

**Title:** Lecture - Statistical Physics, PHYS 602

**Speakers:** Emilie Huffman

**Collection/Series:** Statistical Physics (Core), PHYS 602, October 7 - November 6, 2024

**Subject:** Condensed Matter, Other

**Date:** October 18, 2024 - 10:45 AM

**URL:** <https://pirsa.org/24100012>

## Landau Theory

- Different approach to studying phase transitions
- Ignore the details of the interactions and produce symmetry-based model

-  $f$  is  
per s  
exactly

transitions  
and

Landau free energy:

$$\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$$

$$Z = \sum_{\sigma} e^{-\beta E(\sigma, H)}$$

organized  
by  
order  
parameter

$$\begin{aligned} &= \sum_m \sum_{\sigma \text{ for } m} e^{-\beta E(\sigma, H)} \\ &\approx \sum_m e^{-\beta N f(m, H)} \end{aligned}$$

-  $f$  is the Landau free energy per site  $\rightarrow$  as  $N \rightarrow \infty$ , becomes exactly equal to term with lowest  $f(m, H)$

# Landau-Ginzburg Theory

- Promotion of Landau-type assumptions to a field theory
- Allows us to model correlations/fluctuations

$$F(m(\vec{r})) = \int d\vec{r} \left\{ [am(\vec{r})^2 + bm(\vec{r})^4] + \alpha |\nabla m(\vec{r})|^2 \right\}$$

↑ functional

$$Z = \int P_m(\vec{r}) e^{-\beta F[m(\vec{r})]}$$

- constant  $m(\vec{r}) \rightarrow$  reduces to Landau Theory

# Application to Ising Model

Taylor

- For Ising, we assume  $f(m, H)$   
↑ scalar

$$f(m, 0) = f(-m, 0)$$

- when  $H$  and  $T_c - T$  are small,  
 $m$  is small

-  $m$  varies smoothly with  $T$



Taylor expansion: 0 WLOG

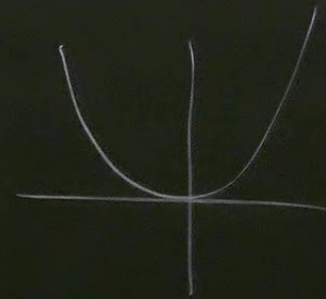
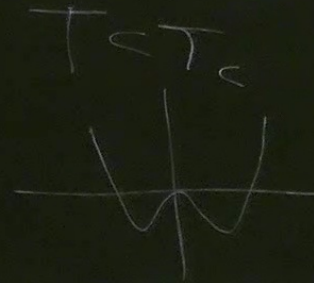
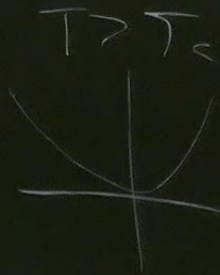
$$f(m, 0) \approx f_0 + r(T)m^2 + u(T)m^4$$

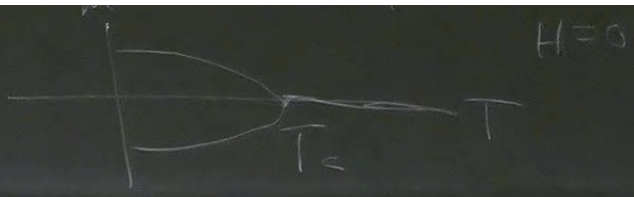
- Near the critical point, we will be minimizing  $f$ , so  $u \approx u_0 > 0$

- Below  $T = T_c$ ,  $r(T)$  should change sign

$$r(T) = r_0 \left( \frac{T - T_c}{T_c} \right)$$

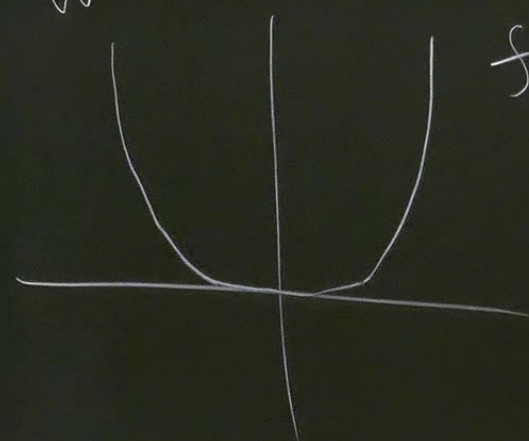
$r > 0$





Geogebra Applet in Notes

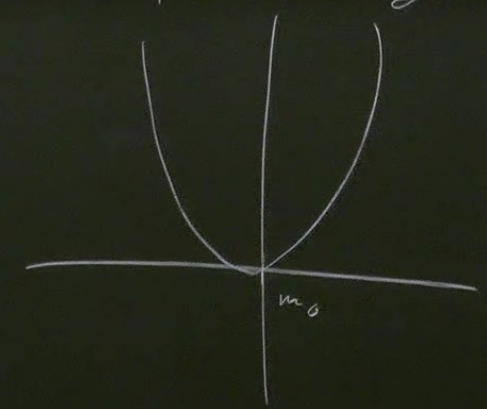
When  $T = T_c$



$f \sim m^4$

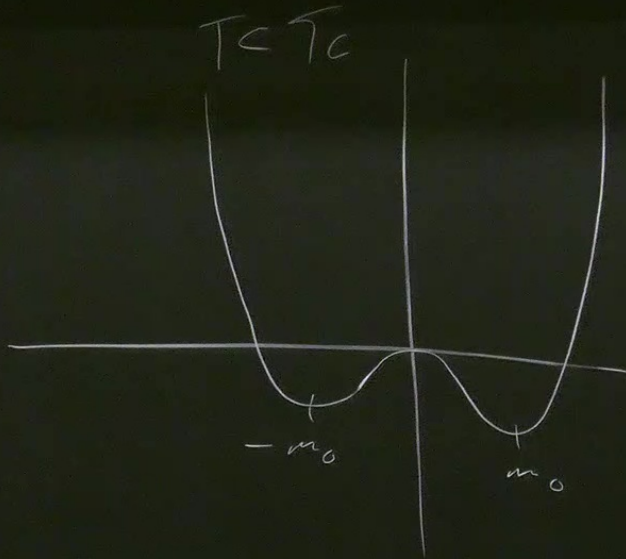
$$\frac{\partial f}{\partial m} = \frac{\partial^2 f}{\partial m^2} = 0$$

$T > T_c$   $\frac{\partial f}{\partial m} = 2r + 4m^3 = 0$



$m_0 = 0$

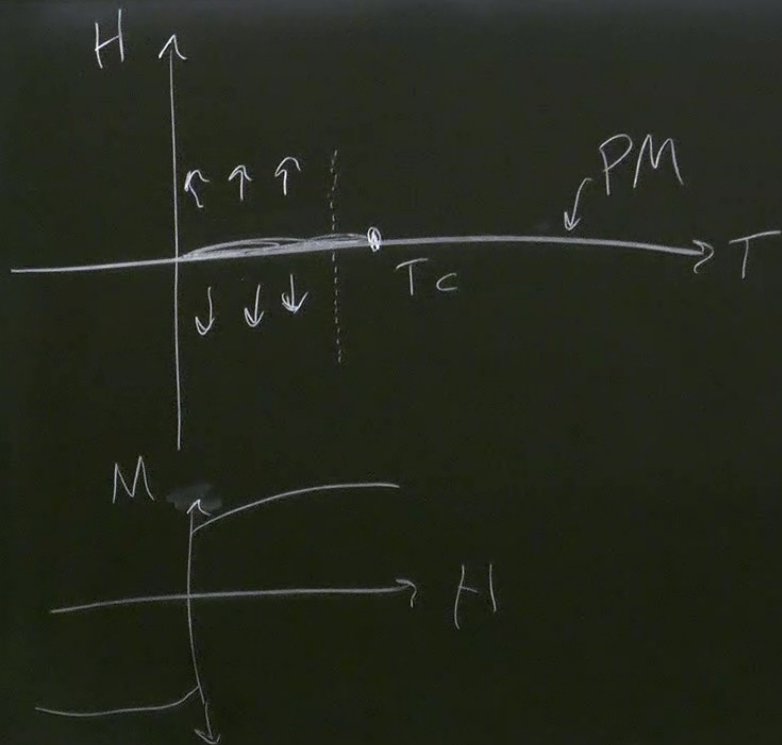
$$m^3 = 0$$



$$2rtm + 4u_0 m^3 = 0$$
$$m(2rt + 4u_0 m^2) = 0$$

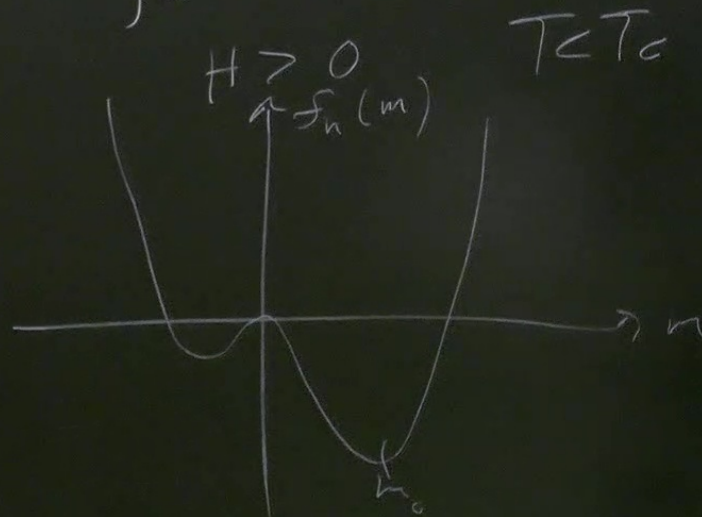
$$m_0 = \left[ \frac{-rt}{4u_0} \right]^{1/2}$$





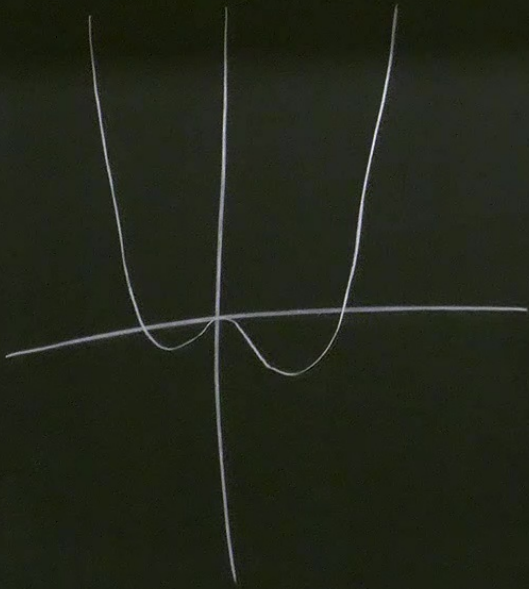
Effect of ordering field:

$$f_0 = -Hm + r_1 m^2 + u_0 m^4$$

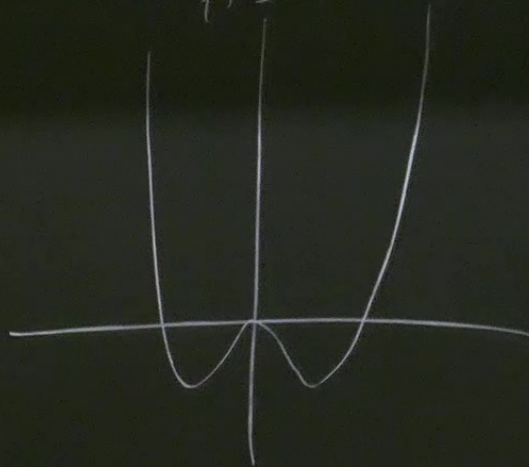




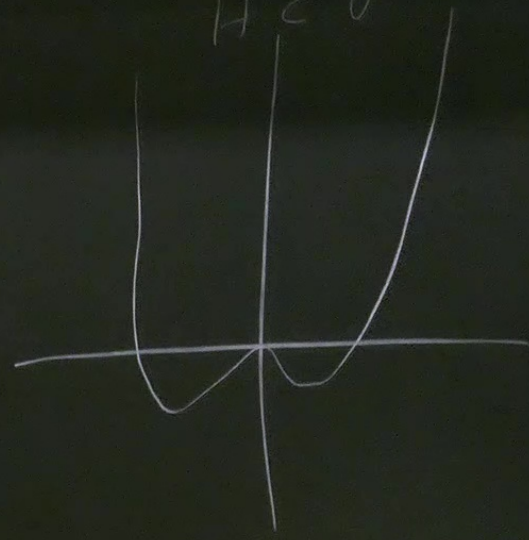
$H > 0$



$H = 0$



$H < 0$



## Critical Exponents:

$$m_0 = \left[ \frac{rt}{u_0^2} \right]^{1/2}, \text{ so } m \sim t^{1/2} \rightarrow \boxed{\beta = \frac{1}{2}}$$

$$\frac{\partial f_h}{\partial m} = -H + 2r_0 t m + 4u_0 m^3 = 0$$

$$A + T = T_c$$

$$H = 4u_0 m^3 \rightarrow m \sim H^{1/3} \rightarrow \boxed{\delta = \frac{1}{3}}$$

Find  $\alpha$ : Need  $C = \frac{\partial \langle E \rangle}{\partial T}$

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{\sum_E E e^{-\beta E}}{Z}$$

$$C = \frac{\partial}{\partial T} \left( kT^2 \frac{\partial \log Z}{\partial T} \right)$$

$$F = -kT \log Z$$

$$C = -T \frac{\partial^2 F}{\partial T^2}$$

When  $T > T_c$ ,  $m_0 = 0$

$$f_0 = rtm^2 + um^4$$

$$C = -T \frac{\partial^2 f}{\partial T^2} = -T(0) = 0$$

$$\boxed{d=0}$$

$$T < T_c$$

$$f_0 = r \frac{(T-T_c)}{T_c} + \frac{r(T_c-T)}{u_0 2 T_c} + \text{higher order}$$

$$C = \frac{r_1^2}{u_0 T_c} > 0$$

$$C \sim |t|^0 \rightarrow \boxed{d=0}$$

$$\gamma = \frac{\partial m}{\partial h} \quad H = 2rtm + 4u_0 m^3$$

$$\chi = \left( \frac{\partial h}{\partial m} \right)^{-1} = (2rt + 12u_0 m^2)^{-1}$$

as  $H \rightarrow 0$ , we get  $m \rightarrow 0$  when  $t > 0$

$$\chi = \frac{1}{2rt} \rightarrow \boxed{\gamma = 1}$$
$$\chi \sim t^{-1}$$

$T < T_c$

$$m \rightarrow \sqrt{(r/2u)h}$$

$$\boxed{\gamma = 1}$$

# Ginzburg Criterion

When is MFT valid?

$$(\Delta M)_{\xi^d}^2 \ll \langle M \rangle_{\xi^d}^2$$

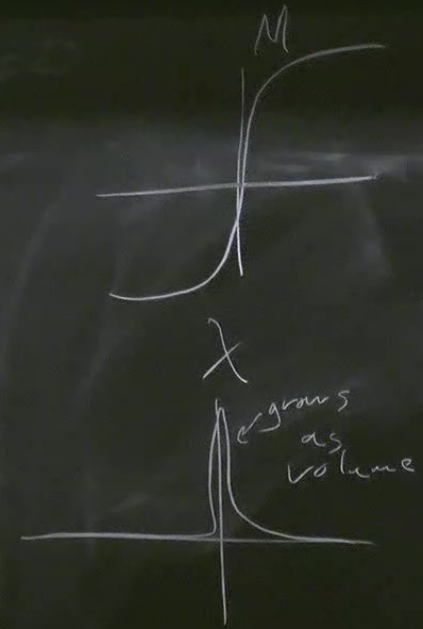
↑  
fluctuations

$$\langle (\sum_i (s_i - \langle s_i \rangle))^2 \rangle_{\xi^d} \ll \langle (\sum_i s_i)^2 \rangle_{\xi^d}$$

$$\sum_{i,j} (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) \ll m^2 \xi^{2d}$$

↑  
intensive

$$\chi = \frac{1}{\beta} \frac{\partial^2}{\partial H^2} \log Z$$





$$\chi \xi^d \ll m^2 \xi^{2d}$$

$$\hookrightarrow \chi \ll m^2 \xi^d$$

$$t = \frac{T_c - T}{T_c}$$

critical exponents:

$$t^{-\gamma} \ll t^{2\beta} t^{-d\nu}$$

$$1 \ll t^{\underbrace{\gamma + 2\beta - d\nu}_{\text{neg}}}$$

$$\gamma + 2\beta - d\nu < 0$$

for MFT  
to be valid

$d > 4$  for Ising MFT

upper critical dimension

$$\chi = \frac{1}{\beta} \int \frac{d^2z}{dH^2} \log Z$$

↑ intensive

## Scaling in our Landau Theory

From Landau theory we have

$$2rtM + 4u_0M^3 = H \quad (*)$$

$$M \sim |t|^{1/2}$$

Motivated by the powers of  $M$ , let's rewrite (\*) by dividing by  $|t|^{3/2}$ :

$$\frac{H}{|t|^{3/2}} = 2r \frac{M}{|t|^{1/2}} \text{sign}(t) + 4u_0 \left( \frac{M}{|t|^{1/2}} \right)^3$$

Landau free energy

$$f = -HM + r t M^2 + u_0 M^4$$

rewrite:

$$f = -H \frac{|t|^{\frac{1}{2}} M}{|t|^{\frac{1}{2}}} + r t \frac{|t|^{\frac{1}{2}} M^2}{|t|^{\frac{1}{2}}} + u_0 |t|^{\frac{1}{2}} \frac{M^4}{|t|^{\frac{1}{2}}}$$

$$= |t|^{\frac{1}{2}} \left[ -\frac{H}{|t|^{\frac{1}{2}}} \text{func.} \left( \frac{H}{|t|^{\frac{1}{2}}} \right) + r \text{sign}(t) \text{func.} \left( \frac{H}{|t|^{\frac{1}{2}}} \right)^2 + u_0 \text{func.} \left( \frac{H}{|t|^{\frac{1}{2}}} \right)^4 \right]$$

=

Landau free energy

$$f = -HM + r t M^2 + u_0 M^4$$

rewrite:

$$f = -H \frac{|t|^{3/2} M}{|t|^{3/2}} + r t \frac{|t| M^2}{|t|} + u_0 |t|^2 \frac{M^4}{|t|^2}$$

$$= |t|^2 \left[ -\frac{H}{|t|^{3/2}} \text{func.} \left( \frac{H}{|t|^{3/2}} \right) + r \text{sign}(t) \text{func.} \left( \frac{H}{|t|^{3/2}} \right)^2 + u_0 \text{func.} \left( \frac{H}{|t|^{3/2}} \right)^4 \right]$$

$$f = |t|^2 \text{func.} \left( \frac{H}{|t|^{3/2}} \right)$$

$$\underline{f = |t|^2 \text{func.}\left(\frac{H}{|t|^{3/2}}\right)}$$

We have shown that  $f$  is dependent on two parameters at criticality:  $H, t$

$$\begin{aligned} f(\lambda |t|, \lambda^{3/2} H) &= \lambda^2 |t|^2 \text{func.}\left(\frac{H}{|t|^{3/2}}\right) \\ &= \lambda^2 f(|t|, H) \end{aligned}$$

## Scaling Hypothesis

Generalize  $f = |t|^\alpha \text{func.} \left( \frac{H}{|t|^{3/2}} \right)$

to  $f = |t|^{2-\Delta} \text{func.} \left( \frac{H}{|t|^\Delta} \right)$

$-\Delta$  is the "gap exponent"

$C =$

$$C = |t|^{-\alpha} = -T \frac{\partial^2 f}{\partial T^2} \Big|_{H \rightarrow 0}$$

Scaling hypothesis  
in Pathria

From here:

$$M = - \frac{\partial f}{\partial H} = - |t|^{2-\alpha} \frac{1}{|t|^\Delta} \text{func.} \left( \frac{H}{|t|^\Delta} \right)$$

$$= - |t|^{2-\alpha-\Delta} \text{func.} \left( \frac{H}{|t|^\Delta} \right)$$

$H \rightarrow 0$

$$M \sim |t|^{2-\alpha-\Delta}$$

$$M \sim |t|^\beta$$

$$\text{so } \beta = 2 - \alpha - \Delta$$

$f = |t| \text{ func. } \left( \frac{T}{|t|^{2+\nu}} \right)$

We can also get

Rushbrook:  $\alpha + 2\beta + \gamma = 2$

$$\gamma = \Delta / \beta$$

Griffiths:  $\alpha + \beta(\gamma + 1) = 2$

→ only two independent exponents



$T = 0$  valid

What about  $\eta$  and  $\nu$ ?  
Add a hyperscaling assumption:

$$L^p = \xi^p$$

Tong, Statistical  
Field Theory  
3.2.1

Fisher:  $\gamma = \nu(2 - \eta)$

Josephson:  $d = 2 - \nu d$

vite