Title: Lecture - Statistical Physics, PHYS 602

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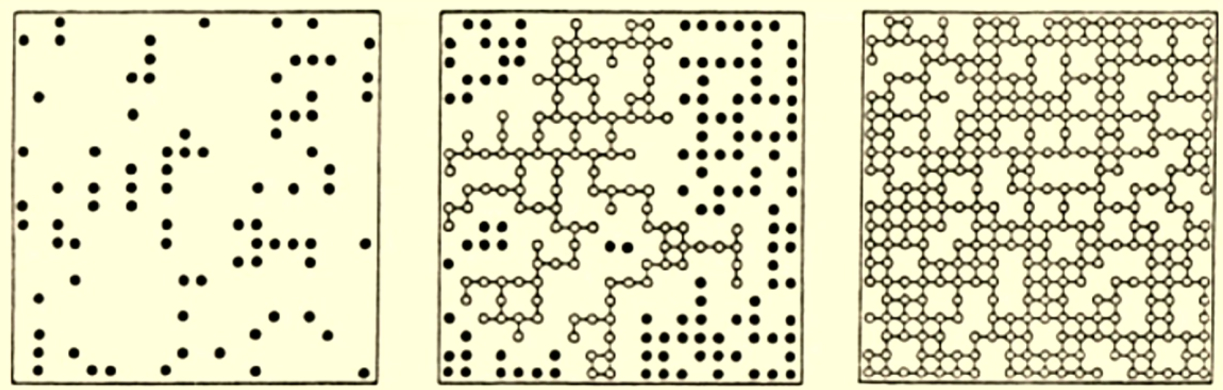
Collection/Series: Statistical Physics (Core), PHYS 602, October 7 - November 6, 2024

Subject: Condensed Matter, Other

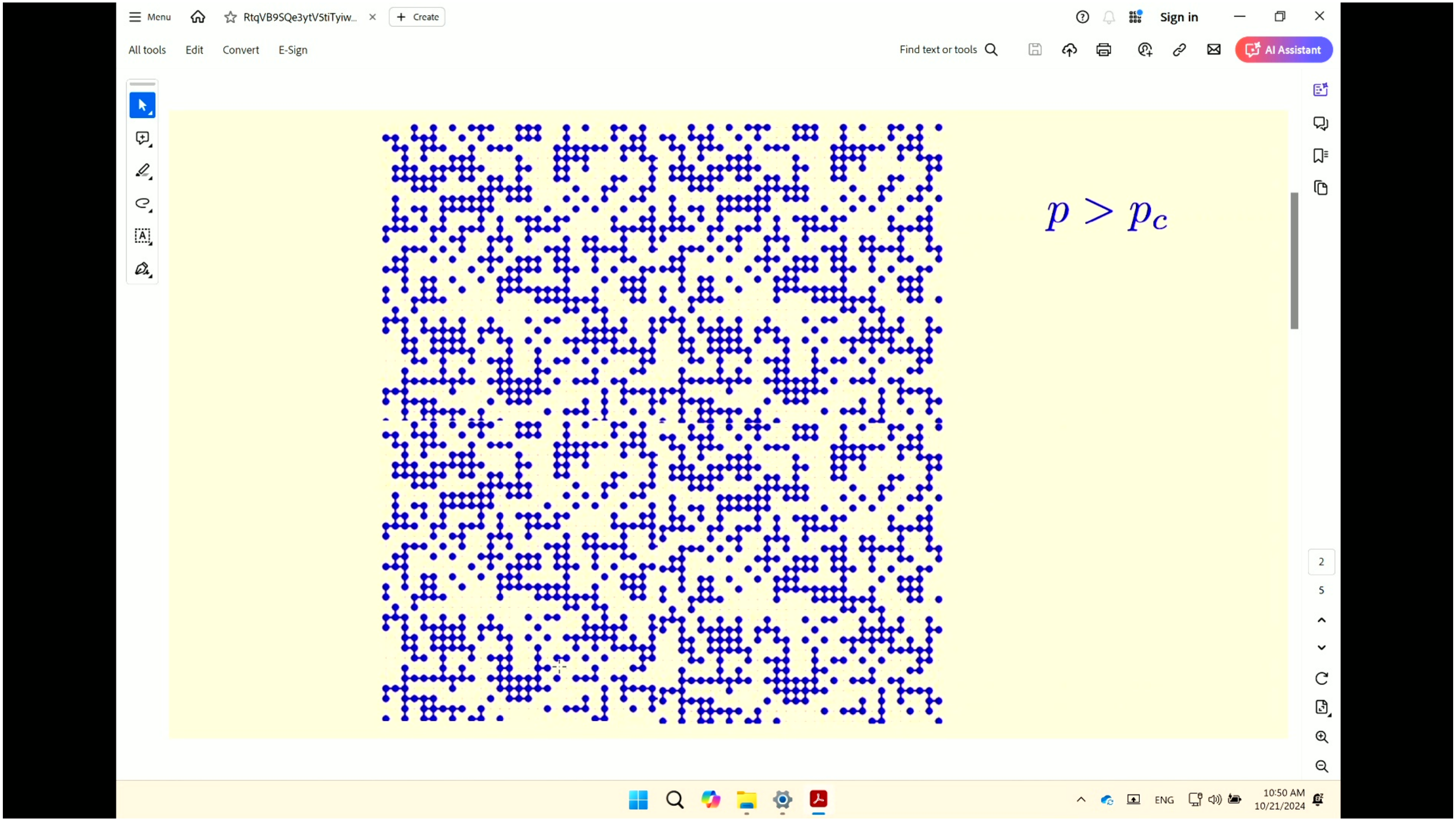
Date: October 21, 2024 - 10:45 AM

URL: https://pirsa.org/24100013

Percolation



a $p = 0.2$ b $p = 0.59$ c $p = 0.8$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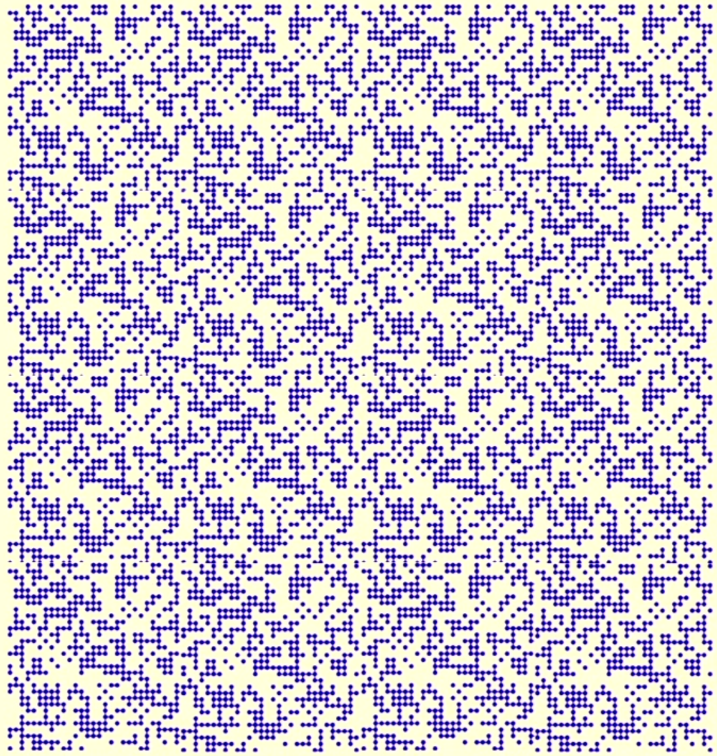
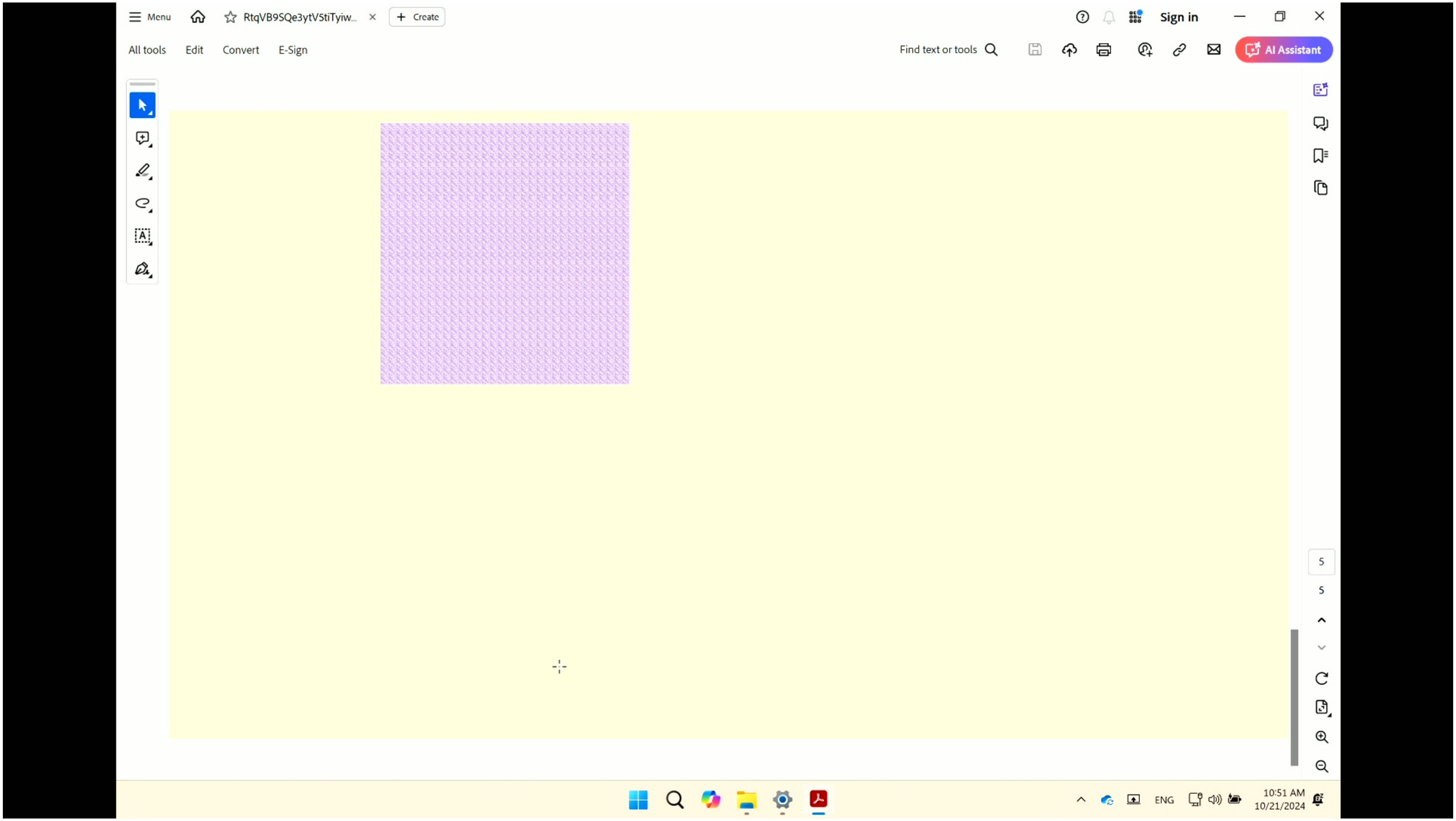


$$p' = R(p)$$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Percolation Theories

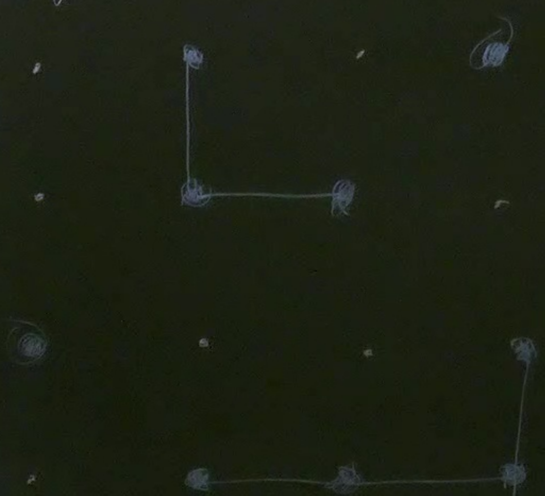
- A family of theories that involve clusters formed from independent probabilities — and some are easily solved with real-space renormalization group (RG
- defined by a lattice and a probability

How do they work?

Square lattice: each site has independent

How do they work?

Square lattice:



four clusters
(sizes 1,1,3,4)

- each site has independent prob. p of being marked
- clusters are groups of contiguous (nearest neighbor) sites
- will assume systems so large that boundary conditions don't matter

four clusters
(sizes 1, 1, 3, 4)

that boundary conditions don't matter

- Each system has a critical probability, p_c

Phases

$p < p_c$ finite clusters
($p_c \simeq 0.59$ for square lattice point model)

$p > p_c$ "infinite" (or cluster that extends across the system)
and complementary finite clusters

$p = p_c$ scale invariance, clusters of all sizes + infinite cluster (incipient cluster)
(power law distribution of cluster sizes)

$\xi \sim (p-p_c)^{-\nu}$

size of finite clusters - first many small clusters, blows up to ∞ at p_c, then the remaining finite clusters get fewer and smaller as p increases more

...up to

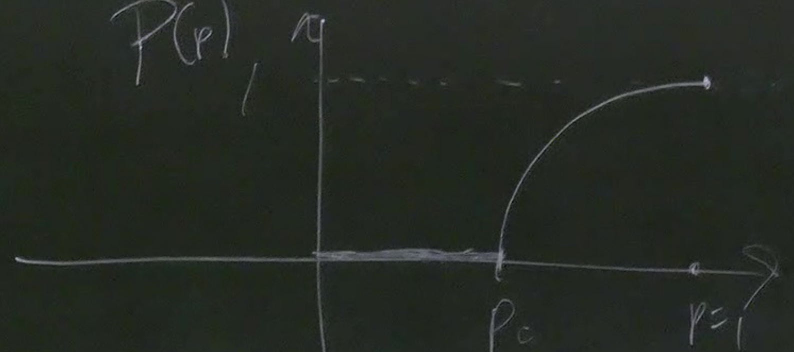
...ining finite

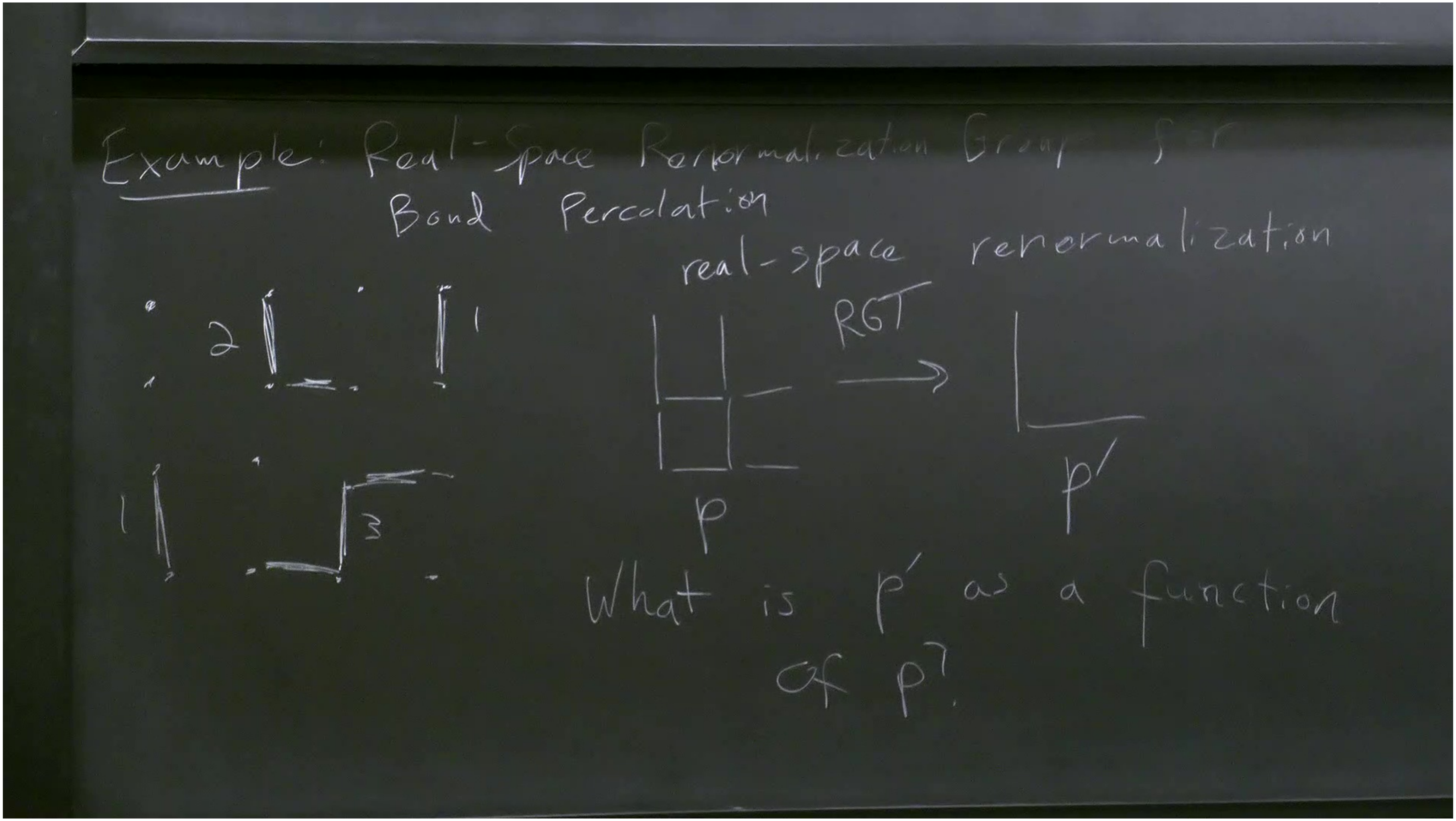
...maller as p

Analog of Magnetization:

$$P(p) \sim (p-p_c)^{\beta} \qquad p \geq p_c$$

prob. of being in the ∞ cluster




Example: Real-Space Renormalization Group for
Bond Percolation
real-space renormalization
2
1
1
3
RGT
p
p'
What is p' as a function of p?

What is p' as a function of p?

Criterion for horizontal coarse graining

- If there is a horizontal path across cell $\rightarrow$ contributes to p'.

Way we can do this for these bonds:

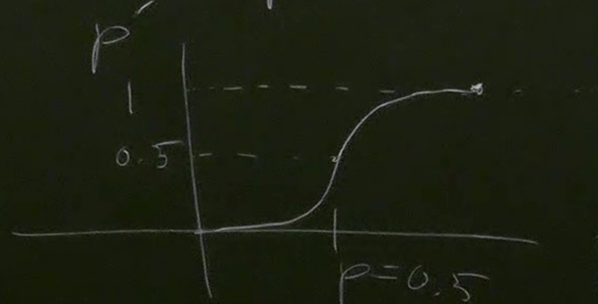
$$p' = p^5 + p^4(1-p) + 4p^4(1-p) + 2p^3(1-p)^2 + 2p^3(1-p)^2 + 4p^3(1-p)^2 + 2p^2(1-p)^3$$

$\uparrow$ RGT(p)

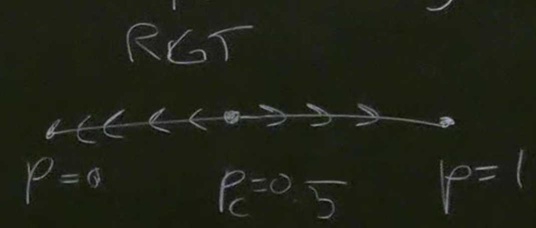
Fixed point: When is $p = RGT(p)$

trivial: $p = 0, 1$

nontrivial: $p = 0.5$



Flow diagram:

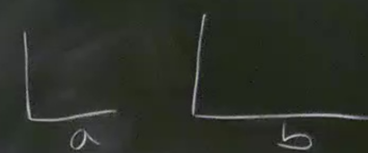


Read More: Percolation Theory Chapters 1 to 4

$p = p_c$ scale invariance, clusters of all sizes + infinite cluster (incipient cluster) (power law distribution of cluster sizes)

Why "Renormalization Group"

- coarse graining is like finding a new unit for the lattice (new norm) each time



- $RGT_a \circ RGT_b = RGT_{ab}$

  semigroup – not a group

  because not reversible

1D Ising Model Block Spin (decimation) Tutorial 5

One-dimension:

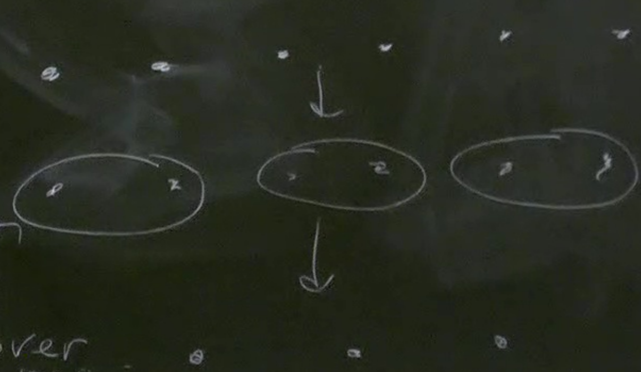
$$E(\sigma) = -J\sum_i \sigma_i \sigma_{i+1} - H\sum_i \sigma_i$$

$$Z = \sum_{\sigma_1,\sigma_2,\ldots\sigma_N} e^{\beta J \sum_i \sigma_i \sigma_{i+1}}$$

Tutorial:

$$Z(N, K) = [a(K')]^{N/2} Z(N/2, K')$$

βJ



Sum up over the parts in the circles

Spin (decimation) Tutorial 5

$$\sigma_i \sigma_{i+1} - H \sum_i \sigma_i$$

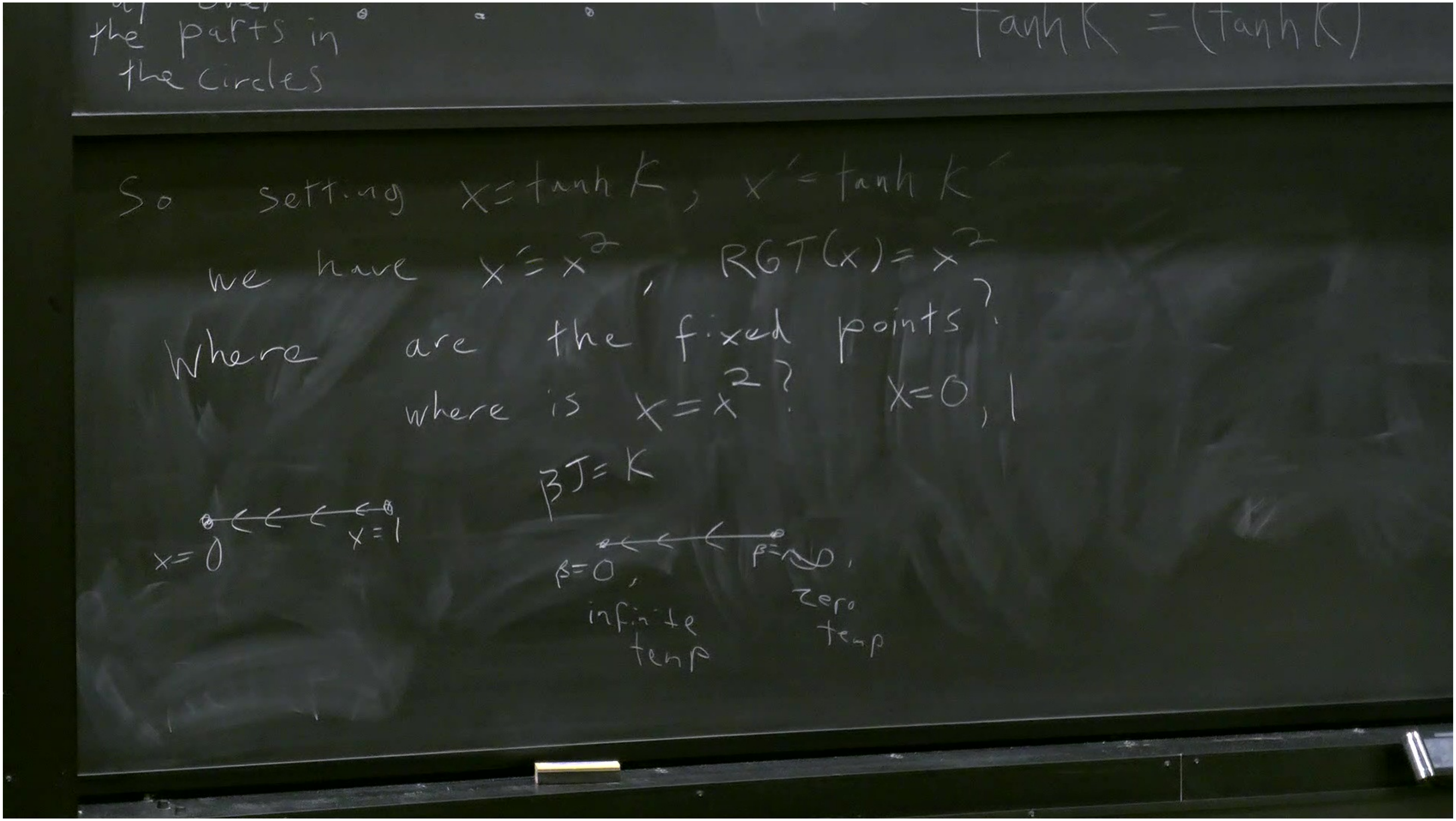
$$Z = \sum_{\sigma_1, \sigma_2, \ldots \sigma_N} e^{\beta J \sum_i \sigma_i \sigma_{i+1}}$$

Tutorial:

$$Z(N, K) = [a(K)]^{N/2} Z(N/2, K')$$

βJ

RGT: $\tanh K' = (\tanh K)^2$


the parts in
the circles
tanh K = (tanh K)
So setting x = tanh K, x' = tanh K'
we have x' = x^2, RGT(x) = x^2
Where are the fixed points?
where is x = x^2? x = 0, 1
βJ = K
x = 0
x = 1
β = 0,
β = ∞,
infinite
temp
zero
temp

When there is a phase transition for the Ising model.

Ising

$T > T_c$ small islands of aligned spins in a sea of disorder

$T < T_c$ small islands of down (up) spins in a sea of up (down) spins

$T = T_c$ Scale invariance, clusters of aligned spins of all sizes

Flow diagrams:

$$\underset{p=0}{\circ} \longleftarrow \underset{p=p_c}{\bullet} \longrightarrow \underset{p=1}{\circ}$$

$$\underset{T=0}{\circ} \longleftarrow \underset{T=T_c}{\bullet} \longrightarrow \underset{T=\infty}{\circ}$$

(power law distribution of cluster sizes)

Critical Exponents

Percolation:

$$\xi(p) \sim (p - p_c)^{-\nu}$$

$$\mathbb{P}(p) \sim (p - p_c)^{\beta} \qquad p \geq p_c$$

↑ prob. of being in ∞ cluster

$$\langle s^2 \rangle \sim (p - p_c)^{-\gamma}$$

↑ cluster size squared

Ising:

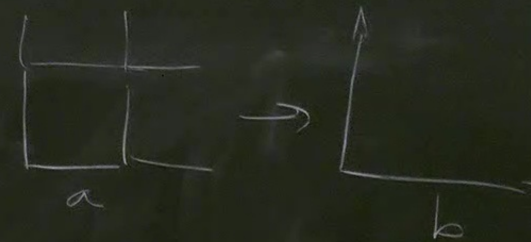
$$\xi(T) \sim (T - T_c)^{-\nu}$$

$$M(T) \sim (T_c - T)^{\beta} \qquad T \leq T_c$$

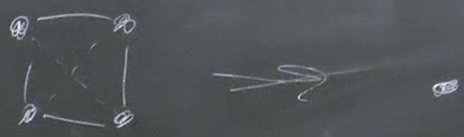
$$\chi(T) \sim (T - T_c)^{-\gamma}$$

Doesn't work nicely for Ising model > 1D

Momentum Space RG



- For lattice spacing a, the maximum momentum in any direction is $\frac{\pi}{a}$. For bigger lattice spacing $b > a$ the maximum momentum is $\frac{\pi}{b} < \frac{\pi}{a}$



Idea:

– "integrate out" higher $k > \frac{\pi}{b}$ and see what we're left with

– to illustrate this technique, we return to the Ising model, but we find a field theory form for it

From Ising model to field theory

- Hubbard–Stratonovich transformation for Ising Model

Gaussian Integral:

$$e^{-\frac{1}{2}\underset{\text{Vector}}{w}^T \overset{\text{Matrix}}{A} w} = \sqrt{\frac{\det A}{(2\pi)^n}} \int_{\mathbb{R}^n} d^n y \, e^{-\frac{1}{2}y^T A y + y^T A w}$$

- $w \in \mathbb{C}^n$
- A is $n \times n$ real, positive-definite, symmetric

Ising Model:

$$E(\sigma) = -\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j - H\sum_{i=1}^{N}\sigma_i$$

$$Z = \sum_{\{\sigma\}} e^{\frac{\beta}{2}\sigma^T J\sigma + \beta\tilde{H}^T\sigma}$$

$$\tilde{H} = \begin{pmatrix} H \\ H \\ H \\ \vdots \\ H \end{pmatrix} \in \mathbb{R}^n$$

real, positive definite, symmetric

Prob: J isn't positive definite $\quad \mathrm{Tr}(J)=0$

Introduce $B = \lambda \mathbb{1} + J$ $\qquad \langle O \rangle_\lambda =$

So physics is unchanged. $\qquad$ finite

$$Z' = e^{\frac{\beta \lambda N}{2}} Z$$

We will now write Z' as Z

...tive-definite, symmetric

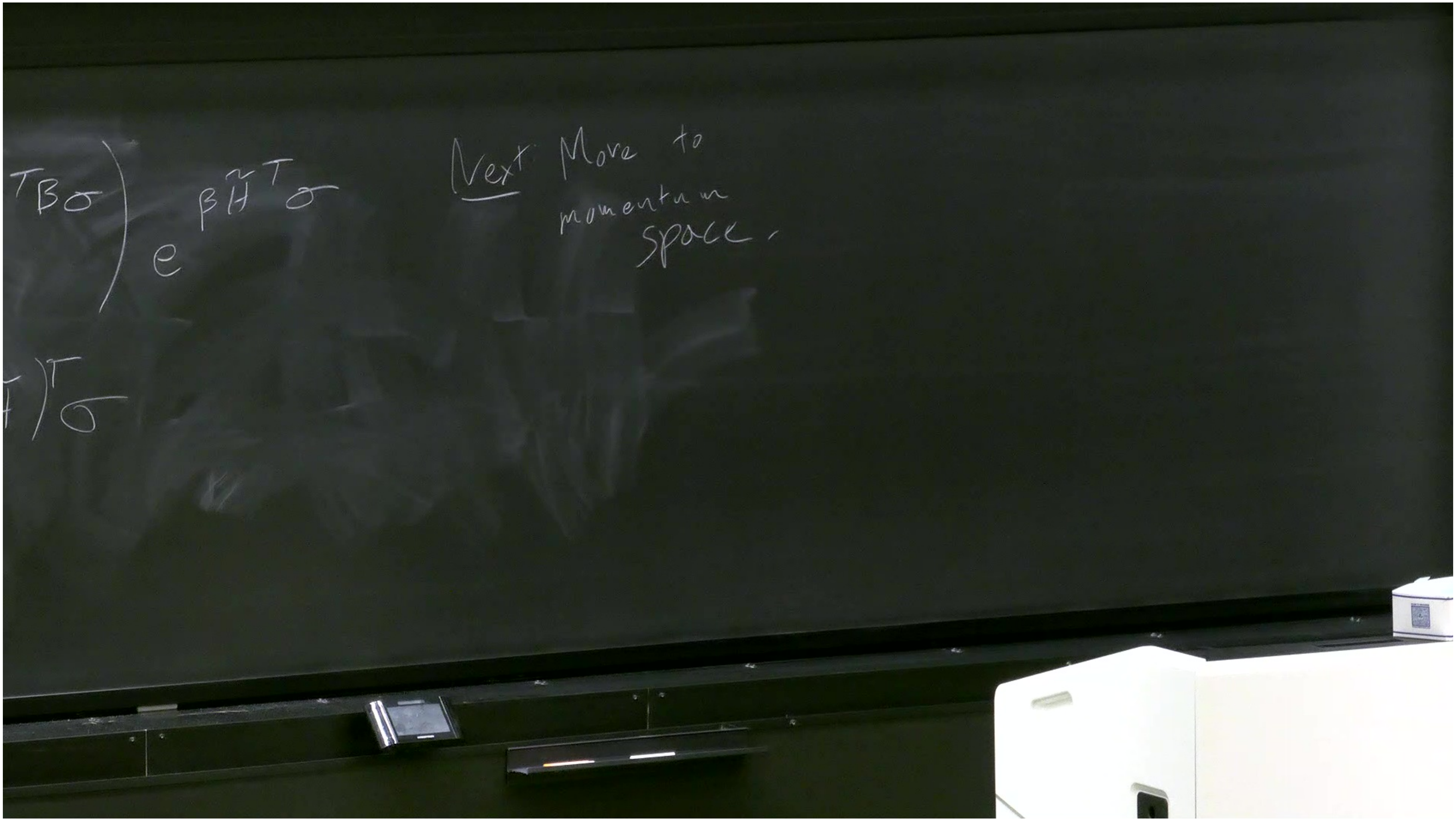
Tr(J) = 0

$$\langle O \rangle_{\underset{\text{finite}}{\lambda}} = \frac{\sum_{\{\sigma\}} O(\sigma)\, e^{-\beta\left(E(\sigma) + \lambda \frac{N}{2}\right)}}{\sum_{\{\sigma\}} e^{-\beta\left(E(\sigma) + \lambda \frac{N}{2}\right)}} = \langle O \rangle_{\lambda = 0}$$

We can rewrite Z as

$$Z = \sum_{\{\sigma\}} \left(\sqrt{\frac{\beta^N \det B}{(2\pi)^N}} \int d^N y \; e^{-\frac{\beta}{2} y^T B y + \beta y^T B \sigma} \right) e^{\beta \tilde{H}^T \sigma}$$

$$= \sqrt{\frac{\beta^N \det B}{(2\pi)^N}} \int d^N y \; e^{-\frac{\beta}{2} y^T B y} \sum_{\{\sigma\}} e^{\beta (B y + \tilde{H})^T \sigma}$$


Next: Move to momentum space.