(x_i)$ is real,

$$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i)\, e^{-ik\cdot x_i}$$

we see that

$$\tilde{f}(k) = \tilde{f}^{*}(-k)$$

So not all values of $\tilde{f}(k)$ are independent

(because $\tilde{f}$ is complex)

Summary:

$$\tilde{f}(k) = \sum_{i=1}^{N} f(x_i)\, e^{-ik\cdot x_i} \longleftrightarrow f(x_i) = \frac{1}{N}\sum_{k} \tilde{f}(k)\, e^{ik\cdot x_i}$$

Generalize:

$$\tilde{f}(k,q) = \sum_{i,j}^{N} f(x_i,x_j)\, e^{-ik\cdot x_i}\, e^{-iq\cdot x_j} \longleftrightarrow f(x_i,x_j) = \frac{1}{N^2}\sum_{kq} \tilde{f}(k,q)\, e^{ik\cdot x_i}\, e^{iq\cdot x_j}$$

$\phi_i$ scalar field, $\varphi_k$ momentum space counterpart

$$\varphi_k = \underbrace{a^D}_{\text{convention}} \sum_i \phi_i\, e^{-ik\cdot x_i} \longleftrightarrow \phi_i = \frac{1}{a^D N}\sum_k \varphi_k\, e^{ik\cdot x_i}$$

we see that

$$S(\phi) = \frac{\beta}{2}\tilde{A}\,\phi^T B\phi - \beta A \tilde{H}^T \phi - \sum_i \ln\left(\cosh\left(\beta A (B\phi)_i\right)\right)$$

Exact

$$S(\phi) \approx \frac{\beta A}{2}$$

## Ising Model Momentum Space Field Theory

For $B$: $\qquad B_{ij} = J_{ij} + \lambda \mathbb{1}$

$$B_{ij} = B(x_i, x_j)$$

$$\tilde{B}_{kq} = \sum_{i,j} B_{ij}\, e^{-ik\cdot x_i}\, e^{-iq\cdot x_j}$$

$$\tilde{B}_{kq} = \sum_i \underbrace{B_{ii}}_{\lambda}\, e^{-i(k+q)\cdot x_i} + \sum_{i,\hat{e}_\mu} B(x_i, x_i + \hat{e}_\mu)\, e^{-i(k+q)x_i}\left(e^{-iq\cdot\hat{e}_\mu a} + e^{iq\cdot\hat{e}_\mu a}\right)$$

$B_{kq} = \sum_i B_{ii} e \quad \ldots \quad \sum_{i,\hat{e}_\mu} B(i, i+\hat{e}_\mu) e$

$\lambda$

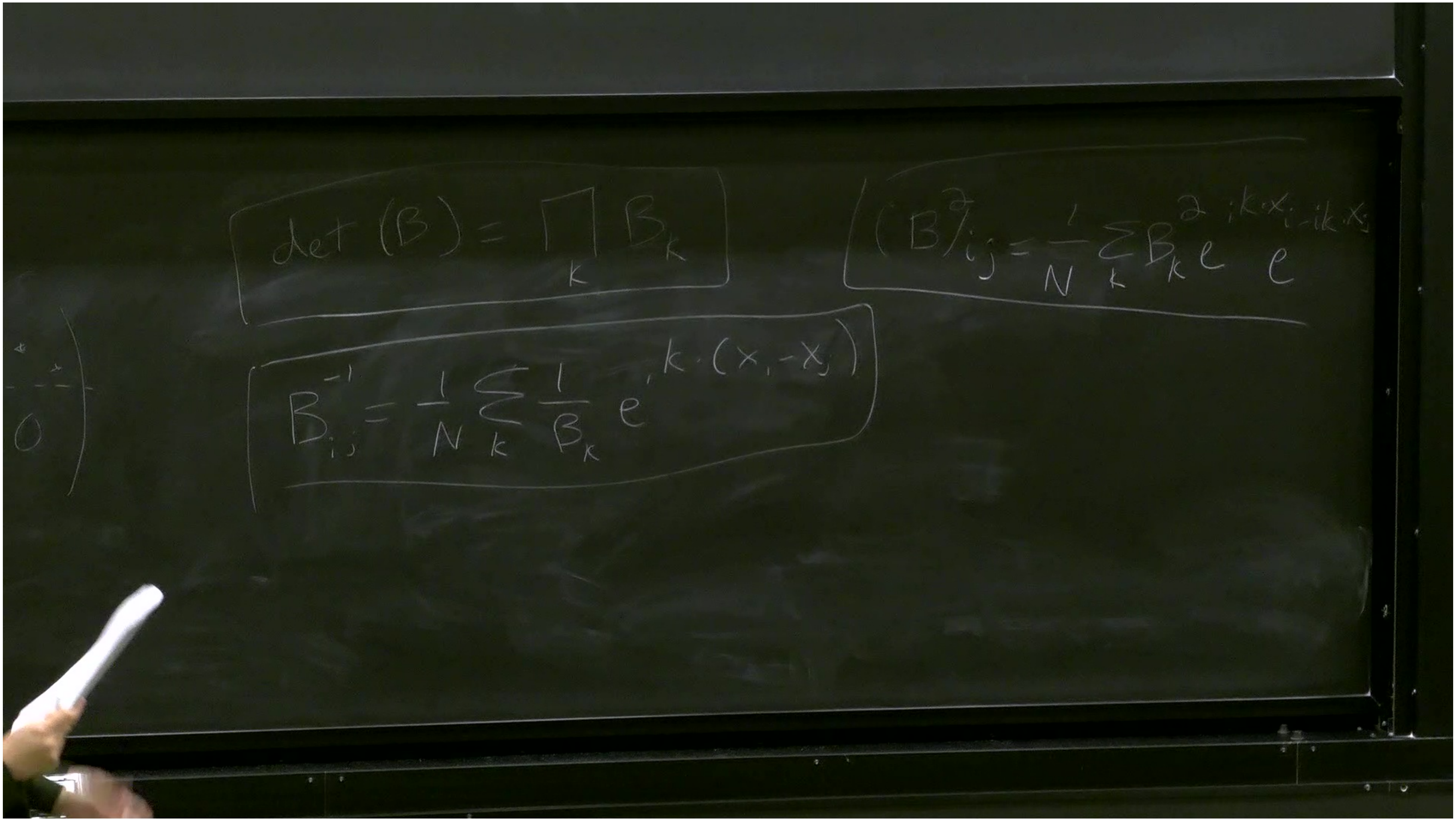
$$\tilde{B}_{kq} = \sum_i \lambda e^{-i(k+q)\cdot x_i} + 2\sum_{i,\mu} J e^{-i(k+q)\cdot x_i} \cos q_\mu a$$

$$= N\lambda\, \delta_{k,-q} + 2NJ \sum_\mu \cos q_\mu a\, \delta_{k,-q}$$

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

$$B_{ij} = \frac{1}{N^2} \sum_{k,q} \tilde{B}_{kq}\, e^{ik\cdot x_i} e^{iq\cdot x_j} = \frac{1}{N} \sum_k \underbrace{\left(\lambda + 2J\sum_\mu \cos k_\mu a\right)}_{B_k}$$

$$\boxed{B_{ij} = \frac{1}{N} \sum_k B_k\, e^{ik\cdot(x_i - x_j)}}$$


$\det(B) = \prod_k B_k$
$(B^2)_{ij} = \frac{1}{N}\sum_k B_k^2 e^{ik\cdot x_i} e^{-ik\cdot x_j}$
$B^{-1}_{ij} = \frac{1}{N}\sum_k \frac{1}{B_k} e^{ik\cdot(x_i - x_j)}$

From $B_{ij}$:

$$\phi^T B \phi = \sum_{ij} \phi_i B_{ij} \phi_j = \frac{1}{V^2 N} \sum_{\substack{i,j \\ k,q,k'}} \varphi_k e^{ik\cdot x_i} B_q e^{iq\cdot(x_i - x_j)} \varphi_{k'} e^{ik'\cdot x_j}$$

$$= \frac{1}{V^2 N} \sum_{\substack{i,j \\ k,q,k'}} \varphi_k B_q \varphi_{k'} \underbrace{e^{i(k+q)\cdot x_i}}_{N\delta_{k,-q}} \underbrace{e^{i(k'-q)x_j}}_{N\delta_{k',q}}$$

$$= \frac{N}{V^2} \sum_k \varphi_k B_{-k} \varphi_{-k} \rightarrow \boxed{\phi^T B \phi = \frac{1}{V a^D} \sum_k B_k |\varphi_k|^2}$$

## Momentum Space Field Theory

$$Z = \frac{1}{\sqrt{2}} \prod_k \sqrt{\frac{\beta A^2 4 B_k}{\pi V a^D}} \int D^N\varphi \; e^{-S(\varphi)}$$

$$S(\varphi) \simeq \underbrace{\underbrace{\frac{A^2\beta}{2}\frac{1}{V a^D}\sum_k B_k(1-\beta B_k)\left|\varphi_k\right|^2 - \beta A \frac{H}{a^D}\varphi_0}_{\text{Gaussian Model}} + \frac{\beta^4 A^4}{12}\frac{1}{V^3 a^D}\sum_{k_1,k_2,k_3,k_4} B_{k_1}B_{k_2}B_{k_3}B_{k_4}\,\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\delta_{k_1+k_2+k_3+k_4,0}}_{\phi^4 \text{ theory}}$$