Title: Lecture - Statistical Physics, PHYS 602

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Subject: Condensed Matter, Other

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Today: Mean Field Theory

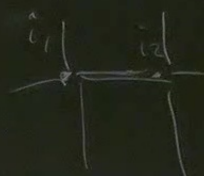
- We'll take an approach that leads to qualitatively correct resulst in systems $> 1D$

- Now from

- We rewrite the model as:

$$E(\sigma) = -\frac{J}{2}\sum_{i}\sum_{j\in n.n.(i)}\sigma_i\sigma_j - H\sum_{i}\sigma_i$$

2D



4 nearest neighbors j

to qualitatively

D

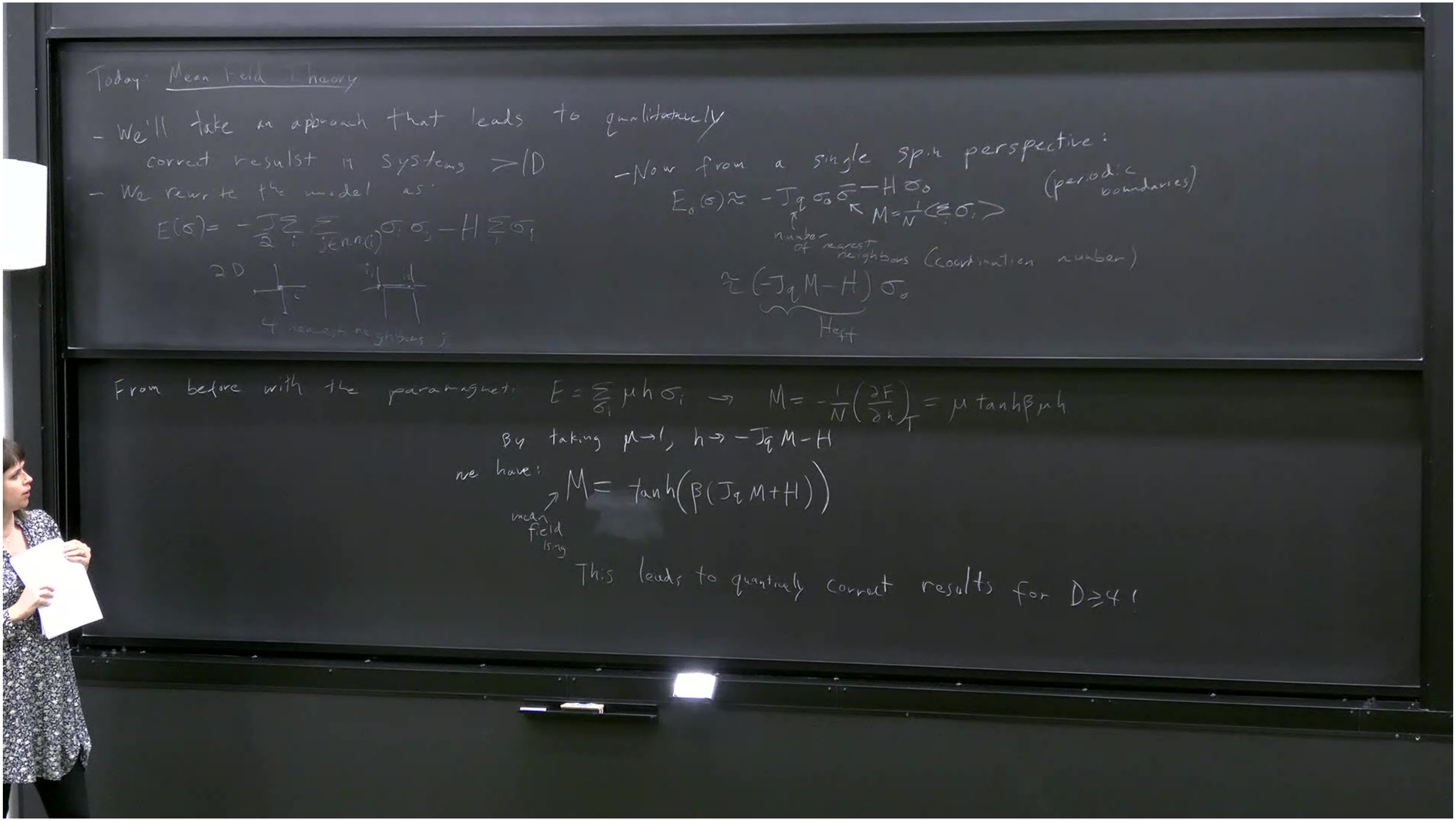
– Now from a single spin perspective:

$$E_0(\sigma) \approx -Jq\,\sigma_0\,\bar{\sigma} - H\sigma_0 \qquad \text{(periodic boundaries)}$$

$$M = \frac{1}{N}\langle \textstyle\sum_i \sigma_i \rangle$$

number of nearest neighbors (coordination number)

$$\approx \underbrace{(-Jq\,M - H)}_{H_{eff}}\,\sigma_0$$


Today: Mean Field Theory
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2D
4 nearest neighbors j
- Now from a single spin perspective:
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(periodic boundaries)
number of nearest neighbors (coordination number)
$\approx \underbrace{(-J_q M - H)}_{H_{eff}} \sigma_0$
From before with the paramagnet: $E = \sum_{\sigma_i} \mu h \sigma_i \rightarrow M = -\frac{1}{N}\left(\frac{\partial F}{\partial h}\right)_T = \mu \tanh \beta \mu h$
By taking $\mu \rightarrow 1$, $h \rightarrow -J_q M - H$
we have: $M = \tanh(\beta(J_q M + H))$
mean field Ising
This leads to quantively correct results for $D \geq 4$!

$$\underbrace{(J_q M + H)}_{H_{eff}} \sigma_i$$

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...have:

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mean field Ising

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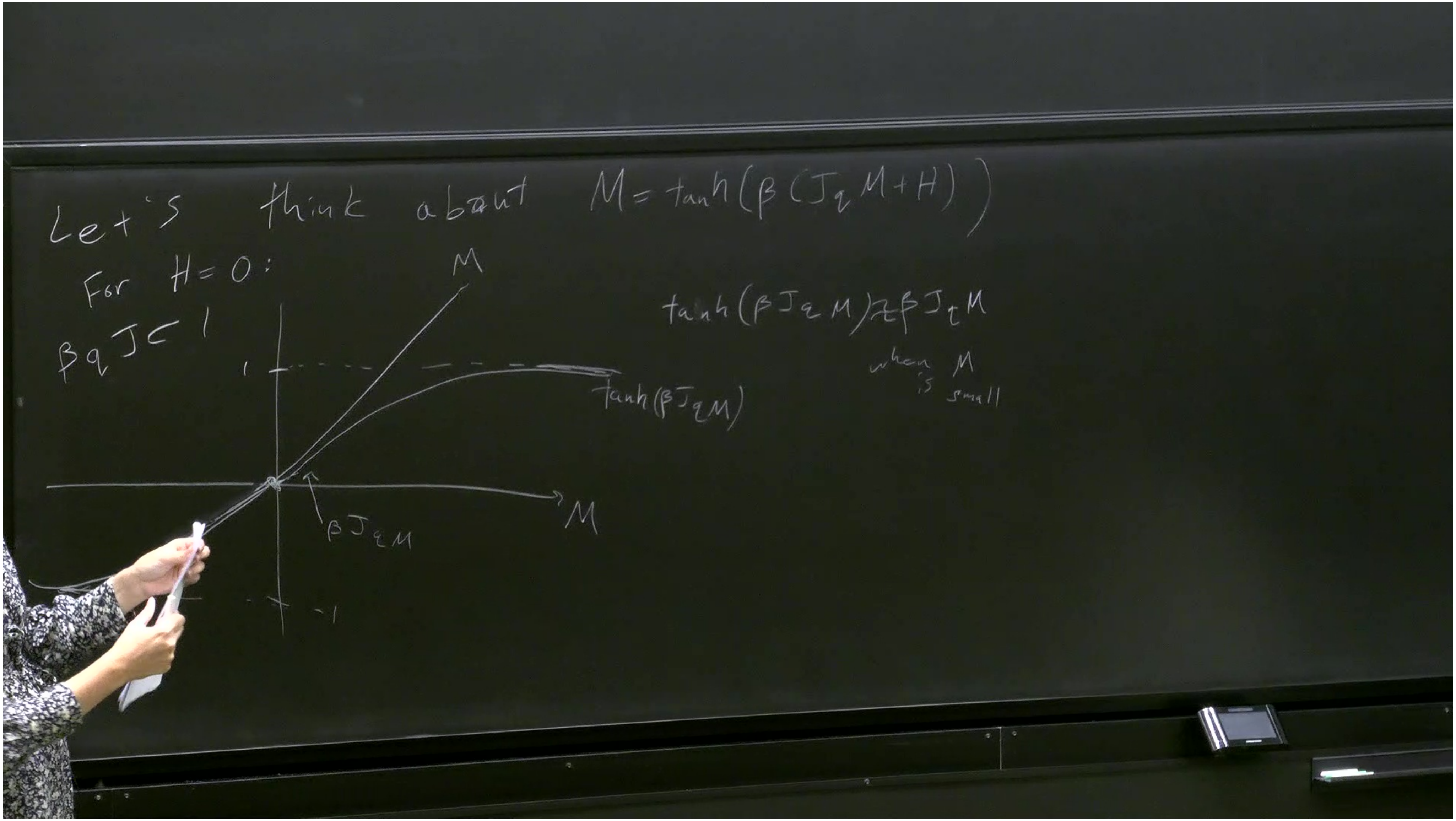
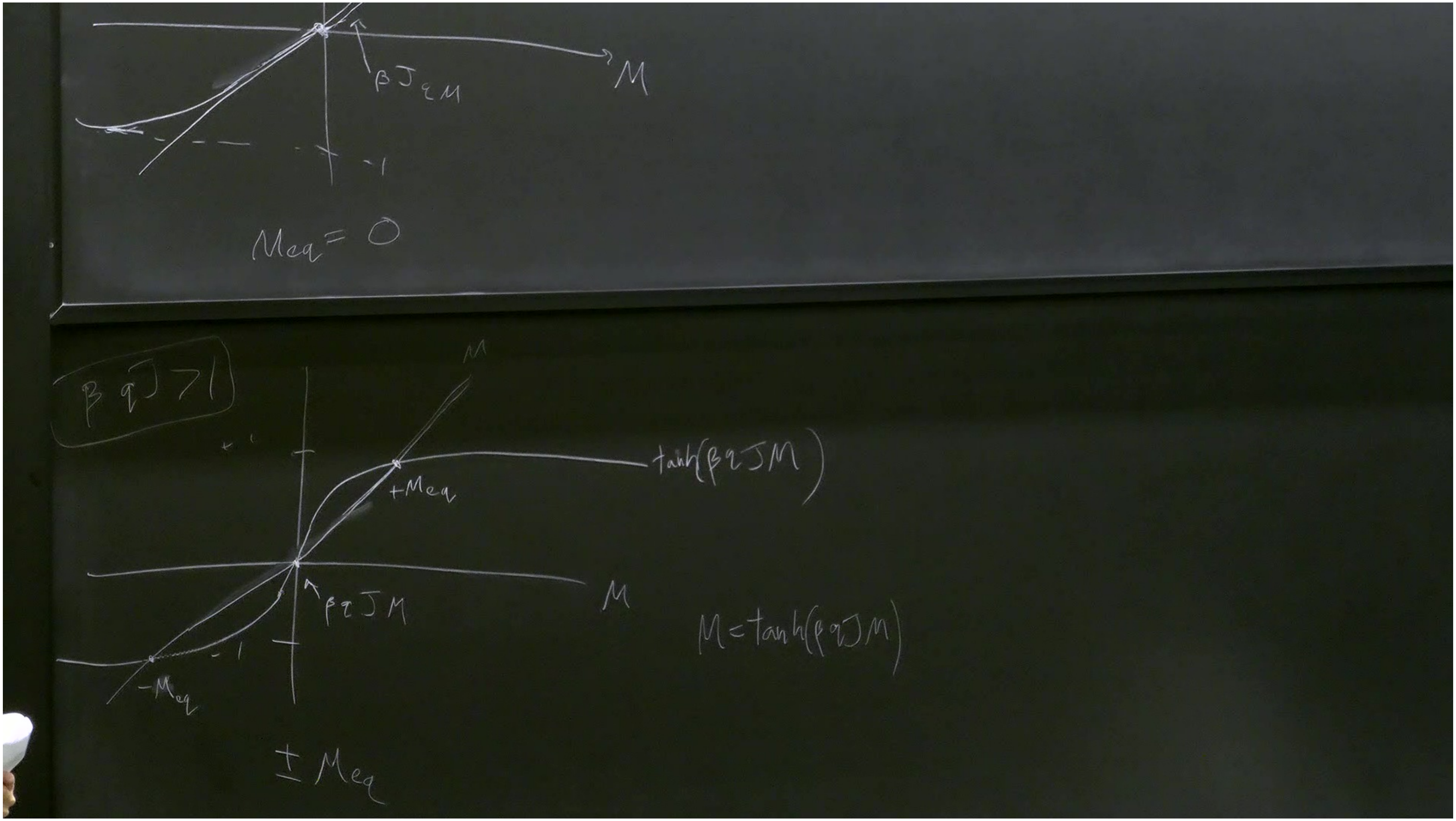
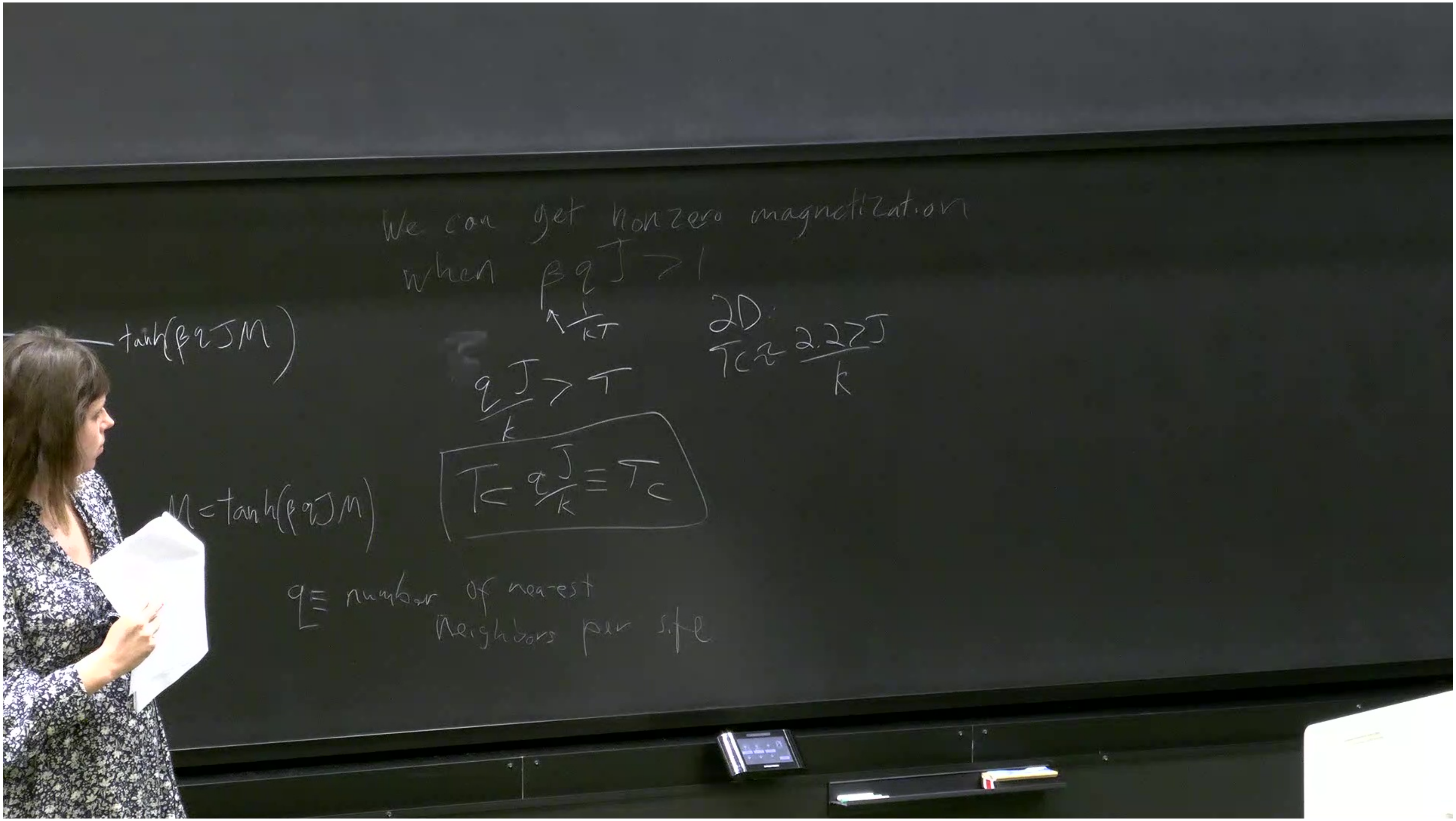

Let's think about $M = \tanh(\beta(J_q M + H))$
For $H = 0$:
$\beta q J < 1$
M
1
$\tanh(\beta J_q M)$
$\beta J_q M$
M
-1
$\tanh(\beta J_q M) \approx \beta J_q M$
when M is small


βJqM
M
-1
Meq = 0
βqJ > 1
M
tanh(βqJM)
+Meq
βqJM
M
M = tanh(βqJM)
-1
−Meq
± Meq


We can get nonzero magnetization
when $\beta q J > 1$
$\frac{1}{kT}$
$\tanh(\beta q J M)$
$\frac{qJ}{k} > T$
$T < \frac{qJ}{k} \equiv T_c$
2D:
$T_c \approx \frac{2.27J}{k}$
$M = \tanh(\beta q J M)$
$q \equiv$ number of nearest
neighbors per site

Critical Exponents:

$$\beta(Jq\,M + H) = \tanh^{-1} M$$

$$\left(\frac{T_c M}{T} + \frac{H}{kT}\right) = \tanh^{-1} M$$

At T_c we get $\left(\tanh^{-1} M \approx M + \frac{1}{3} M^3\right)$

$$M + \frac{H}{kT_c} \approx M + \frac{1}{3} M^3$$

$$H \approx \frac{kT_c}{3} M^3 \longrightarrow \boxed{\delta = 3}$$

get β: we want how does M go with T?

$(H \to 0)$

$$M \approx \beta J q M - \beta^3 J^3 q^3 M^3 \frac{1}{3}$$

$$M \approx \frac{T_c}{T} M - \frac{T_c^3}{T^3} M^3 \frac{1}{3}$$

$$\frac{1}{3} \frac{T_c^3}{T^3} M^3 = \left(\frac{T_c}{T} - 1\right) M$$

$$M \approx \frac{T}{T_c} \sqrt{3\left(1 - \frac{T}{T_c}\right)} \approx \sqrt{\frac{3}{T_c}(T_c - T)} \quad (\star)$$

$T \approx T_c$

$$\boxed{\beta = \frac{1}{2}}$$

$$\chi = \frac{\partial M}{\partial H} = \frac{1}{k} \frac{1 - M^2}{T - T_c(1 - M^2)}$$

$M \to 0$

$$\chi \sim \begin{cases} \frac{1}{k}(T - T_c)^{-1}, & T \geq T_c \\ \frac{1}{2k}(T_c - T)^{-1}, & T < T_c \end{cases}$$

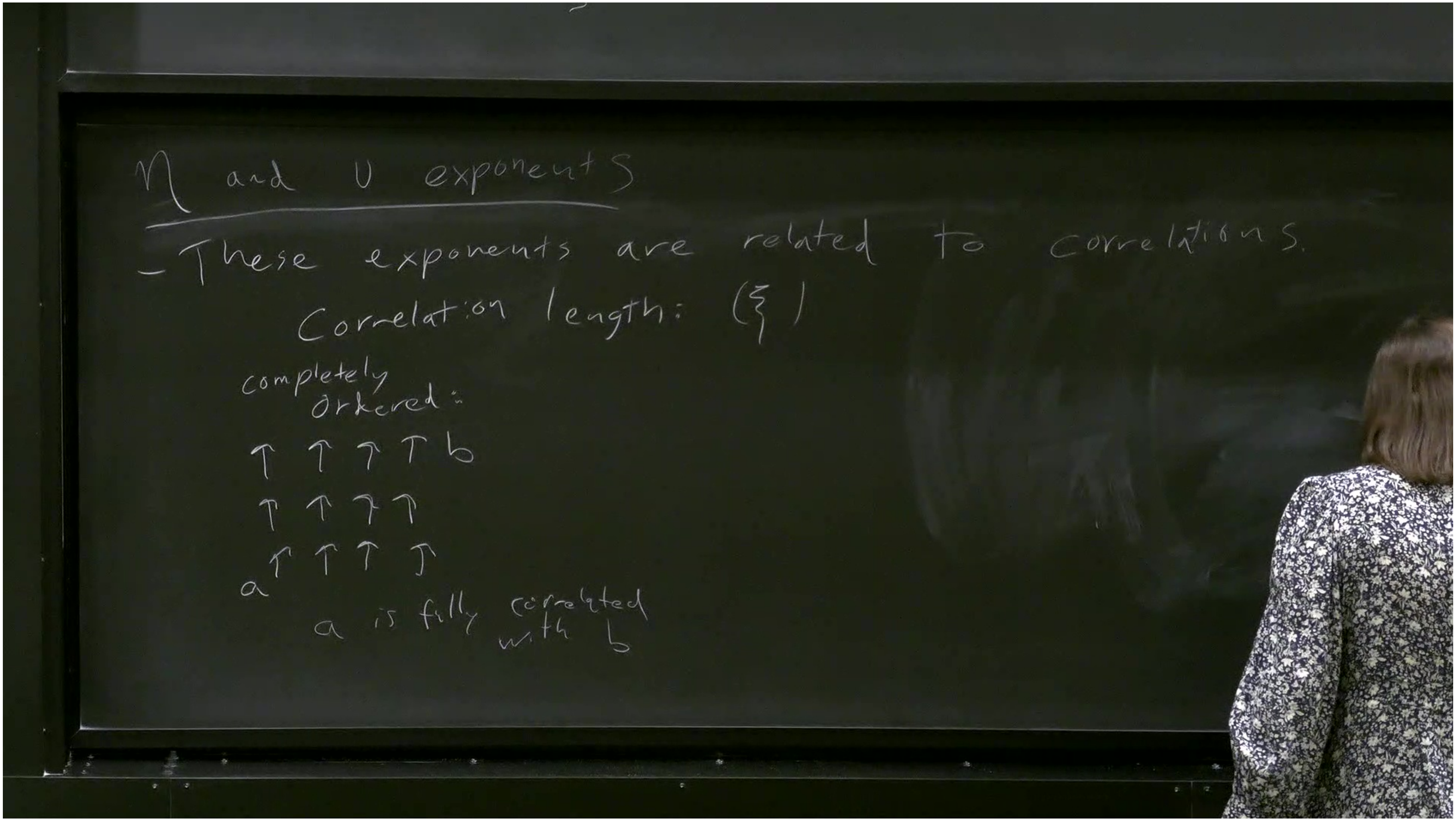
$$\boxed{\gamma = 1}$$

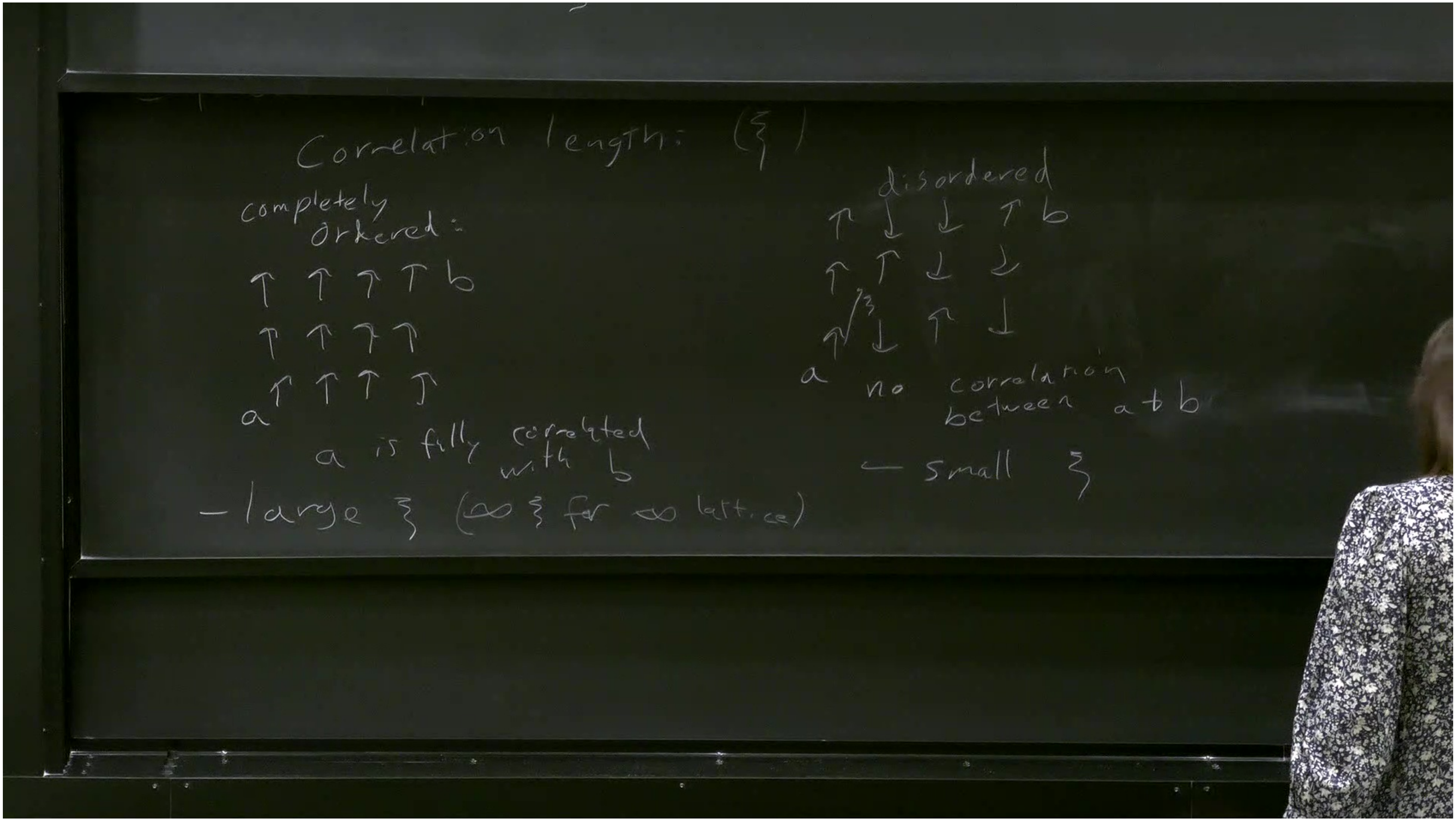
use $\star$

Pathria "Ising Model in zeroth approximation"

$\mp M_{eq}$ neighbors per site

- MFT is qualitatively correct in 2D & 3D
- quantitatively correct when $D \geq 4$
- wrong for 1D


η and ν exponents
- These exponents are related to correlations.
Correlation length: (ξ)
completely ordered:
b
a
a is fully correlated with b


Correlation length: (ξ)
completely ordered:
b
a
a is fully correlated with b
— large ξ (∞ ξ for ∞ lattice)
disordered
b
ξ
a
no correlation between a + b
— small ξ

$\beta = \frac{1}{2}$

5. Measure with correlation functions:

$$g_{ij} = \langle \sigma_i \sigma_j \rangle$$

For a continuous phase transition,

$$g(r) \sim |r|^{-\tau} e^{-|r|/\xi}$$

↑ distance between i and j

At $T = T_c$, correlation length becomes infinite system

$$\boxed{\xi(T) \sim |t|^{-\nu}}$$

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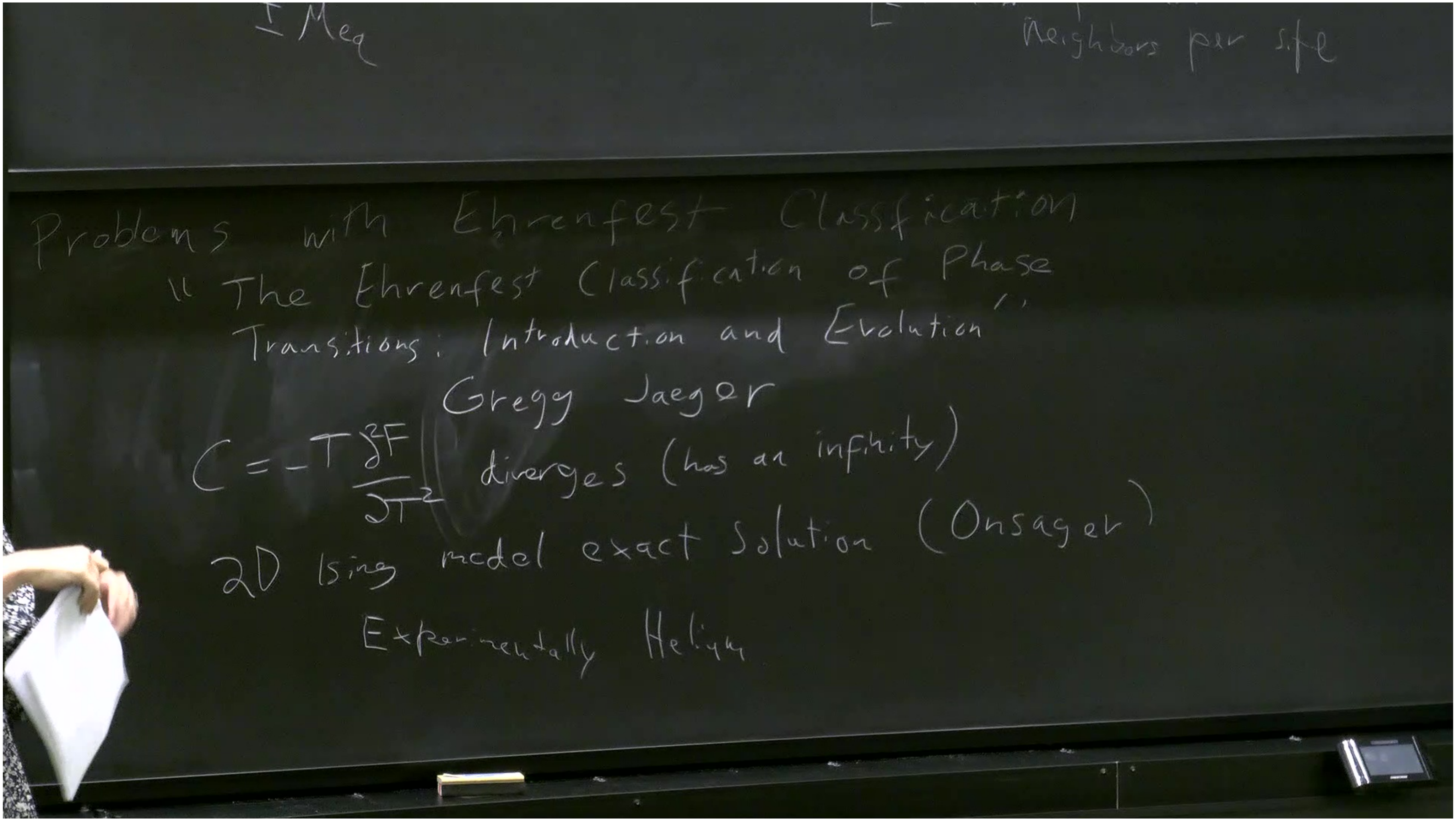
At $T = T_c$,

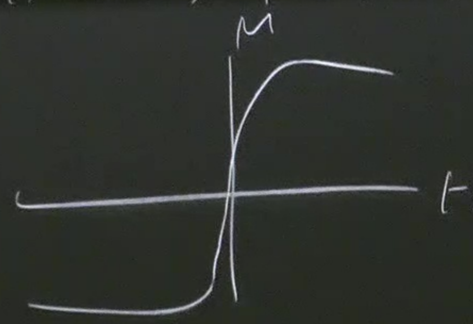
$$\boxed{g(r) \sim |r|^{2-D-\eta}}$$ ← scale invariance


neighbors per site
Problems with Ehrenfest Classification
"The Ehrenfest Classification of Phase Transitions: Introduction and Evolution"
Gregg Jaeger
$C = -T \frac{\partial^2 F}{\partial T^2}$ diverges (has an infinity)
2D Ising model exact Solution (Onsager)
Experimentally Helium

Experimentally Helium

"Modern" classification of continuous Phase Transitions

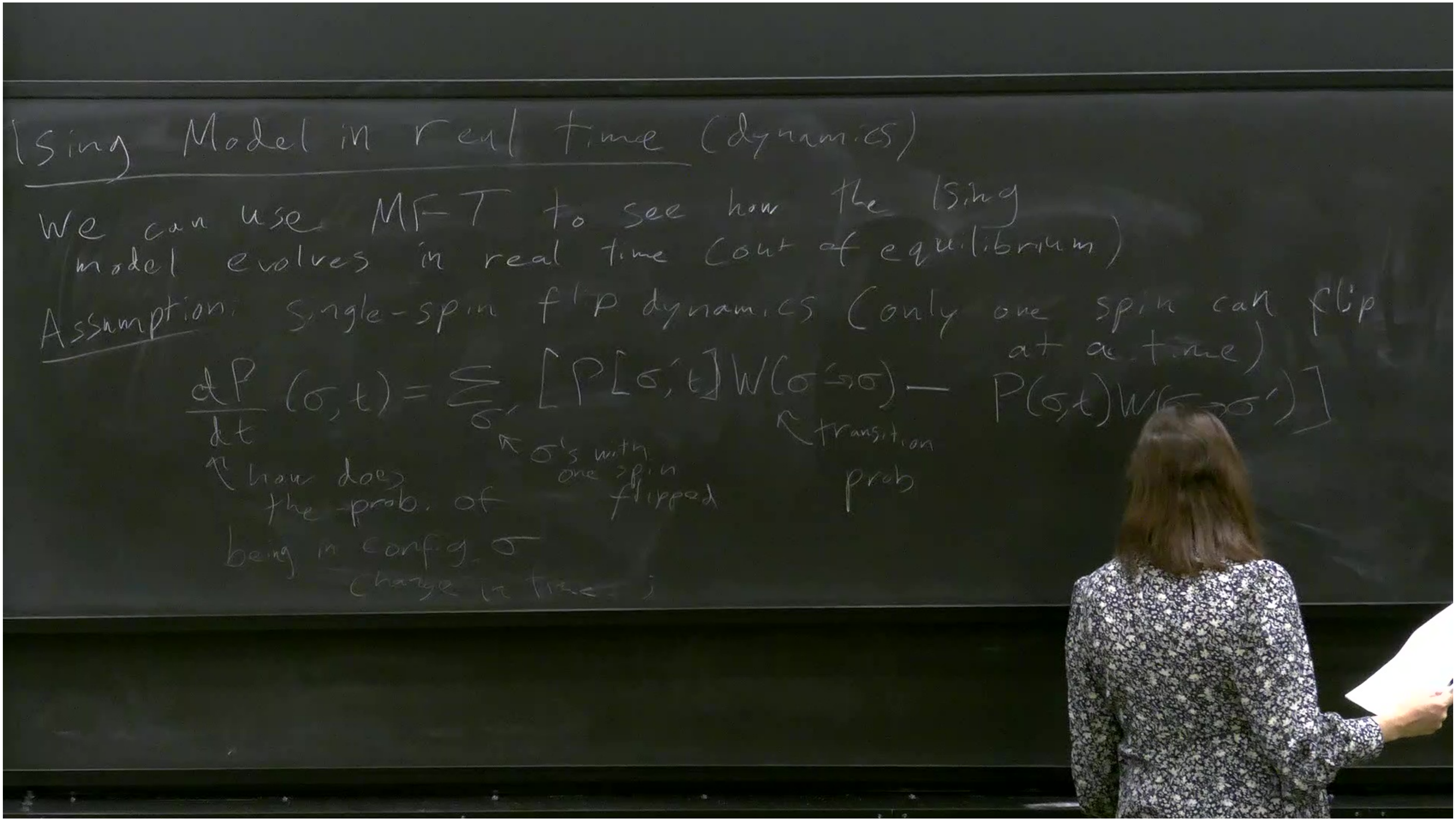
- divergent susceptibility $\frac{\partial M}{\partial H}$



- infinite correlation length

$$\xi \sim |t|^{-\nu}$$

- power law behavior in correlations at critical point: $g(r) \sim |r|^{2-D-\eta}$


Ising Model in real time (dynamics)
We can use MFT to see how the Ising model evolves in real time (out of equilibrium)
Assumption: single-spin flip dynamics (only one spin can flip at a time)
$\frac{dP}{dt}(\sigma,t) = \sum_{\sigma'} [P[\sigma',t] W(\sigma'\to\sigma) - P(\sigma,t) W(\sigma\to\sigma')]$
how does the prob. of being in config σ change in time?
σ's with one spin flipped
transition prob

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how does the prob. of being in config. σ change in time

σ's with one spin flipped

transition prob

n)

spin can flip
a time)

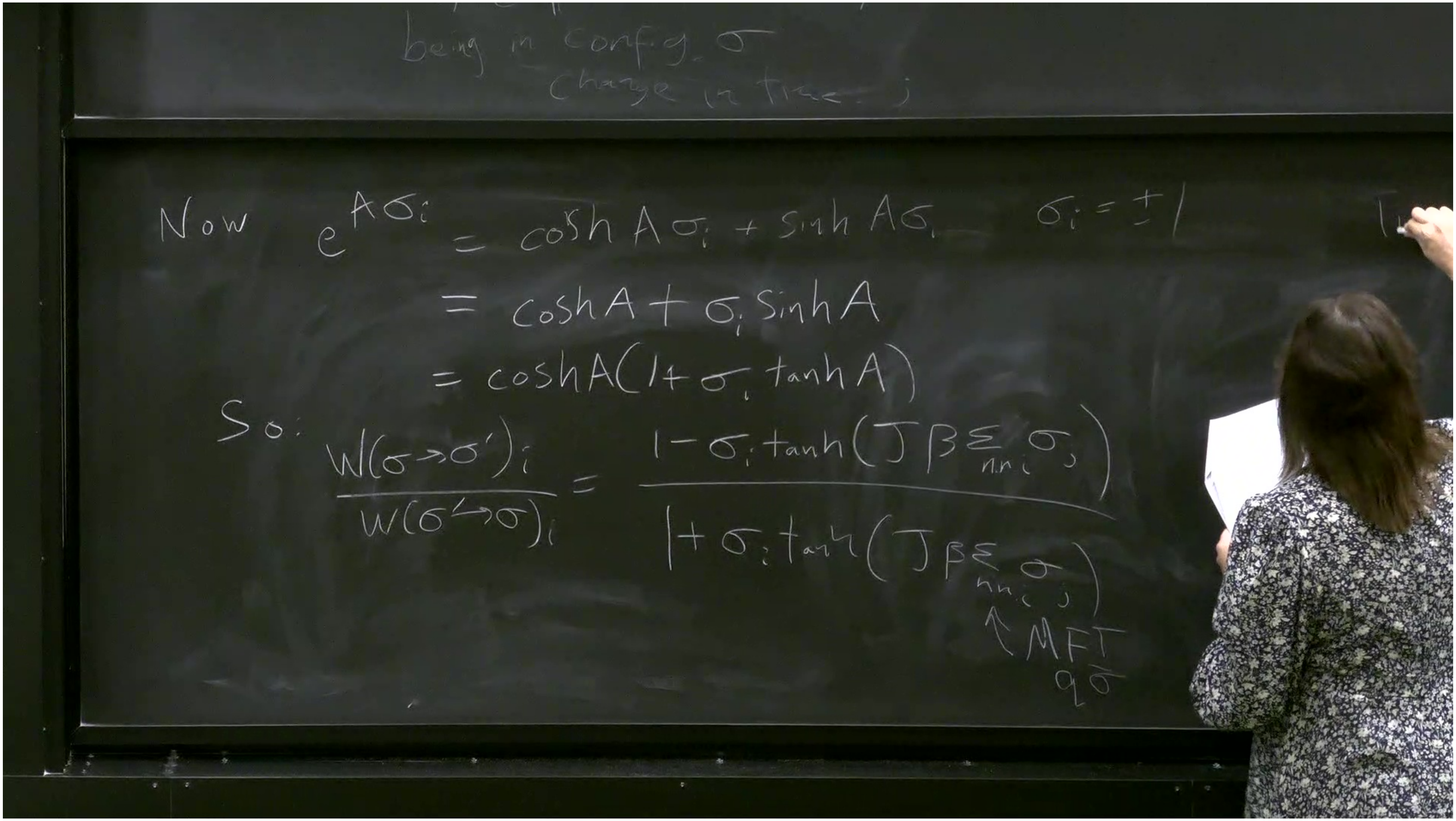
$t)W(\sigma\to\sigma')\Big]\to 0$

Equilibrium: $\frac{dP}{dt}=0$

Condition on transition probabilities to reach equilibrium called

Detailed balance:

$$\frac{W(\sigma\to\sigma')_i}{W(\sigma'\to\sigma)_i}=\frac{P(\sigma')_i}{P(\sigma)_i}=\frac{e^{-\beta J\sum_{j \text{ nn of } i}\sigma_i\sigma_j}}{e^{\beta J\sum_{j \text{ nn of } i}\sigma_i\sigma_j}}$$


being in config. σ
change in time
Now $e^{A\sigma_i} = \cosh A\sigma_i + \sinh A\sigma_i$ $\quad \sigma_i = \pm 1$
$= \cosh A + \sigma_i \sinh A$
$= \cosh A(1+\sigma_i \tanh A)$
So: $\dfrac{W(\sigma\to\sigma')_i}{W(\sigma'\to\sigma)_i} = \dfrac{1-\sigma_i \tanh(J\beta\sum_{n.n.i}\sigma_j)}{1+\sigma_i \tanh(J\beta\sum_{n.n.i}\sigma_j)}$
MFT
$q\bar{\sigma}$

being in config. σ

change in time ;

Now $e^{A\sigma_i} = \cosh A\sigma_i + \sinh A\sigma_i$ $\quad \sigma_i = \pm 1$ $\quad$ In

$$= \cosh A + \sigma_i \sinh A$$

$$= \cosh A (1 + \sigma_i \tanh A)$$

So:

$$\frac{W(\sigma \to \sigma')_i}{W(\sigma' \to \sigma)_i} = \frac{1 - \sigma_i \tanh\left(J\beta \sum_{n.n.i} \sigma_j\right)}{1 + \sigma_i \tanh\left(J\beta \sum_{n.n.i} \sigma_j\right)}$$

MFT

$q\bar{\sigma}$

$\sigma_i = \pm 1$

In MFT

$$\frac{W(\sigma\to\sigma')_i}{W(\sigma'\to\sigma)_i} \approx \frac{1-\sigma_i \tanh(J\beta q M)}{1+\sigma_i \tanh(J\beta q M)}$$

To normalize as probabilities:

$$W(\sigma\to\sigma')_i = \frac{1}{2}\left[1-\sigma_i \tanh(J\beta q M)\right]$$

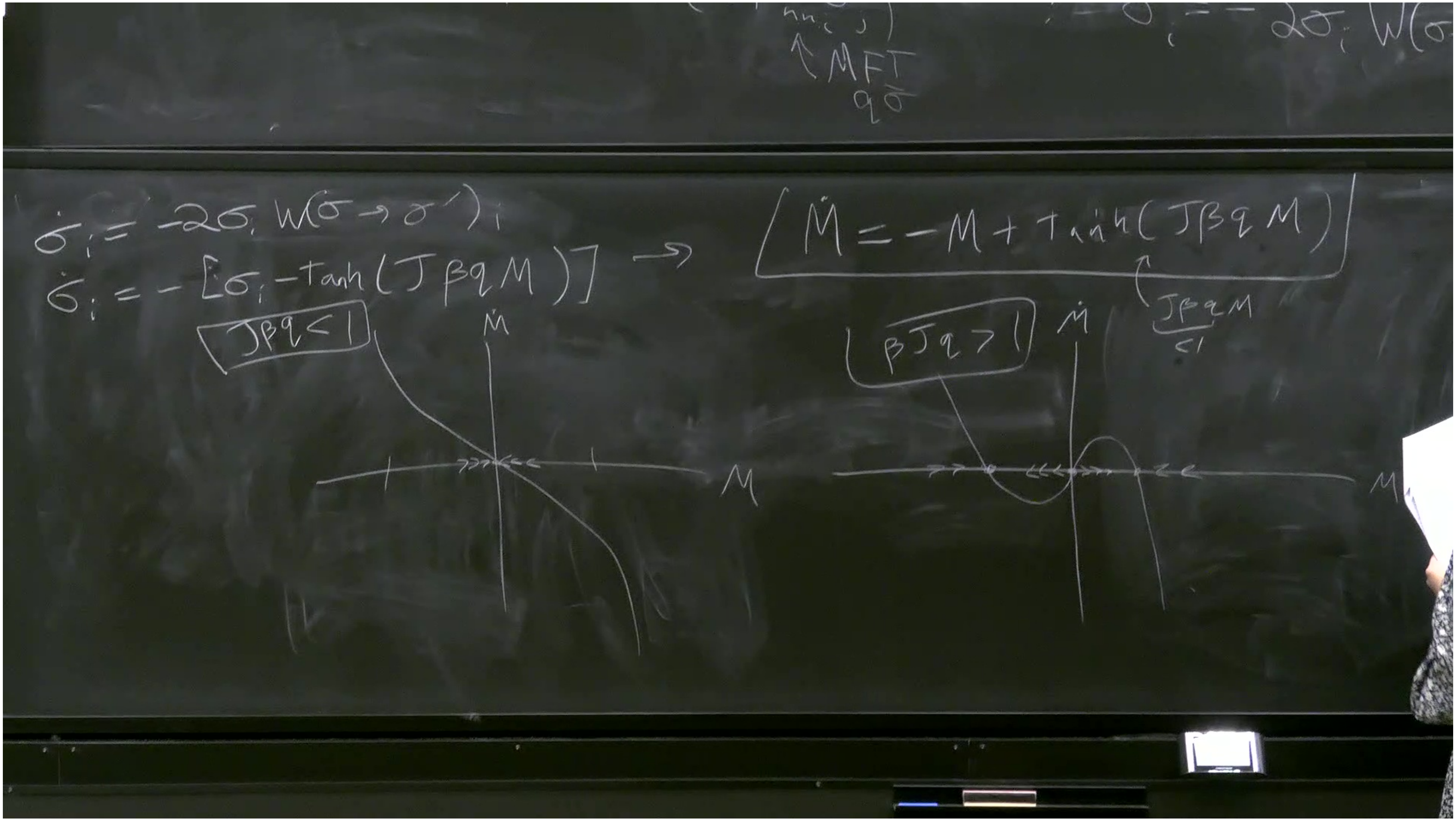
$$\therefore\ \dot{\sigma}_i = -2\sigma_i W(\sigma\to\sigma')_i$$

(σ_i)

(σ_i)

MFT

$q\bar{\sigma}$


MFT
qσ
σ̇_i = −2σ_i W(σ → σ')_i
σ̇_i = −[σ_i − tanh(JβqM)] →
Ṁ = −M + tanh(JβqM)
JβqM
<1
Jβq<1
βJq>1
Ṁ
M
Ṁ
M

$\dot{\sigma}_i = -2\sigma_i W(\sigma$

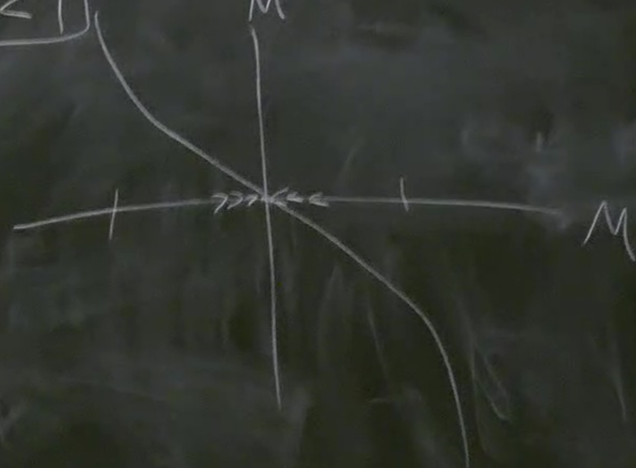
$\uparrow$ MFT

$$\dot{\sigma}_i = -2\sigma_i W(\sigma \to \sigma')_i$$

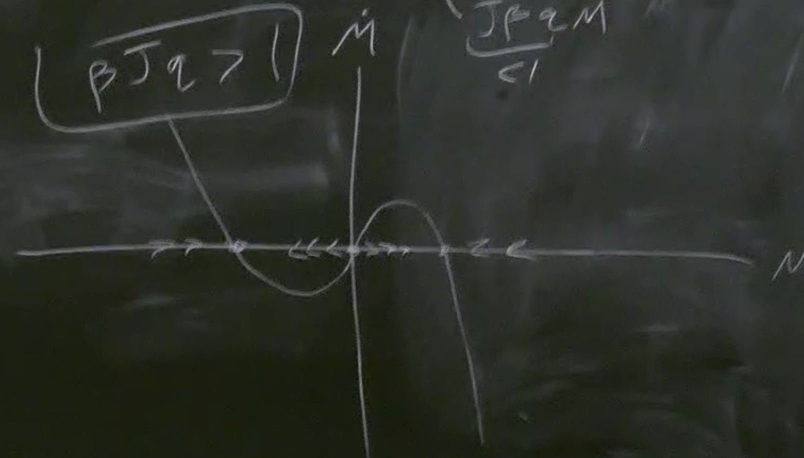
$$\dot{\sigma}_i = -[\sigma_i - \tanh(J\beta q M)] \to \boxed{\dot{M} = -M + \tanh(J\beta q M)}$$

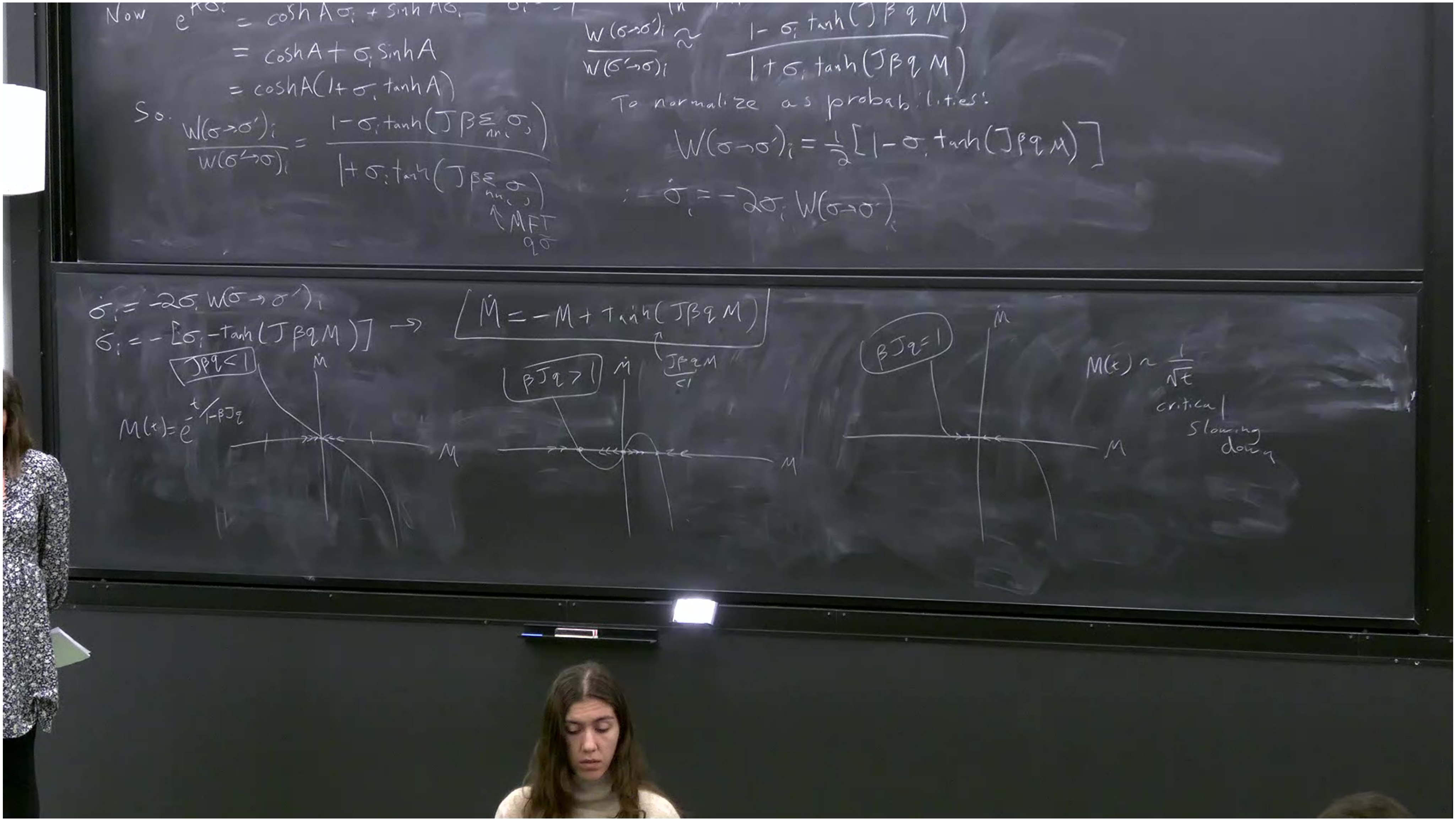
$\underline{J\beta q M}$ <1

$\boxed{J\beta q < 1}$



$\boxed{\beta J q > 1}$

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MFT $q\bar{\sigma}$
$\frac{W(\sigma\to\sigma')_i}{W(\sigma'\to\sigma)_i} \approx \frac{1-\sigma_i \tanh(J\beta q M)}{1+\sigma_i \tanh(J\beta q M)}$
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$\dot{\sigma}_i = -2\sigma_i W(\sigma\to\sigma')_i$
$\dot{\sigma}_i = -2\sigma_i W(\sigma\to\sigma')_i$
$\dot{\sigma}_i = -[\sigma_i - \tanh(J\beta q M)] \to$
$\boxed{\dot{M} = -M + \tanh(J\beta q M)}$
$J\beta q M < 1$
$J\beta q < 1$
$M(t) = e^{-t/1-\beta J q}$
$\beta J q > 1$
$\beta J q = 1$
$\dot{M}$
M
$M(t) \sim \frac{1}{\sqrt{t}}$
critical slowing down