Title: Lecture - Statistical Physics, PHYS 602

Speakers: Emilie Huffman

Collection/Series: Statistical Physics (Core), PHYS 602, October 7 - November 6, 2024

Subject: Condensed Matter, Other

Date: October 11, 2024 - 10:45 AM

URL: https://pirsa.org/24100009

Quantum Gases

We can model fermionic and bosonic gases in a semiclassical way using the grand canonical ensemble.

$E \leftarrow$ total energy

$N \leftarrow$ total number particles

$$E = \sum_{\lambda} n_\lambda \varepsilon_\lambda \,, \quad N = \sum_{\lambda} n_\lambda$$

λ: single particle states

ses

canonical

rticles

$\sum_\lambda n_\lambda$

Particle function:

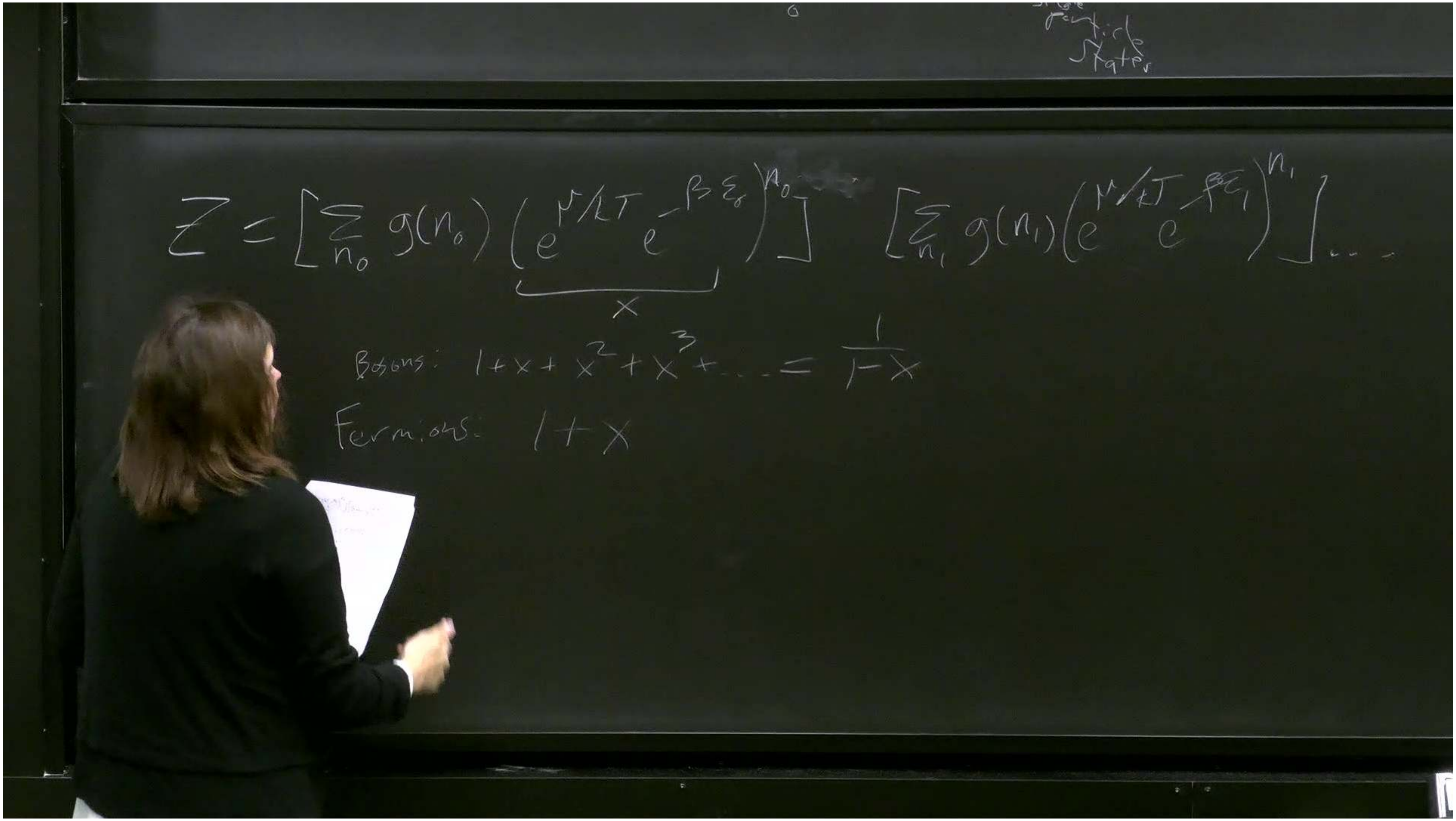
$$Z = \sum_{N,E(N)} e^{\mu N/kT} e^{-\beta E(N)}$$

$$= \sum_{N=0}^{\infty}\left[\sum_{\{n_\lambda\}} g\{n_\lambda\}\prod_\lambda \left(e^{\mu/kT}e^{-\beta\epsilon_\lambda}\right)^{n_\lambda}\right]$$

↑such that $\sum_\lambda n_\lambda = N$

$g\{n_\lambda\} = 1$ for bosons

$$g\{n_\lambda\} = \begin{cases} 1 & \text{if all } n_\lambda = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$ fermions



$$Z = \left[\sum_{n_0} g(n_0)\,\underbrace{\left(e^{\mu/kT}e^{-\beta\varepsilon_0}\right)}_{x}{}^{n_0}\right]\left[\sum_{n_1} g(n_1)\left(e^{\mu/kT}e^{-\beta\varepsilon_1}\right)^{n_1}\right]\ldots$$
Bosons: $1+x+x^2+x^3+\ldots = \frac{1}{1-x}$
Fermions: $1+x$

$$Z = \sum_N e^{\mu N/kT} Z_N$$

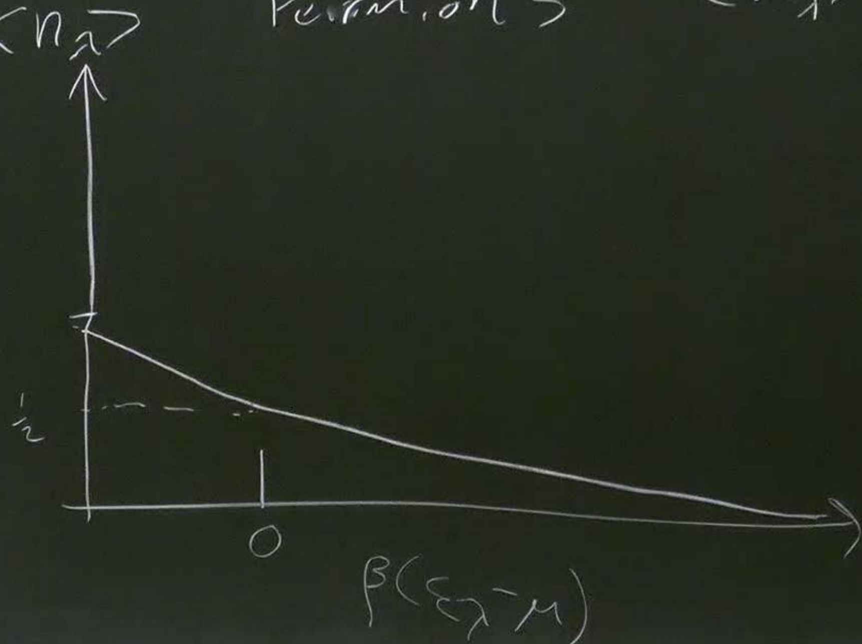
$$Z = \begin{cases} \prod_\lambda \dfrac{1}{1 - e^{\beta\mu - \beta\varepsilon_\lambda}} & \text{Bosons} \\ \prod_\lambda \left(1 + e^{\beta\mu - \beta\varepsilon_\lambda}\right) & \text{Fermions} \end{cases}$$

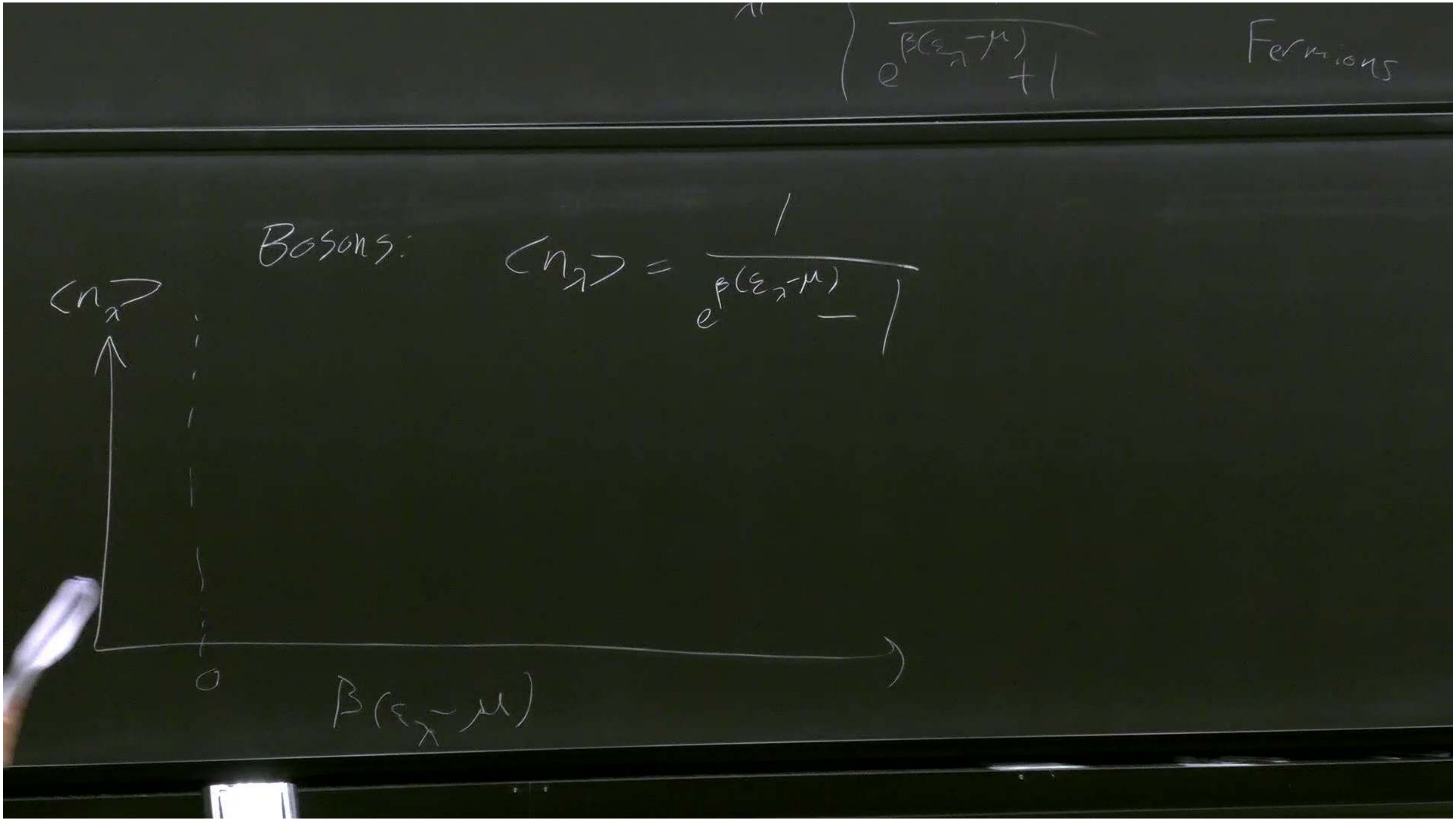
$$\langle n_\lambda \rangle = -\frac{1}{\beta}\frac{\partial}{\partial \varepsilon_\lambda} \ln(Z)$$

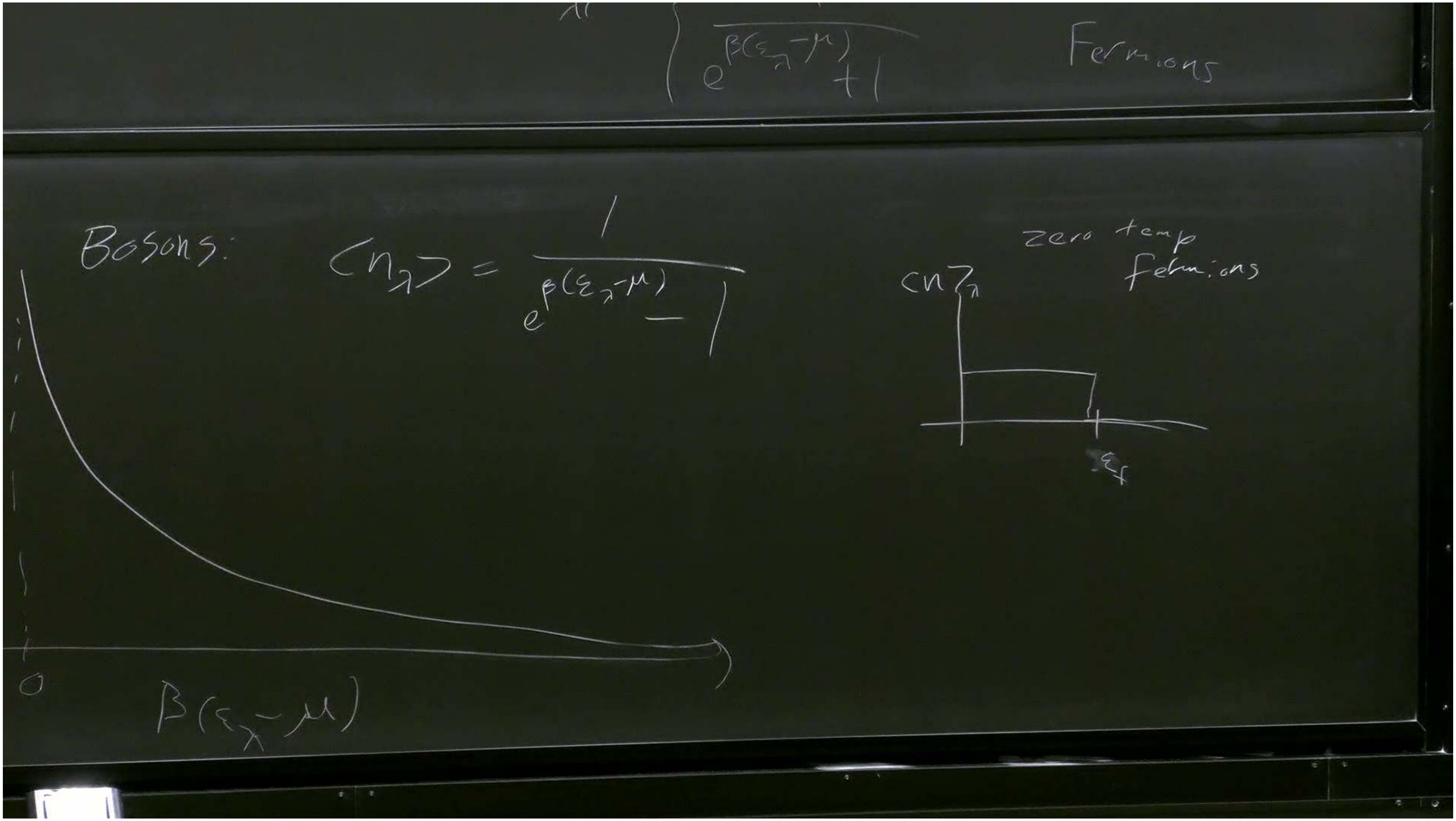
$$\langle n_\lambda \rangle = \begin{cases} \dfrac{1}{e^{\beta(\varepsilon_\lambda - \mu)} - 1} & \text{Bosons} \\ \dfrac{1}{e^{\beta(\varepsilon_\lambda - \mu)} + 1} & \text{Fermions} \end{cases}$$

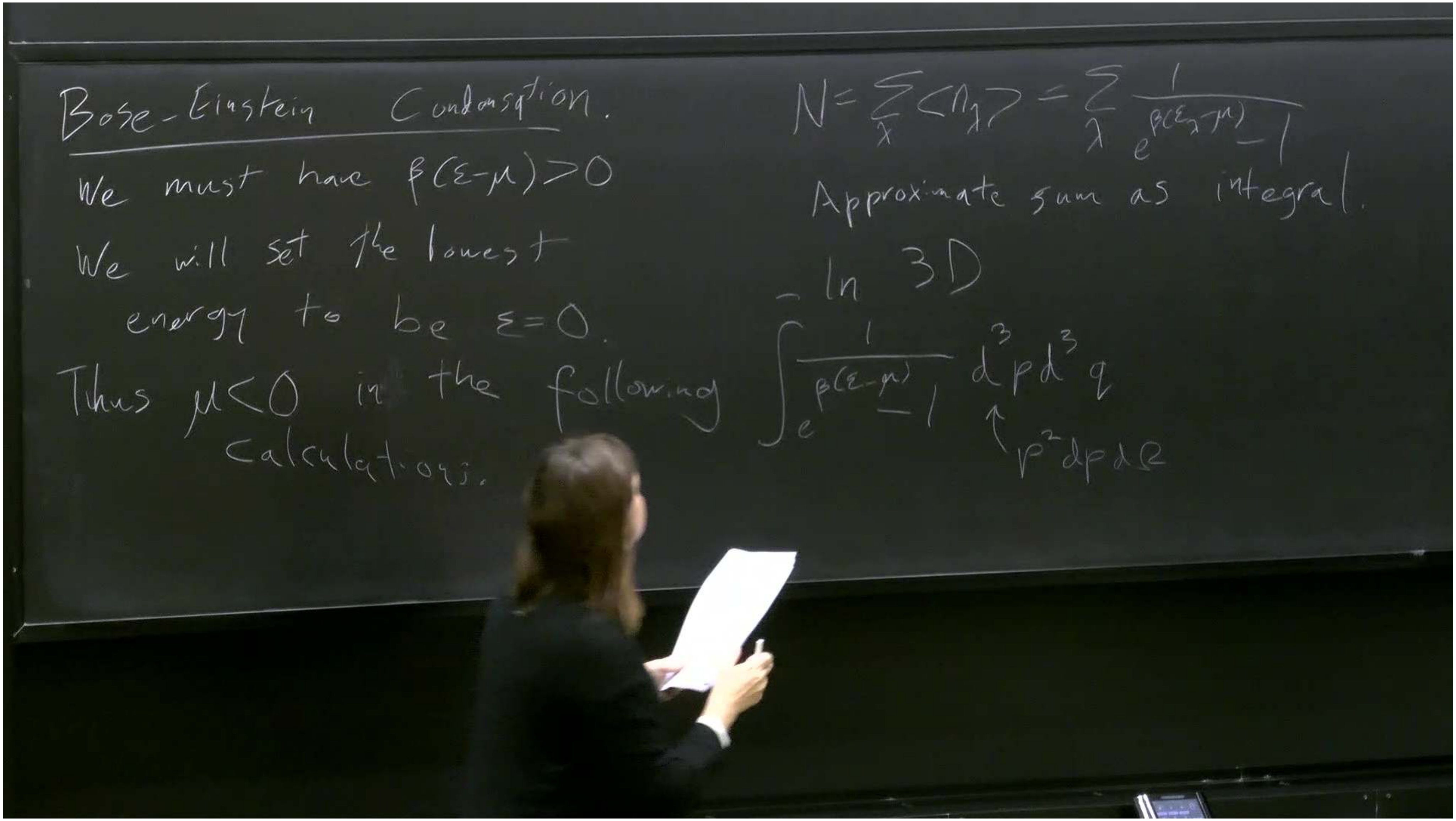
Occupation Numbers

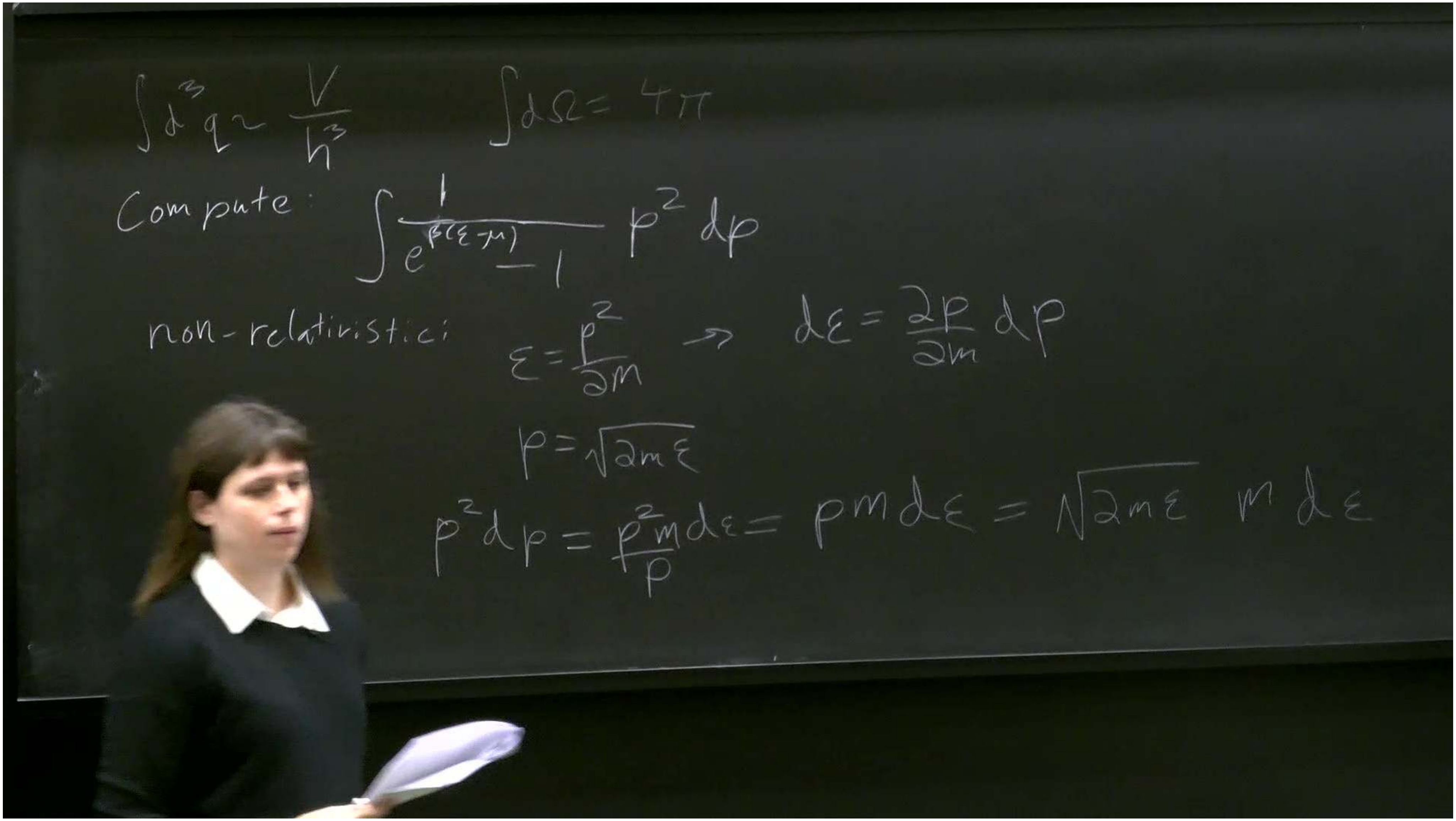
Fermions $\langle n_\lambda \rangle = \dfrac{1}{e^{\beta(\varepsilon_\lambda - \mu)} + 1}$

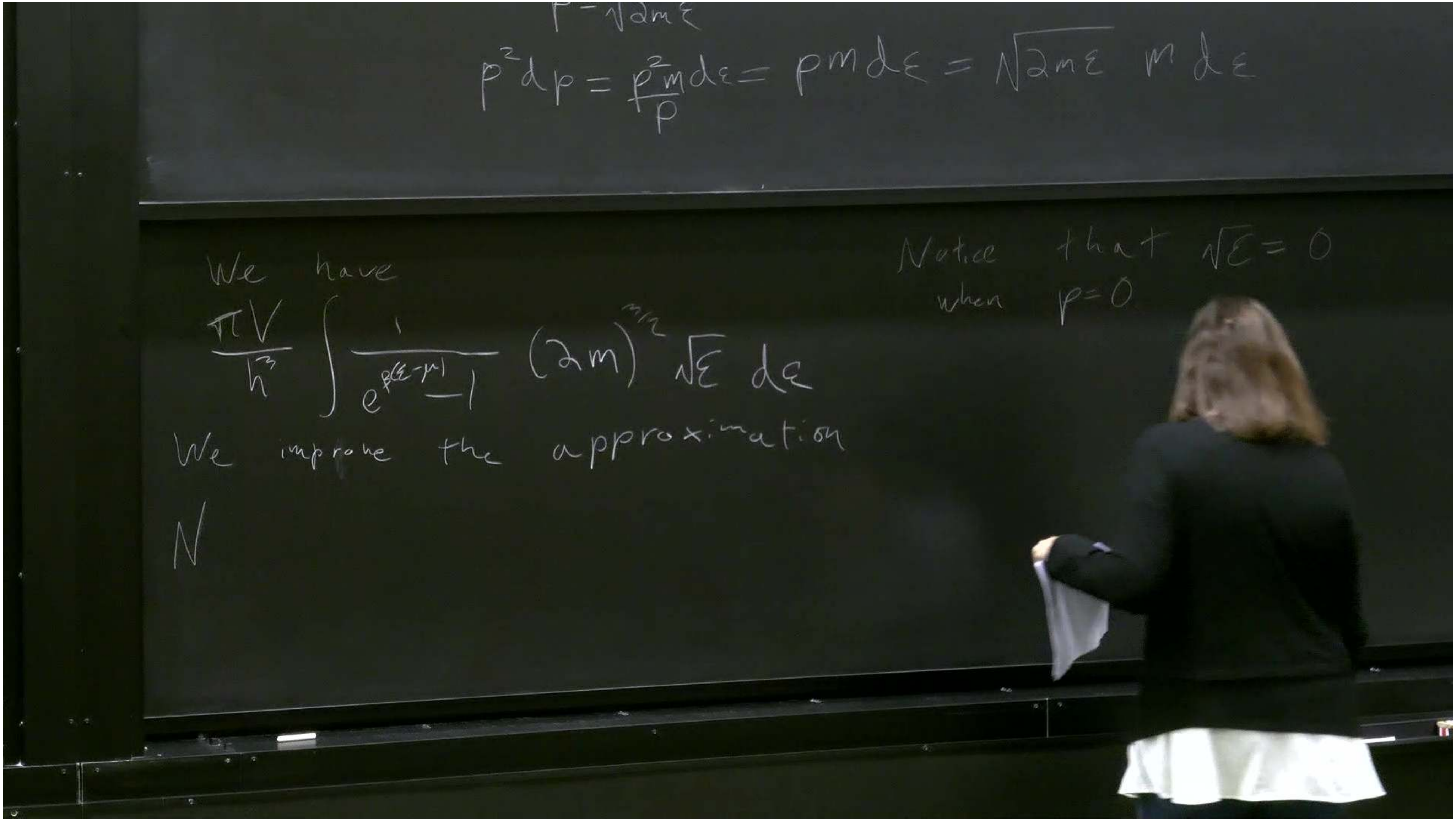



$e^{\beta(\varepsilon_\lambda-\mu)}+1$
Fermions
Bosons:
$\langle n_\lambda \rangle = \frac{1}{e^{\beta(\varepsilon_\lambda-\mu)}-1}$
$\langle n_\lambda \rangle$
0
$\beta(\varepsilon_\lambda-\mu)$


$\frac{1}{e^{\beta(\varepsilon_\lambda-\mu)}+1}$ Fermions
Bosons: $\langle n_\lambda \rangle = \frac{1}{e^{\beta(\varepsilon_\lambda-\mu)}-1}$
zero temp fermions
$\langle n \rangle_\lambda$
ε_f
0
$\beta(\varepsilon_\lambda-\mu)$


Bose-Einstein Condensation.
We must have $\beta(\varepsilon-\mu)>0$
We will set the lowest
energy to be $\varepsilon=0$.
Thus $\mu<0$ in the following
calculations.
$N=\sum_\lambda \langle n_\lambda \rangle = \sum_\lambda \frac{1}{e^{\beta(\varepsilon_\lambda-\mu)}-1}$
Approximate sum as integral.
In 3D
$\int \frac{1}{e^{\beta(\varepsilon-\mu)}-1} d^3p\, d^3q$
$p^2 dp\, d\Omega$


$\int d^3q \approx \frac{V}{h^3}$ $\int d\Omega = 4\pi$
Compute: $\int \frac{1}{e^{\beta(\varepsilon-\mu)}-1} p^2 \, dp$
non-relativistic: $\varepsilon = \frac{p^2}{2m} \rightarrow d\varepsilon = \frac{2p}{2m} dp$
$p = \sqrt{2m\varepsilon}$
$p^2 dp = \frac{p^2 m}{p} d\varepsilon = pm \, d\varepsilon = \sqrt{2m\varepsilon} \; m \, d\varepsilon$


$p = \sqrt{2m\varepsilon}$
$p^2 dp = \frac{p^2 m}{p} d\varepsilon = p m d\varepsilon = \sqrt{2m\varepsilon}\, m\, d\varepsilon$
We have
$\frac{\pi V}{h^3} \int \frac{1}{e^{\beta(\varepsilon-\mu)}-1} (2m)^{3/2} \sqrt{\varepsilon}\, d\varepsilon$
We improve the approximation
N
Notice that $\sqrt{\varepsilon} = 0$ when $p = 0$

$$p^2 dp = \frac{p^2 m}{p} d\varepsilon = pm\, d\varepsilon = \sqrt{2m\varepsilon}\, m\, d\varepsilon$$

when $p = 0$

$$\frac{\pi V}{h^3} \int \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} (2m)^{3/2} \sqrt{\varepsilon}\, d\varepsilon$$

We improve the approximation

$$N = \underbrace{\frac{\pi V}{h^3} \int \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} (2m)^{3/2} \sqrt{\varepsilon}\, d\varepsilon}_{N_e} + \underbrace{\frac{1}{e^{-\beta\mu} - 1}}_{N_0}$$

Mathemitical Identity: $\mu < 0$

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{-\beta\mu}e^{x}-1} = \Gamma(\nu)\left(e^{\beta\mu} + \frac{e^{2\beta\mu}}{2^\nu} + \frac{e^{3\beta\mu}}{3^\nu} + \dots\right)$$

zeta-function: $\zeta(\nu) = 1 + \frac{1}{2^\nu} + \frac{1}{3^\nu} + \dots$

$$x = \beta\varepsilon, \quad dx = \beta\, d\varepsilon$$

$$N_e = \frac{\pi V}{h^3}(2m)^{3/2}\int \frac{\varepsilon^{1/2}}{e^{-\beta\mu}e^{\beta\varepsilon}-1}\, d\varepsilon = \frac{\pi V}{h^3\beta^{3/2}}(2m)^{3/2}\int \frac{x^{1/2}\, dx}{e^{-\beta\mu}e^{x}-1}$$

$\boxed{3D}$ $\quad \nu = 3/2$

$$N_e \leq \frac{\pi V}{h^3}(2mkT)^{3/2}\underbrace{\Gamma(3/2)}_{\pi^{1/2}/2}\,\zeta\left(\frac{3}{2}\right)$$

$x = \beta\varepsilon, \quad dx = \beta\, d\varepsilon$

$$N_e = \frac{\pi V}{h^3}(2m)^{3/2}\int \frac{\varepsilon^{1/2}}{e^{-\beta\mu}e^{\beta\varepsilon}-1}\, d\varepsilon = \frac{\pi V}{h^3\beta^{3/2}}(2m)^{3/2}\int \frac{x^{1/2}\, dx}{e^{-\beta\mu}e^{x}-1}$$

$\boxed{3D}$ $\quad U = 3/2$

$$N_e \leq \frac{\pi V}{h^3}(2mkT)^{3/2}\,\Gamma(3/2)\,\zeta\left(\frac{3}{2}\right)$$

$\Gamma(3/2) = \frac{\pi^{1/2}}{2}$

The number of particles in the $\varepsilon = 0$ state is

$$N - N_{e,max} = N - \frac{V}{2h^3}(2mkT\pi)^{3/2}\,\zeta\left(\frac{3}{2}\right)$$

$$N_e = \frac{\pi V}{h^3}(2m)^{3/2}\int \frac{\varepsilon^{1/2}}{e^{-\beta\mu}e^{\beta\varepsilon}-1}\,d\varepsilon = \frac{\pi V}{h^3\beta^{3/2}}(2m)^{3/2}\int \frac{x^{1/2}\,dx}{e^{-\beta\mu}e^{x}-1}$$

$x = \beta\varepsilon,\ dx = \beta\, d\varepsilon$

$\boxed{3D}$ $\quad \nu = 3/2$

$$N_e \leq \frac{\pi V}{h^3}(2mkT)^{3/2}\,\Gamma\left(\tfrac{3}{2}\right)\zeta\left(\tfrac{3}{2}\right)$$

$\frac{\pi^{1/2}}{2}$

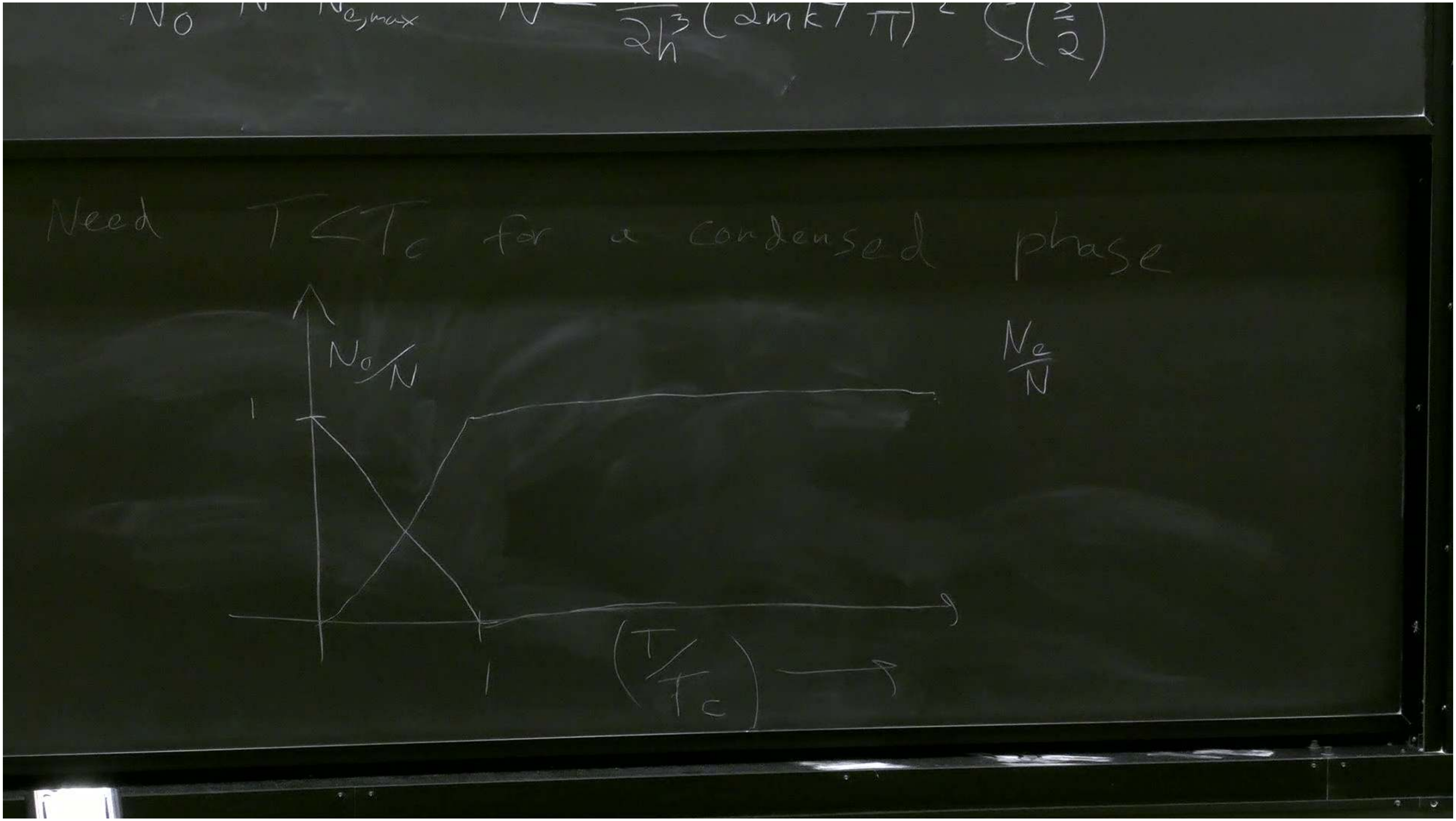
The number of particles in the $\varepsilon = 0$ state is

$$N_0 = N - N_{e,max} = N - \frac{V}{2h^3}(2mkT\pi)^{3/2}\,\zeta\left(\tfrac{3}{2}\right)$$

At what $T = T_c$ do we get BEC?

$$N = N_{e,\,max} = \frac{V}{2h^3}\left(2mkT_c\pi\right)^{3/2}\zeta\left(\tfrac{3}{2}\right)$$

$$T_c = \left(\frac{N\,2h^3}{\zeta\left(\frac{3}{2}\right)V}\right)^{2/3}\frac{1}{2mk\pi}$$


Need T < T_c for a condensed phase
N_0/N
N_e/N
1
1
(T/T_c)

$$p\,dp = \frac{p\,m\,d\varepsilon}{p}$$

Two dimensions.

$$N_e = \frac{A}{h^2} 2m\pi \int \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1} = \frac{A}{h^2} 2m\pi kT \int \frac{x^0\,dx}{e^{\beta\varepsilon} e^{-x} - 1}$$

$\nu = 1$, $\zeta(1)$ diverges

N_e is unbounded.

$T < T_c = 0$

No cond phase i

$p\,dp = \frac{p\,m\,d\varepsilon}{p} = \ldots$

wo dimensions:

$$N_e = \frac{A}{h^2}\, 2m\pi \int \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)}-1} = \frac{A}{h^2}\, 2m\pi kT \int \frac{x^0\,dx}{e^{\beta\varepsilon}e^{-x}-1}$$

$\nu = 1$, $\zeta(1)$ diverges

N_e is unbounded.

$T < T_c = 0$

No condensed phase in 2D.

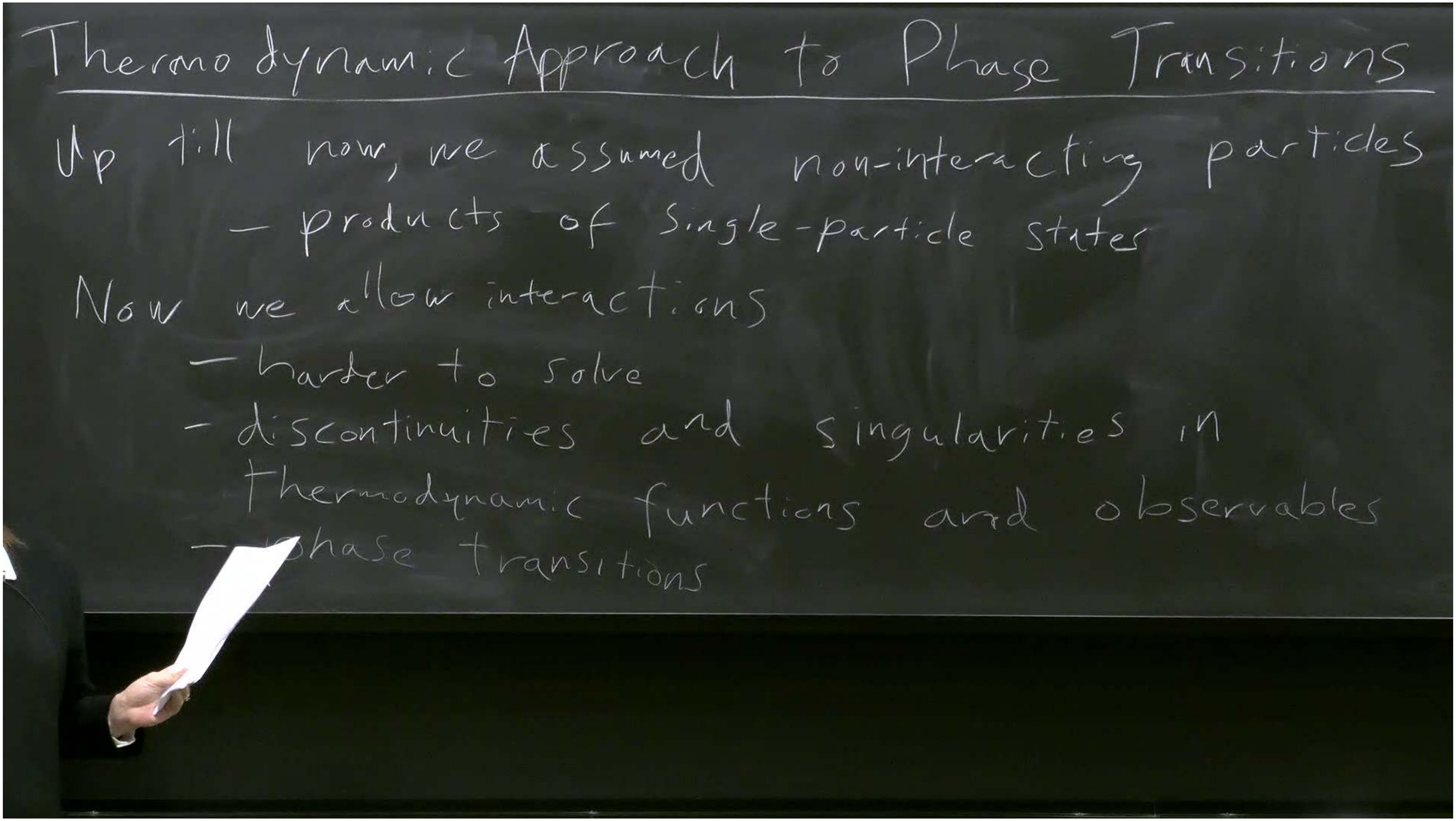
Thermodynamic Approach to Phase Transitions

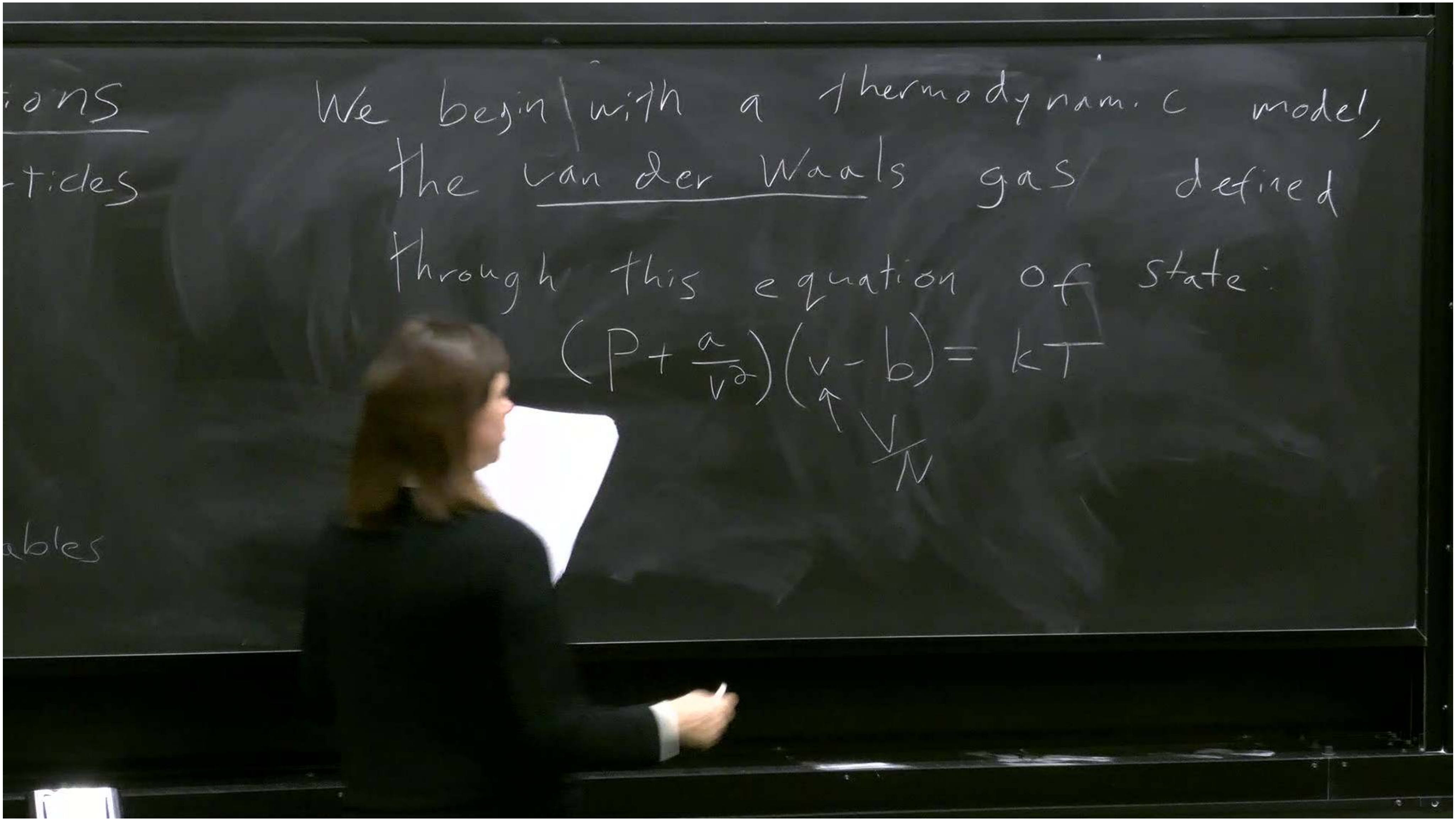
Up till now, we assumed non-interacting particles
– products of single-particle states

ions

rticles

$$Z_N = (Z_1)^N$$


Thermodynamic Approach to Phase Transitions
Up till now, we assumed non-interacting particles
– products of single-particle states
Now we allow interactions
– harder to solve
– discontinuities and singularities in
thermodynamic functions and observables
– phase transitions


We begin with a thermodynamic model,
the van der Waals gas defined
through this equation of state:
$(P + \frac{a}{v^2})(v - b) = kT$
V/N

-ticles

the van der Waals gas defined through this equation of state:

$$\left(P + \frac{a}{v^2}\right)(v - b) = kT$$

$v = V/N$

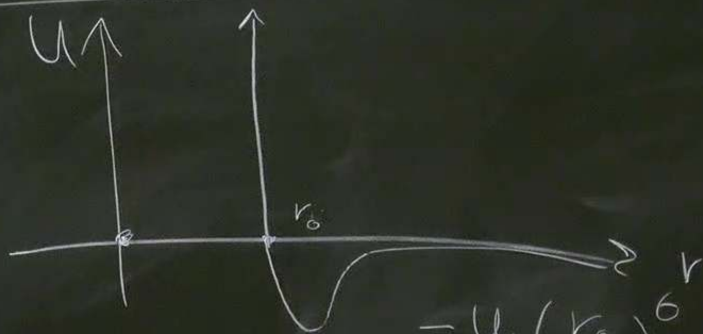
ables

David Tong Statistical Physics 2.5

Pathria, 10.1–10.3

– phase transitions

First-ord improvement to ideal gas law

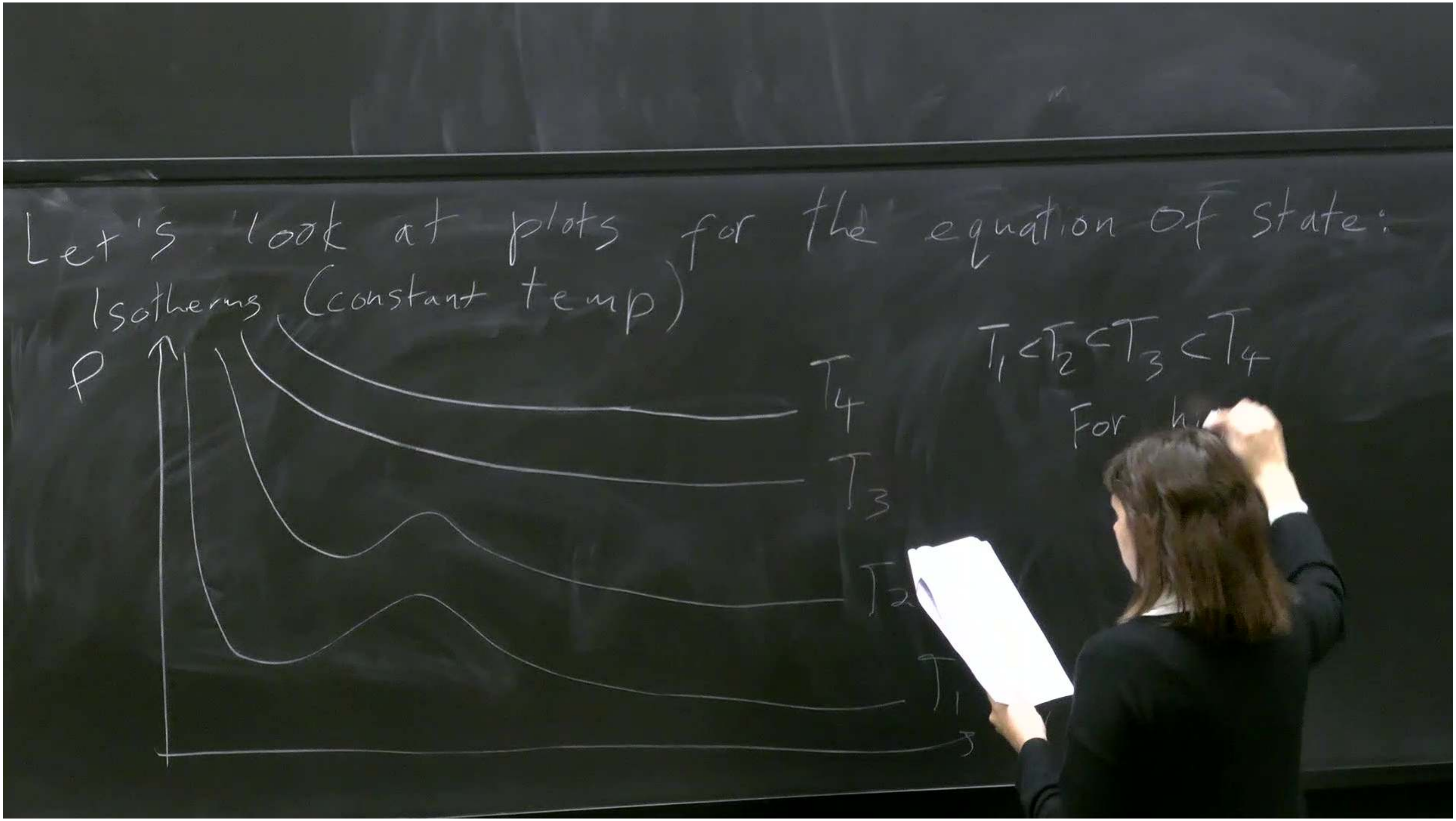


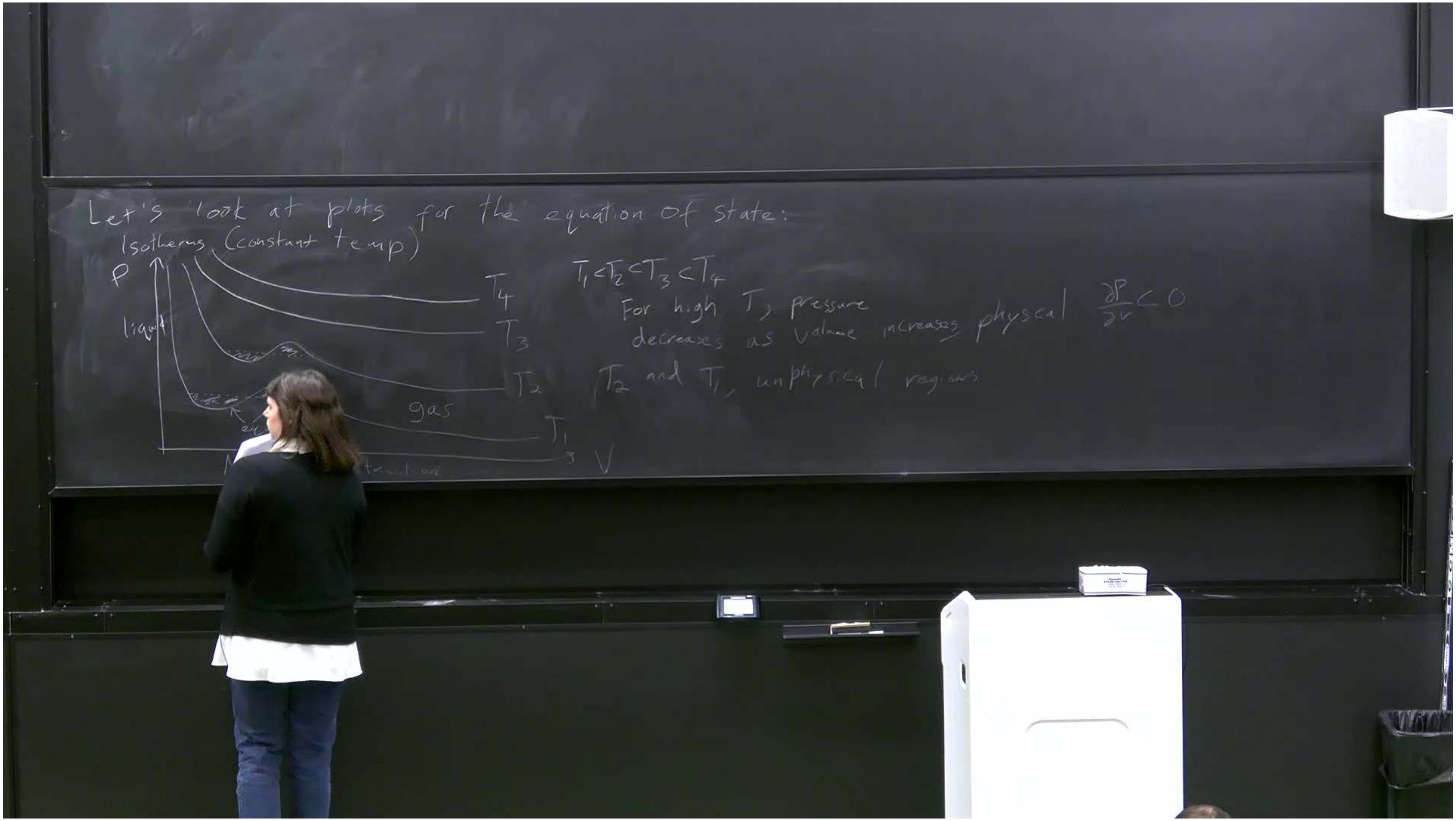
$-U_0\left(\frac{r_0}{r}\right)^6$ ← two dipoles

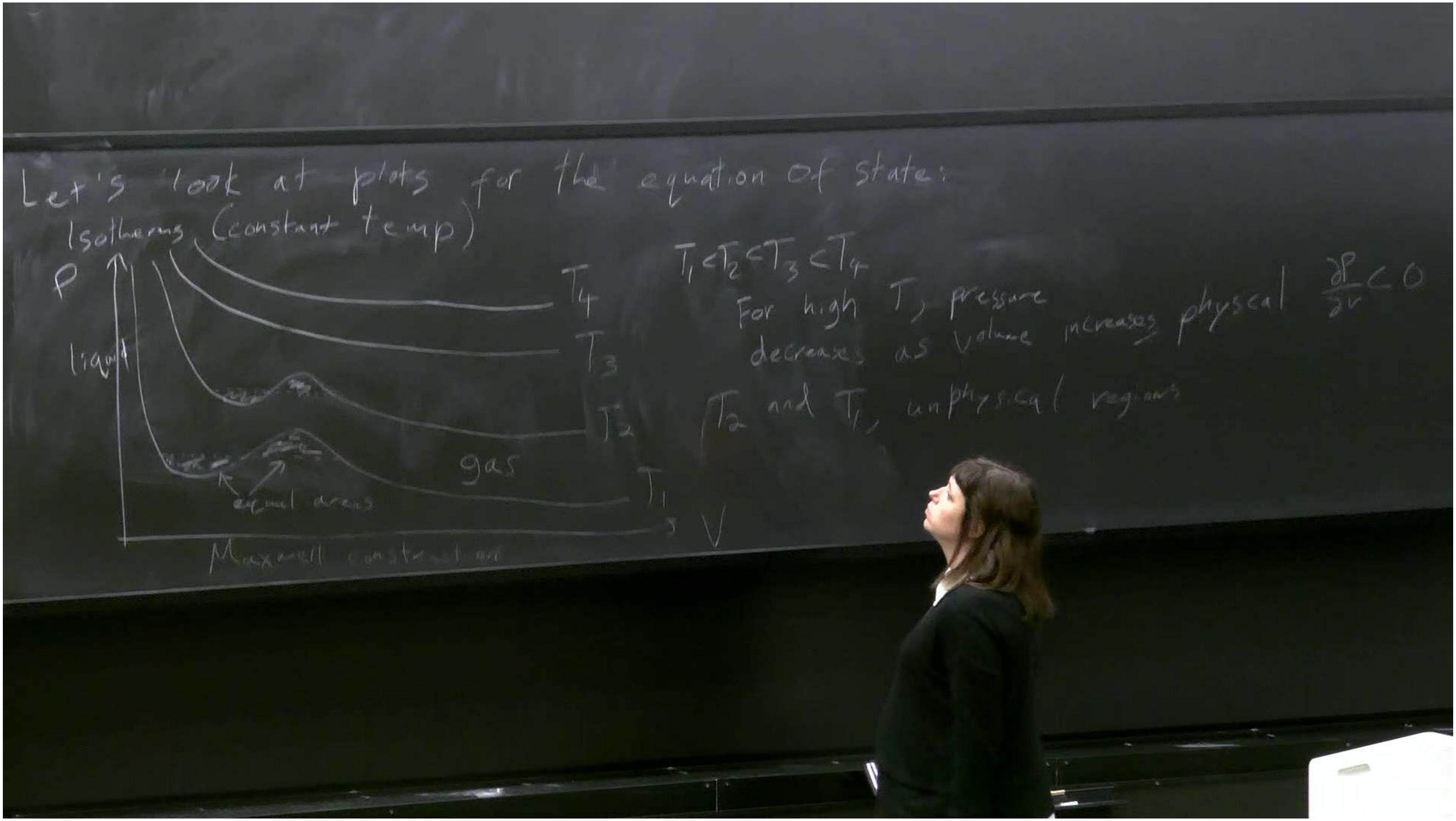
Pathria, 10.1-10.3

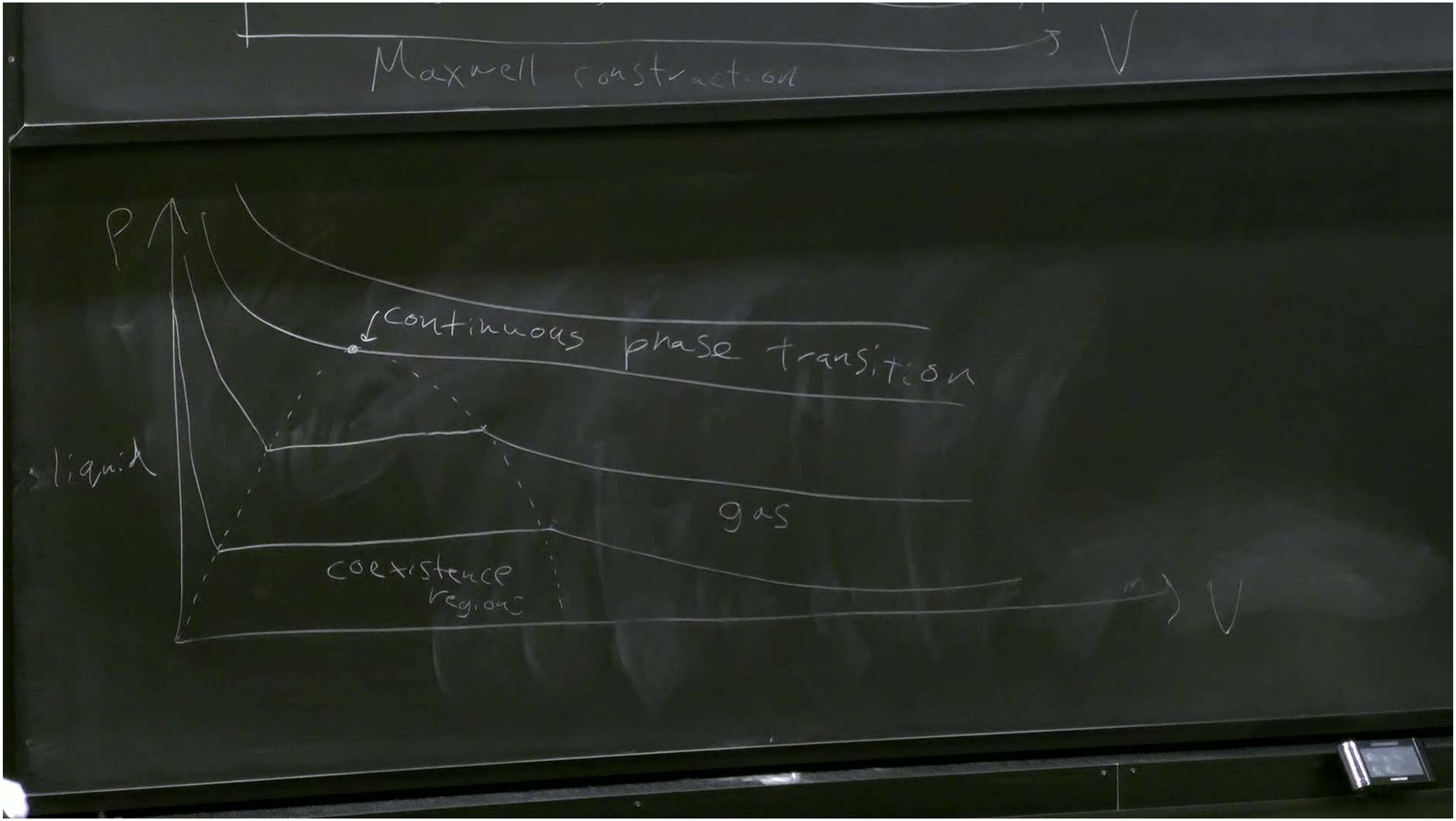
Empirically:

- when $a=0$, $b=0$, $Pv = kT$ ideal gas
- a increases $P \rightarrow$ interactions between particles should increase pressure
- b reduces $v \rightarrow$ particles now have some volume - reduces overall volume available


Let's look at plots for the equation of state:
Isotherms (constant temp)
P
T_4
T_3
T_2
T_1
$T_1 < T_2 < T_3 < T_4$
For


Let's look at plots for the equation of state:
Isotherms (constant temp)
P
liquid
gas
V
T1 < T2 < T3 < T4
T4
T3
T2
T1
For high T, pressure decreases as volume increases physical ∂P/∂V < 0
T2 and T1, unphysical regions


Let's look at plots for the equation of state:
Isotherms (constant temp)
P
liquid
gas
equal areas
Maxwell construction
V
T4
T3
T2
T1
T1 < T2 < T3 < T4
For high T, pressure decreases as volume increases physical ∂P/∂V < 0
T2 and T1, unphysical regions


Maxwell construction
V
P
continuous phase transition
liquid
gas
coexistence
region
V