Title: Lecture - Statistical Physics, PHYS 602

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Subject: Condensed Matter, Other

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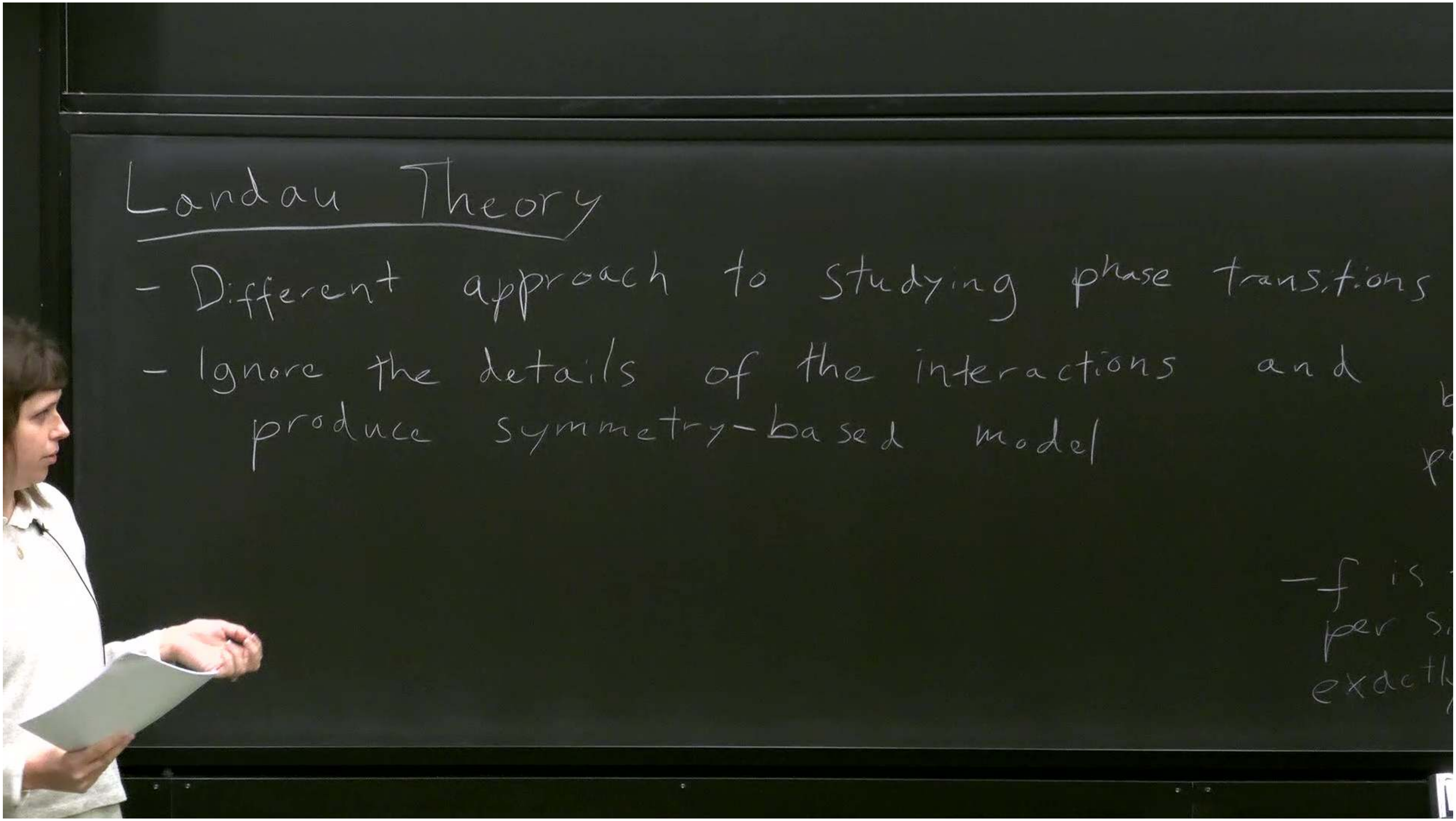

Landau Theory
- Different approach to studying phase transitions
- Ignore the details of the interactions and
produce symmetry-based model

ransitions

and

Landau free energy:

$$Z = \sum_{\sigma} e^{-\beta E(\sigma, H)}$$

$$\sigma = \{\sigma_1, \sigma_2, \ldots \sigma_N\}$$

organized by order parameter

$$= \sum_{m} \sum_{\sigma \text{ for } m} e^{-\beta E(\sigma, H)}$$

$$\approx \sum_{m} e^{-\beta N f(m, H)}$$

– f is the Landau free energy per site → as $N \to \infty$, becomes exactly equal to term with lowest $f(m, H)$

Landau-Ginzburg Theory

- Promotion of Landau-type assumptions to a field theory
- Allows us to model correlations/fluctuations

$$F(m(\vec{r})) = \int d\vec{r} \left\{ \left[a m(\vec{r})^2 + b m(\vec{r})^4 + \dots \right] + \alpha |\nabla m(\vec{r})|^2 + \dots \right\}$$

functional

$$Z = \int \mathcal{D} m(\vec{r})\, e^{-\beta F[m(\vec{r})]}$$

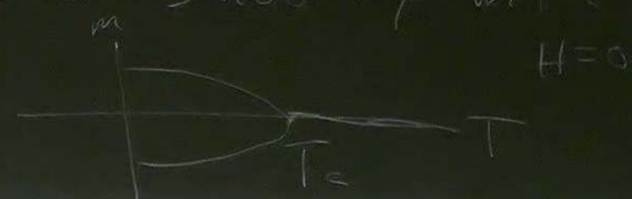
- constant $m(\vec{r}) \rightarrow$ reduces to Landau Theory

Application to Ising Model

- For Ising, we assume $f(m, H)$ ← scalar

$$f(m,0) = f(-m,0)$$

- when H and $T_c - T$ are small, m is small
- m varies smoothly with T

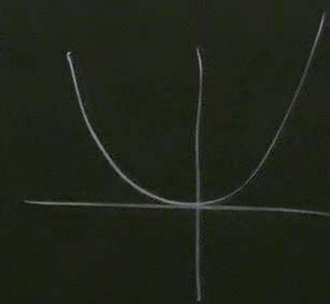


Tayl

Taylor expansion: 0 WLOG

$$f(m,0) \approx \cancel{f_0} + r(T)\, m^2 + u(T)\, m^4$$

- Near the critical point, we will be minimizing f, so $u \approx u_0 > 0$



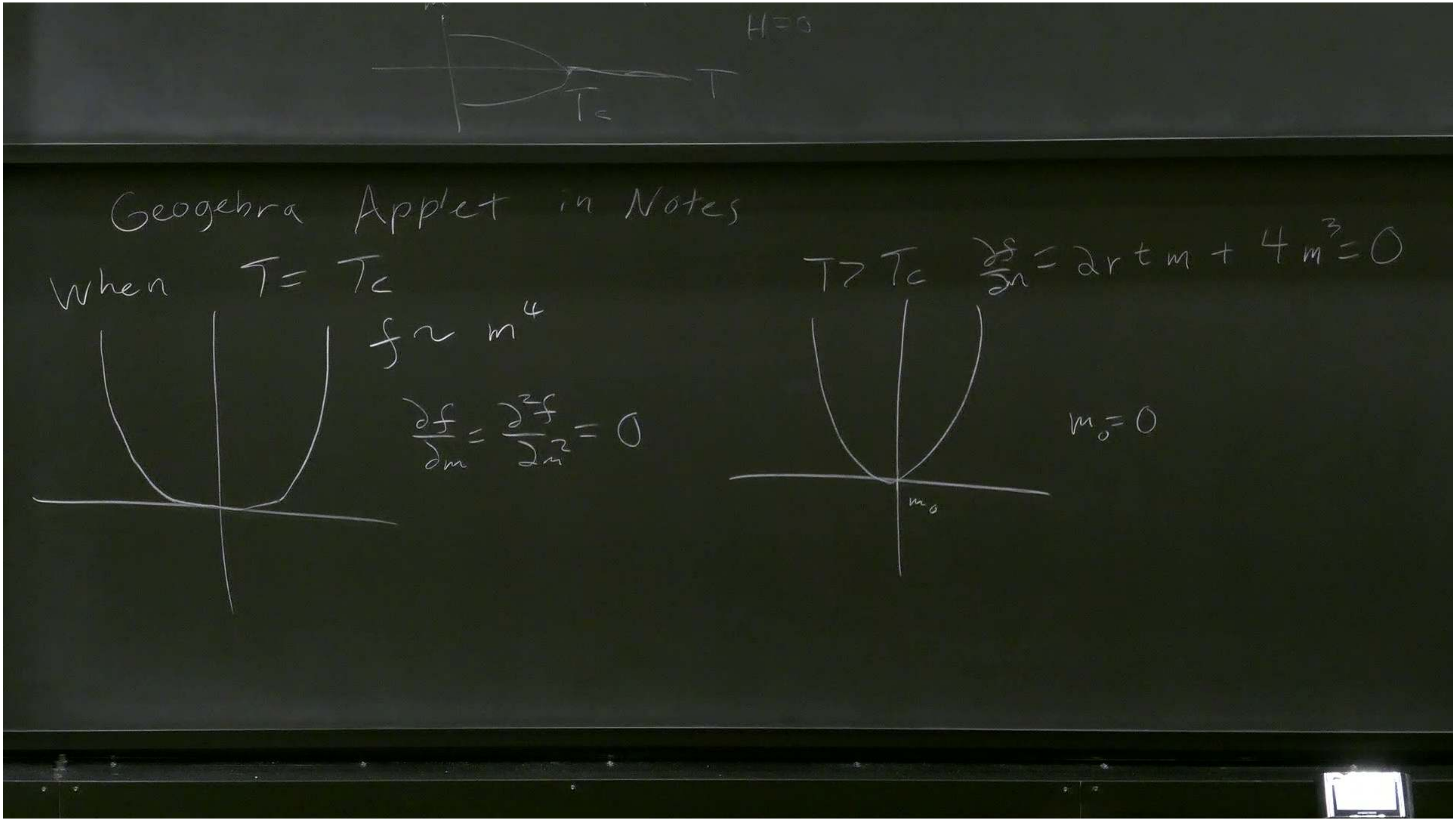
- Below $T = T_c$, $r(T)$ should change sign

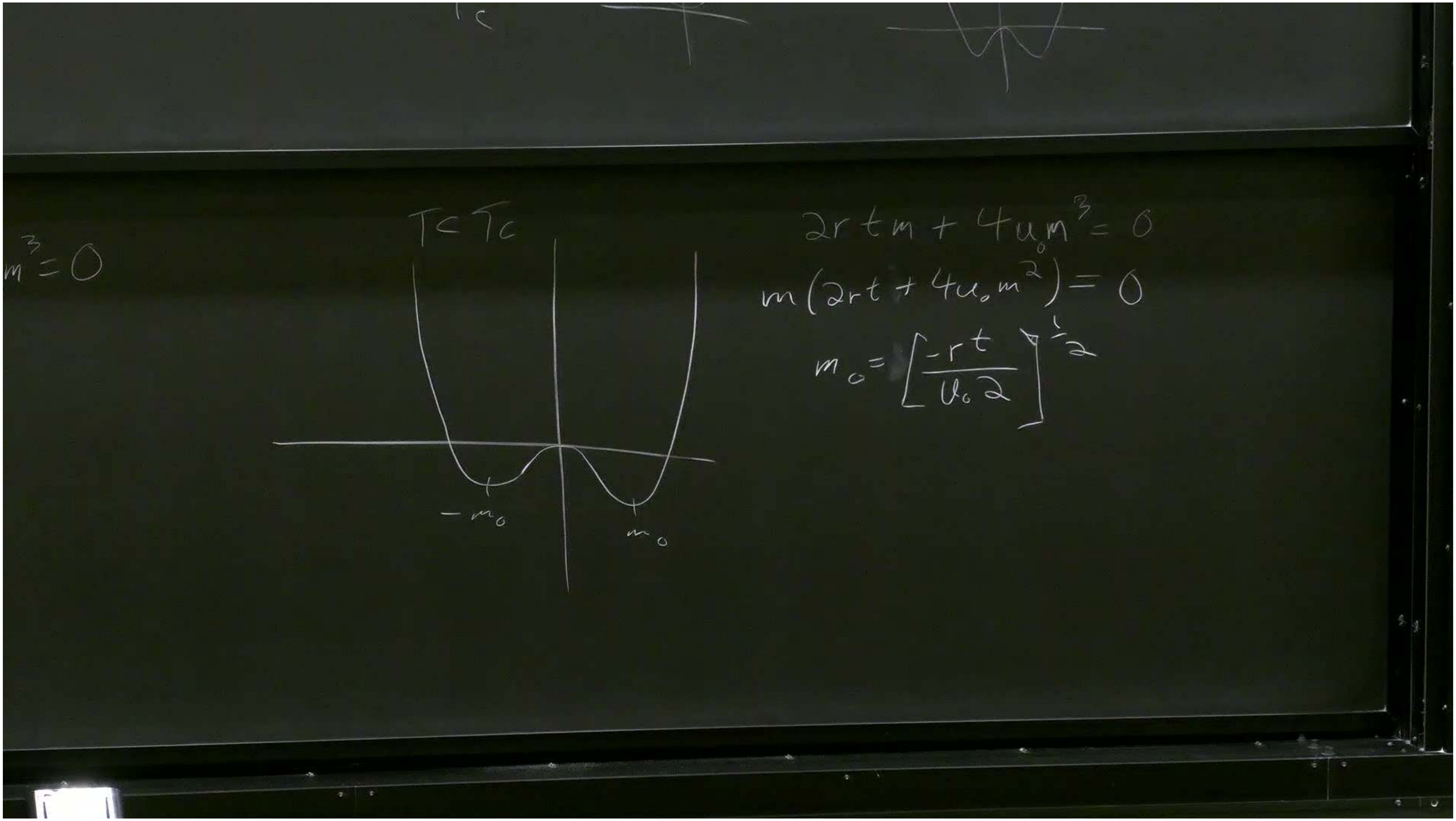
$$r(T) = r\,t \quad \nwarrow \frac{T-T_c}{T_c}$$

$$r > 0$$

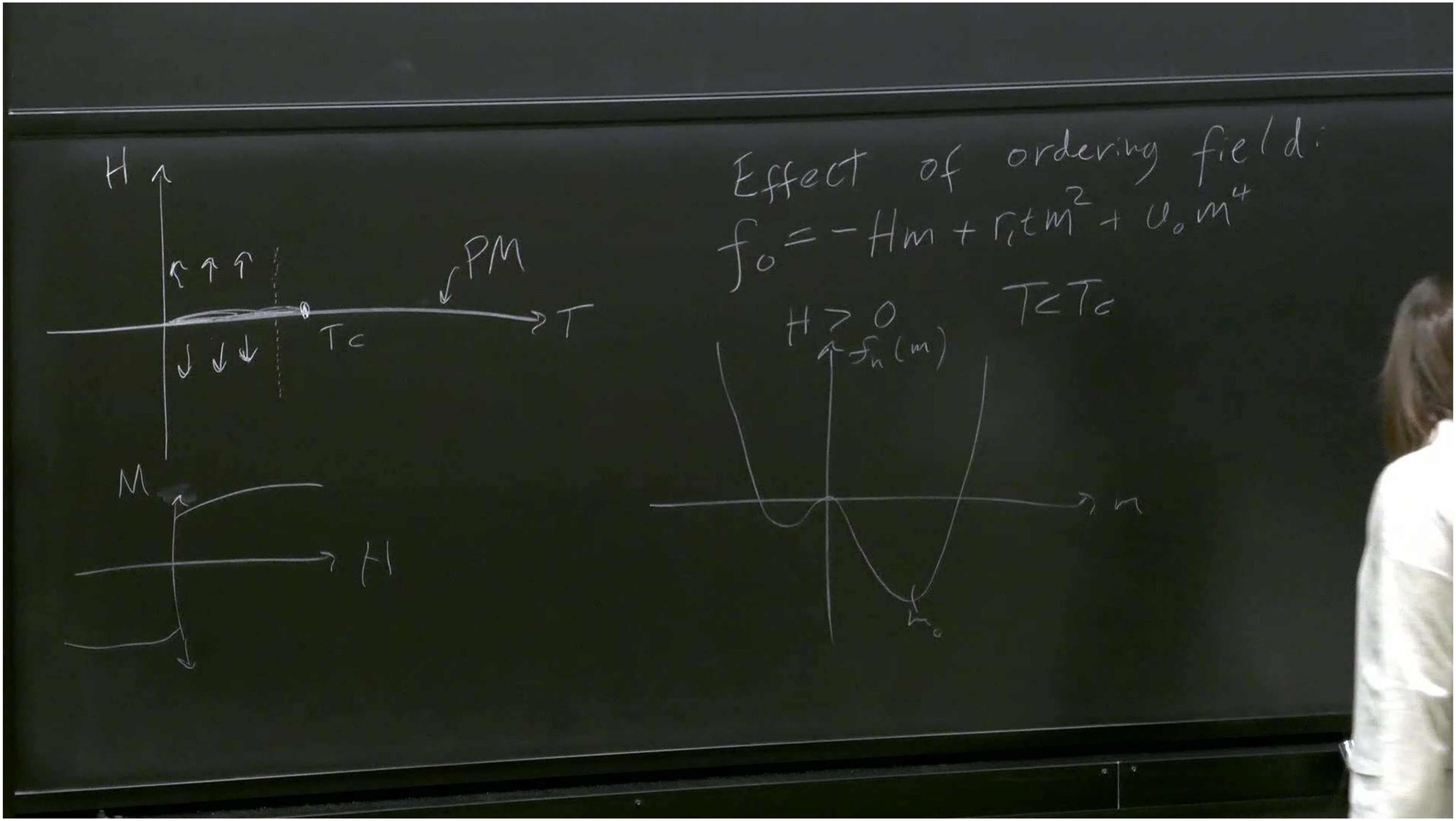
$T > T_c$

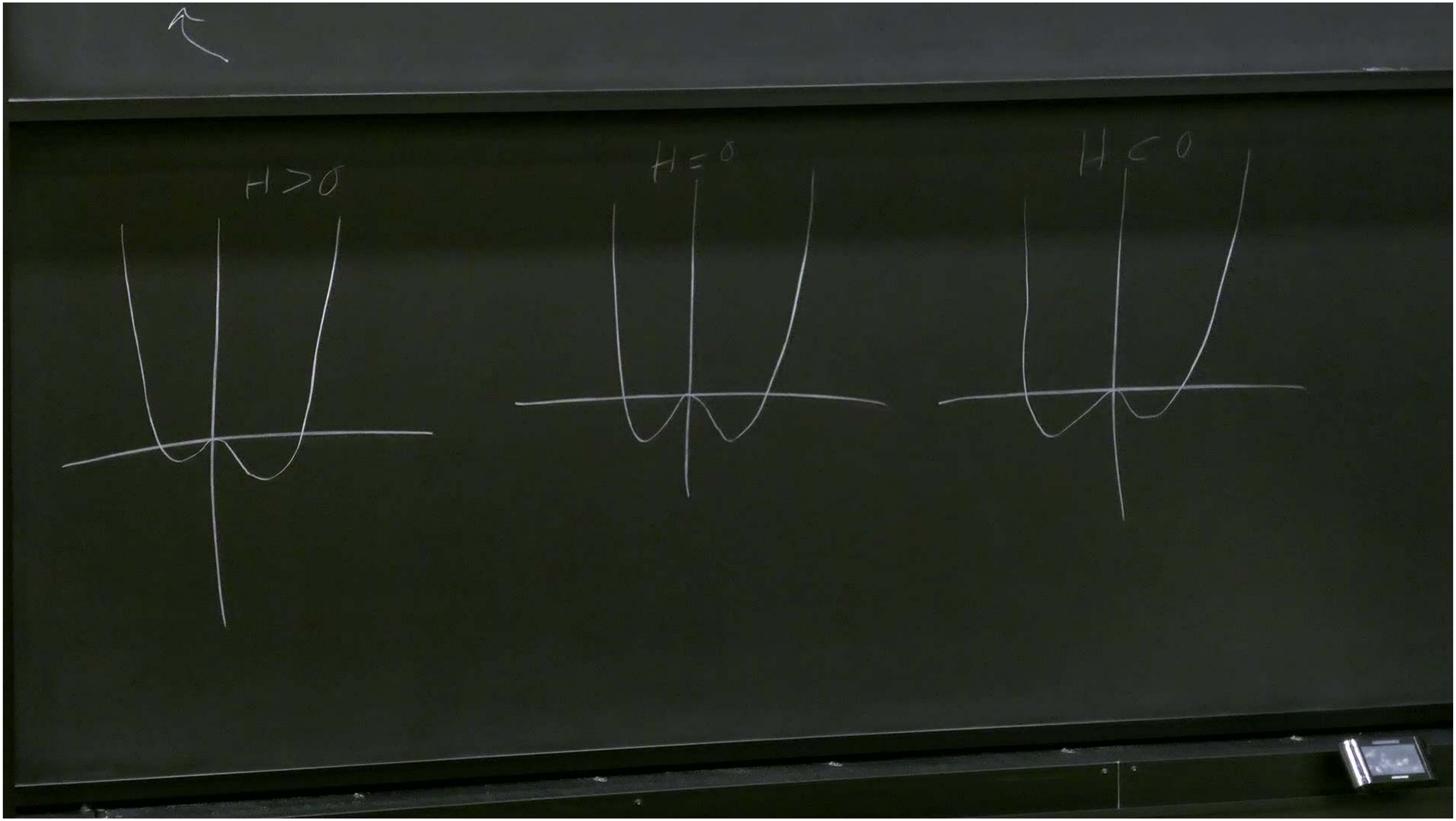
$T < T_c$


H=0
T
T_c
Geogebra Applet in Notes
When $T = T_c$
$f \sim m^4$
$\frac{\partial f}{\partial m} = \frac{\partial^2 f}{\partial m^2} = 0$
$T > T_c$ $\frac{\partial f}{\partial m} = 2rtm + 4m^3 = 0$
$m_0 = 0$
m_0



$m^3 = 0$
$T < T_c$
$2rtm + 4u_0 m^3 = 0$
$m(2rt + 4u_0 m^2) = 0$
$m_0 = \left[\frac{-rt}{u_0 2}\right]^{\frac{1}{2}}$
$-m_0$
m_0


Effect of ordering field:
$f_0 = -Hm + r_1 t m^2 + u_0 m^4$
$H > 0$
$T < T_c$
$f_n(m)$
m
m_0
H
T
T_c
PM
M
H


H > 0
H = 0
H < 0

Critical Exponents:

$$m_0 = \left[\frac{-rt}{u_0 2}\right]^{1/2}, \text{ so } m \sim t^{1/2} \rightarrow \boxed{\beta = \frac{1}{2}}$$

$$\frac{\partial f_h}{\partial m} = -H + 2 r_1 t m + 4 u_0 m^3 = 0$$

At $T = T_c$

$$H = 4 u_0 m^3 \rightarrow m \sim H^{1/3} \rightarrow \boxed{\delta = \frac{1}{3}}$$

Find α: Need $C = \frac{\partial \langle E \rangle}{\partial T}$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z = \frac{\sum_E E e^{-\beta E}}{Z}$$

$$C = \frac{\partial}{\partial T}\left(k T^2 \frac{\partial \log Z}{\partial T}\right)$$

$$F = -kT \log Z$$

$$C = -T \frac{\partial^2 F}{\partial T^2}$$

When $\underline{T > T_c}$, $m_0 = 0$

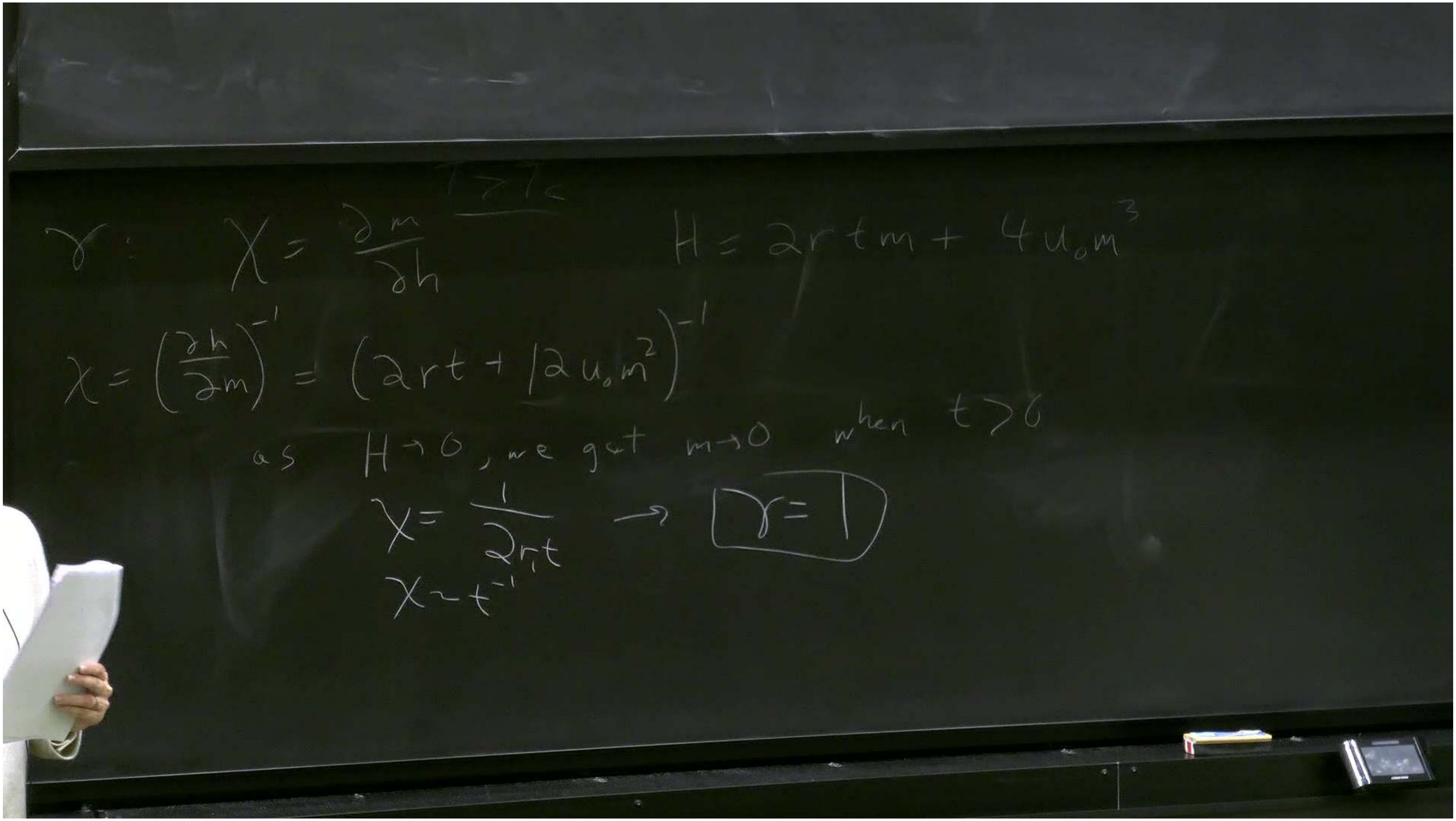
$$f_0 = r t m^2 + u m^4$$

$$C = -T\frac{\partial^2 f}{\partial T^2} = -T(0) = 0 \qquad \boxed{\alpha = 0}$$

$\underline{T < T_c}$

$$f_0 = r\frac{(T - T_c)}{T_c}\,\frac{r(T_c - T)}{u_0 2 T_c} + \text{higher order}$$

$$C = \frac{r_1^2}{u_0 T_c} > 0 \qquad C \sim |t|^0 \rightarrow \boxed{\alpha = 0}$$



$T > T_c$
γ: $\chi = \frac{\partial m}{\partial h}$
$H = 2rtm + 4u_0 m^3$
$\chi = \left(\frac{\partial h}{\partial m}\right)^{-1} = \left(2rt + 12u_0 m^2\right)^{-1}$
as $H \to 0$, we get $m \to 0$ when $t > 0$
$\chi = \frac{1}{2r_1 t} \to \boxed{\gamma = 1}$
$\chi \sim t^{-1}$

$T < T_c$

$m \simeq \sqrt{(r/2u_0)|t|}$

$\boxed{\gamma = 1}$

Ginzburg Criterion

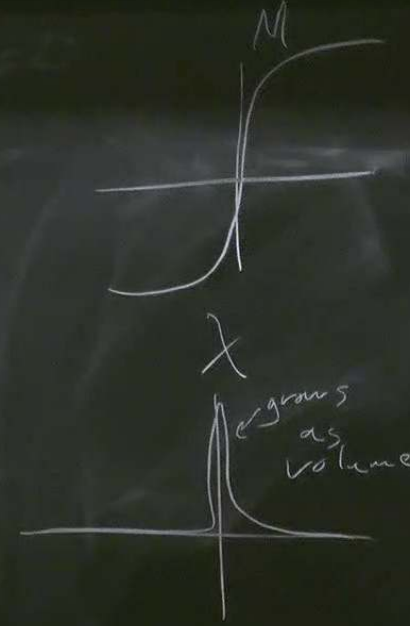
When is MFT valid?

$$\underbrace{(\Delta M)^2_{\xi^d}}_{\text{fluctuations}} \ll \langle M \rangle^2_{\xi^d}$$

$$\left\langle \left(\sum_i (s_i - \langle s_i \rangle) \right)^2 \right\rangle_{\xi^d} \ll \left\langle \left(\sum_i s_i \right) \right\rangle^2_{\xi^d}$$

$$\sum_{i,j} \left(\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right)_{\xi^d} \ll \underbrace{m^2}_{\text{intensive}} \xi^{2d}$$

$$\chi = \frac{1}{\beta} \frac{\partial^2}{\partial H^2} \log Z$$



$$\chi \xi^{d} \ll m^{2} \xi^{2d}$$

$$\hookrightarrow \chi \ll m^{2} \xi^{d}$$

$$t = \frac{T_c - T}{T_c}$$

critical exponents:

$$t^{-\gamma} \ll t^{2\beta} t^{-d\nu}$$

$$1 \ll t^{\underbrace{\gamma + 2\beta - d\nu}_{neg}}$$

$$\gamma + 2\beta - d\nu < 0$$ for MFT to be valid

$\underline{d > 4}$ for Ising MFT

upper critical dimension

$\uparrow$ intensive

$\uparrow\ \chi = \frac{1}{\beta}\frac{\partial^2}{\partial H^2}\log Z$

γ

Scaling in our Landau Theory

From Landau theory we have

$$2rtM + 4u_0M^3 = H \quad (\triangle)$$

$$M \sim |t|^{\frac{1}{2}}$$

Motivated by the powers of M, let's rewrite $(\triangle)$ by dividing by $|t|^{\frac{3}{2}}$:

$$\frac{H}{|t|^{\frac{3}{2}}} = 2r\frac{M}{|t|^{\frac{1}{2}}}\mathrm{sign}(t) + 4u_0\left(\frac{M}{|t|^{\frac{1}{2}}}\right)^3$$

Landau free energy

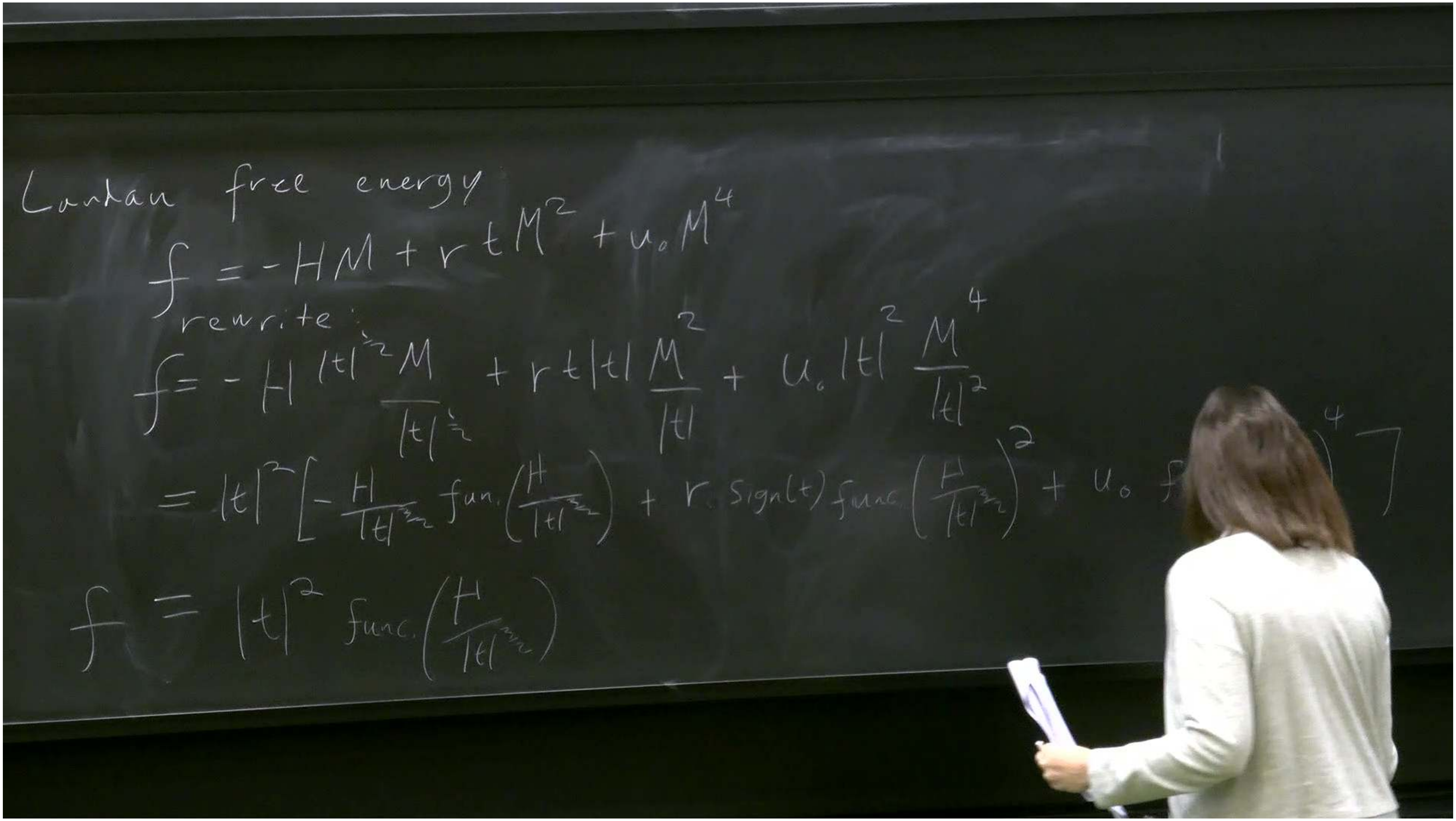
$$f = -HM + r\,t\,M^2 + u_0 M^4$$

rewrite:

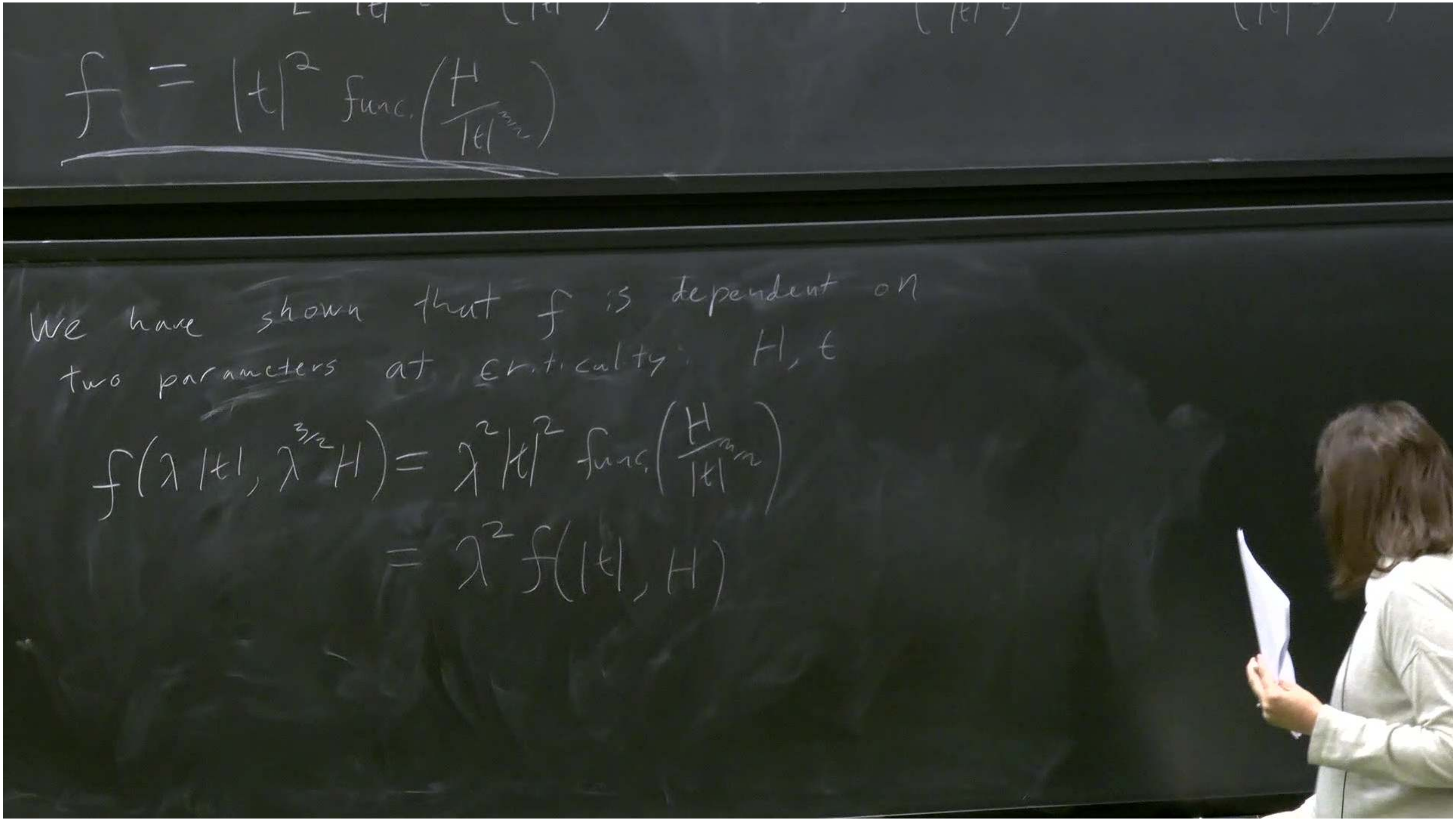
$$f = -H\,|t|^{3/2}\frac{M}{|t|^{1/2}} + r\,t\,|t|\frac{M^2}{|t|} + u_0\,|t|^2\frac{M^4}{|t|^2}$$

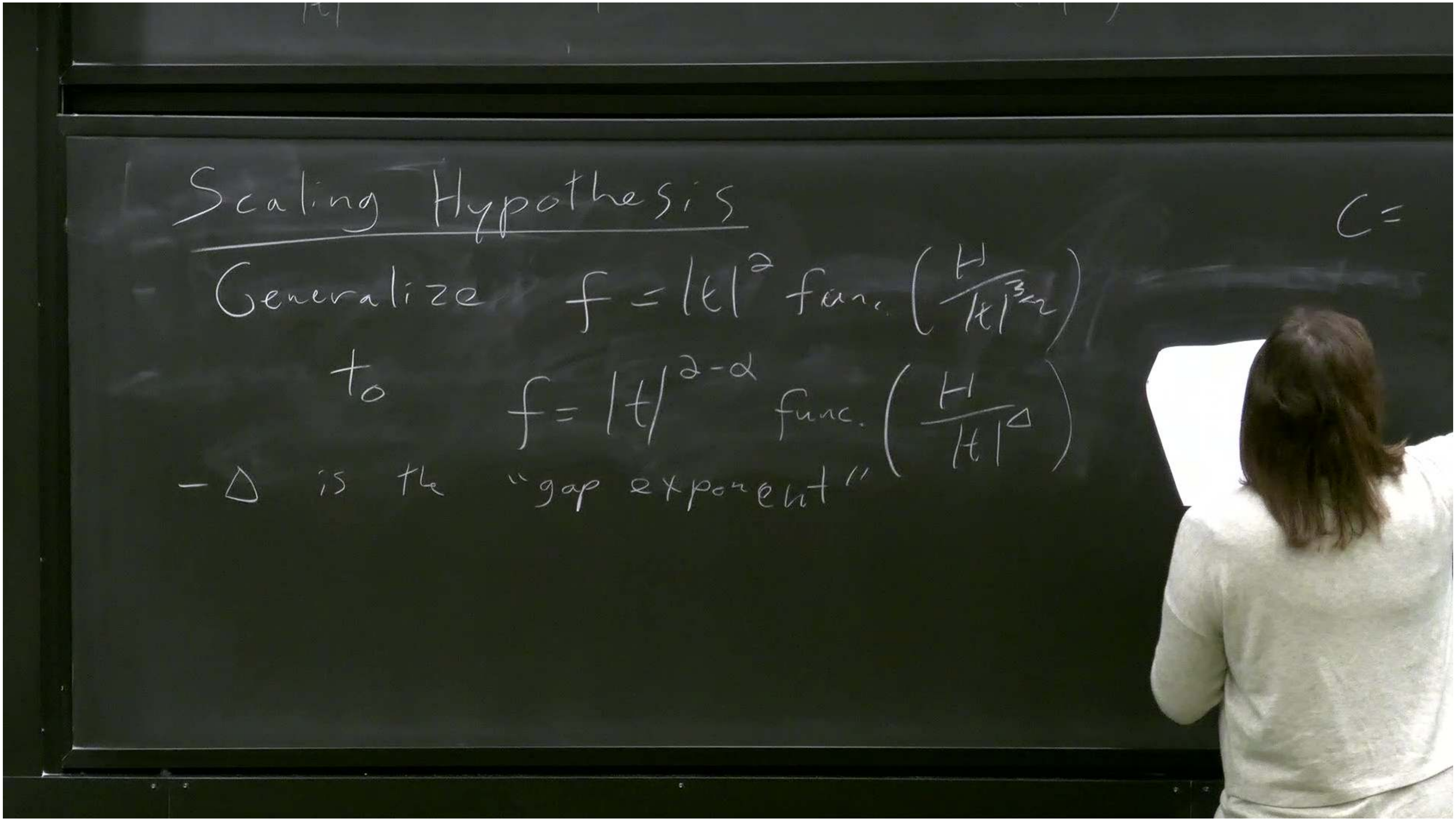
$$= |t|^2\left[-\frac{H}{|t|^{3/2}}\,\text{fun.}\left(\frac{H}{|t|^{3/2}}\right) + r\,\text{sign}(t)\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^2 + u_0\,\text{func.}\left(\frac{H}{|t|^{3/2}}\right)^4\right]$$

$$=$$



Landau free energy:
$f = -HM + rtM^2 + u_0 M^4$
rewrite:
$f = -H|t|^{1/2}\frac{M}{|t|^{1/2}} + rt|t|\frac{M^2}{|t|} + u_0|t|^2\frac{M^4}{|t|^2}$
$= |t|^2\left[-\frac{H}{|t|^{3/2}}\,\mathrm{fun.}\left(\frac{H}{|t|^{3/2}}\right) + r\,\mathrm{sign}(t)\,\mathrm{func.}\left(\frac{H}{|t|^{3/2}}\right)^2 + u_0\,f\ \ )^4\right]$
$f = |t|^2\,\mathrm{func.}\left(\frac{H}{|t|^{3/2}}\right)$


$f = |t|^2 \text{ func.}\left(\frac{H}{|t|^{3/2}}\right)$
We have shown that f is dependent on two parameters at criticality: H, t
$f(\lambda |t|, \lambda^{3/2} H) = \lambda^2 |t|^2 \text{ func.}\left(\frac{H}{|t|^{3/2}}\right)$
$= \lambda^2 f(|t|, H)$


Scaling Hypothesis
Generalize $f = |t|^2 \text{ func.}\left(\frac{H}{|t|^{3/2}}\right)$
to $f = |t|^{2-\alpha} \text{ func.}\left(\frac{H}{|t|^{\Delta}}\right)$
- Δ is the "gap exponent"
$C =$

$$C = |t|^{-\alpha} = -T \frac{\partial^2 f}{\partial T^2}\Big|_{H \to 0}$$

Scaling hypothesis in Pathria

From here:

$$M = -\frac{\partial f}{\partial H} = -|t|^{2-\alpha} \frac{1}{|t|^{\Delta}} \text{func}\left(\frac{H}{|t|^{\Delta}}\right)$$

$$= -|t|^{2-\alpha-\Delta} \text{func}\left(\frac{H}{|t|^{\Delta}}\right)$$

$H \to 0$

$M \sim |t|^{2-\alpha-\Delta}$ $\quad M \sim |t|^{\beta}$ $\quad$ so $\beta = 2 - \alpha - \Delta$

$$f = |t|\ \text{func.}\left(\frac{T}{|t|^{3/2}}\right)$$

We can also get:

Rushbrook: $\alpha + 2\beta + \gamma = 2$

$\delta = \Delta / \beta$

only two idependent exponents

Griffiths: $\alpha + \beta(\delta + 1) = 2$

What about η and ν?

Add a hyperscaling assumption:

$$L^{D} = \xi^{D}$$

Tong, Statistical Field Theory 3.2.1

Fisher: $\gamma = \nu(2-\eta)$

Josephson: $\alpha = 2 - \nu d$

rite