

Title: Lecture - Statistical Physics, PHYS 602

Speakers: Emilie Huffman

Collection/Series: Statistical Physics (Core), PHYS 602, October 7 - November 6, 2024

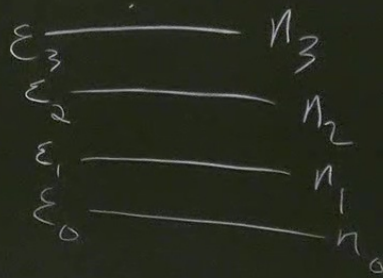
Subject: Condensed Matter, Other

Date: October 11, 2024 - 10:45 AM

URL: <https://pirsa.org/24100009>

Quantum Gases

We can model fermionic and bosonic gases in a semiclassical way using the grand canonical ensemble.



$E \leftarrow$ total energy

$N \leftarrow$ total number particles

$$E = \sum_{\lambda} n_{\lambda} E_{\lambda}, \quad N = \sum_{\lambda} n_{\lambda}$$

λ single particle states

Particle function:

$$Z = \sum_{N, E(N)} e^{\frac{\mu N}{kT}} e^{-\beta E(N)}$$

$$= \sum_{N=0}^{\infty} \left[\sum_{\{n_{\lambda}\}} g\{n_{\lambda}\} \prod_{\lambda} \left(e^{\frac{\mu}{kT}} e^{-\beta E_{\lambda}} \right)^{n_{\lambda}} \right]$$

↑ such that $\sum_{\lambda} n_{\lambda} = N$

$g\{n_{\lambda}\} = 1$ for bosons

$g\{n_{\lambda}\} = \begin{cases} 1 & \text{if all } n_{\lambda} = 0, 1 \\ 0 & \text{otherwise} \end{cases}$ fermions

particle states

$$Z = \left[\sum_{n_0} g(n_0) \underbrace{\left(e^{\mu/kT} e^{-\beta \epsilon_0} \right)^{n_0}}_x \right] \left[\sum_{n_1} g(n_1) \left(e^{\mu/kT} e^{-\beta \epsilon_1} \right)^{n_1} \right] \dots$$

Bosons: $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

Fermions: $1 + x$

$$Z = \sum_N e^{\frac{\mu N}{kT}} Z_N$$

$$\rightarrow Z = \prod_{\lambda} \frac{1}{1 - e^{\beta\mu - \beta\epsilon_{\lambda}}} \quad \text{Bosons}$$

$$\left(\prod_{\lambda} (1 + e^{\beta\mu - \beta\epsilon_{\lambda}}) \right) \quad \text{Fermions}$$

$$\langle N_{\lambda} \rangle = - \frac{1}{\beta} \frac{\partial}{\partial \epsilon_{\lambda}} \ln(Z)$$

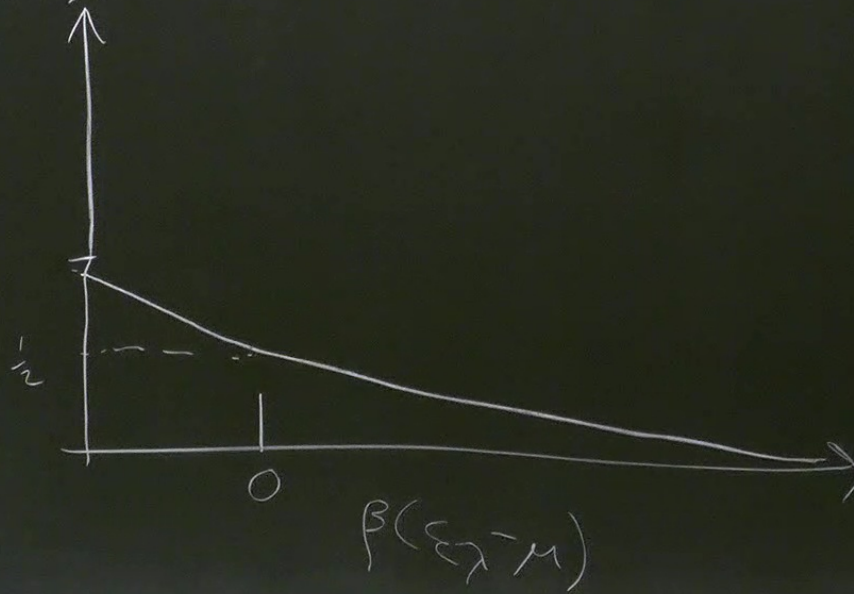
$$\langle N_{\lambda} \rangle = \begin{cases} \frac{1}{e^{\beta(\epsilon_{\lambda} - \mu)} - 1} & \text{Bosons} \\ \frac{1}{e^{\beta(\epsilon_{\lambda} - \mu)} + 1} & \text{Fermions} \end{cases}$$

Occupation Numbers

Fermions

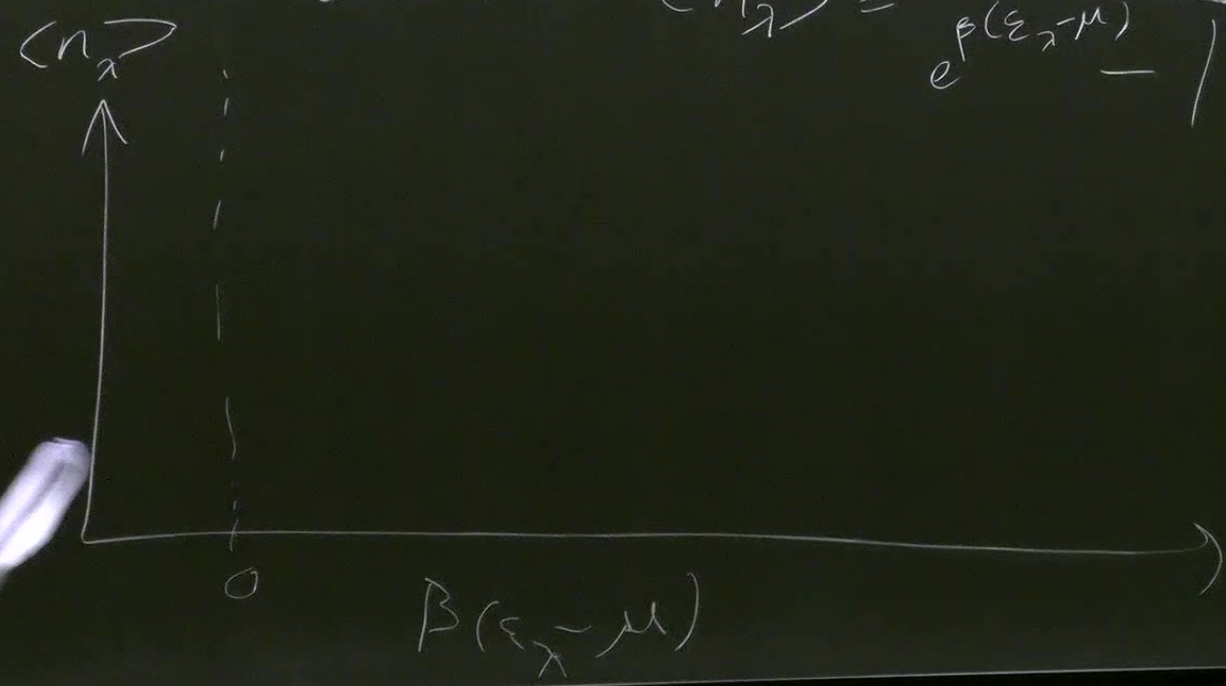
$$\langle n_\lambda \rangle = \frac{1}{e^{\beta(\epsilon_\lambda - \mu)} + 1}$$

$\langle n_\lambda \rangle$



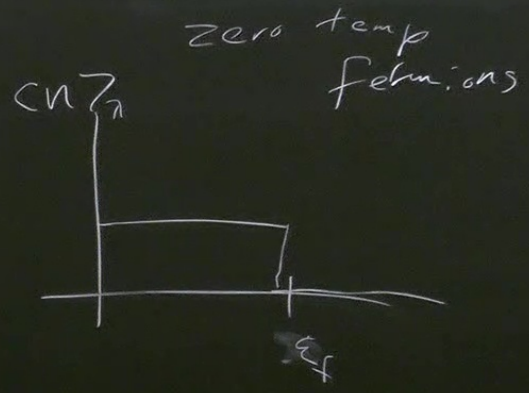
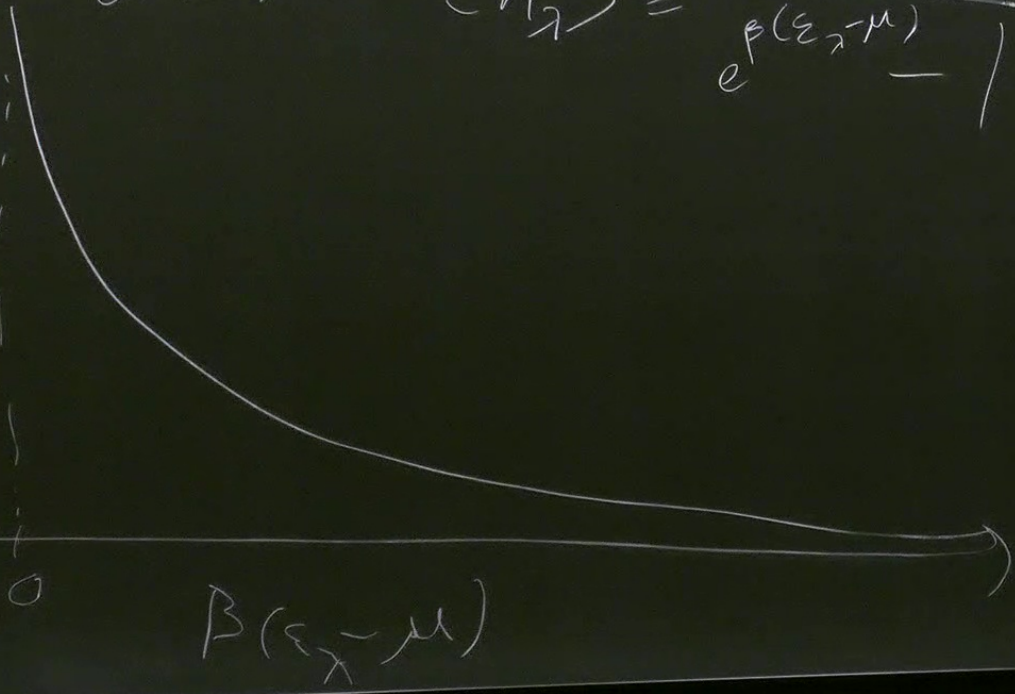
$$\left. \begin{matrix} \pi \\ e^{\beta(\epsilon_{\lambda} - \mu)} + 1 \end{matrix} \right\} \text{Fermions}$$

Bosons: $\langle n_{\lambda} \rangle = \frac{1}{e^{\beta(\epsilon_{\lambda} - \mu)} - 1}$



$$\left. \begin{matrix} \text{Fermions} \\ e^{\beta(\epsilon_i - \mu)} + 1 \end{matrix} \right\}$$

Bosons: $\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$



Bose-Einstein Condensation.

We must have $\beta(\epsilon - \mu) > 0$

We will set the lowest energy to be $\epsilon = 0$.

Thus $\mu < 0$ in the following calculations.

$$N = \sum_x \langle n_x \rangle = \sum_x \frac{1}{e^{\beta(\epsilon_x - \mu)} - 1}$$

Approximate sum as integral.

- ln 3D

$$\int \frac{1}{e^{\beta(\epsilon - \mu)} - 1} d^3 p d^3 q$$

\uparrow
 $v^2 dp dR$

$$\int d^3q \sim \frac{V}{h^3}$$

$$\int d\Omega = 4\pi$$

Compute:

$$\int \frac{1}{e^{\beta(\epsilon - \mu)} - 1} p^2 dp$$

non-relativistic:

$$\epsilon = \frac{p^2}{2m} \rightarrow d\epsilon = \frac{2p}{2m} dp$$

$$p = \sqrt{2m\epsilon}$$

$$p^2 dp = \frac{p^2}{p} m d\epsilon = \sqrt{2m\epsilon} m d\epsilon$$

$$p = \sqrt{2m\varepsilon}$$

$$p^2 dp = \frac{p^2 m d\varepsilon}{p} = p m d\varepsilon = \sqrt{2m\varepsilon} m d\varepsilon$$

We have

$$\frac{\pi V}{h^3} \int \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} (2m)^{3/2} \sqrt{\varepsilon} d\varepsilon$$

We improve the approximation

N

Notice that $\sqrt{\varepsilon} = 0$
when $p = 0$

$$p^2 dp = \frac{p^2}{p} m d\varepsilon = p m d\varepsilon = N 2m\varepsilon^{1/2} d\varepsilon$$

$$\frac{\pi V}{h^3} \int \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} (2m)^{3/2} \sqrt{\varepsilon} d\varepsilon$$

when $\mu = 0$

We improve the approximation

$$N = \underbrace{\frac{\pi V}{h^3} \int \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} (2m)^{3/2} \sqrt{\varepsilon} d\varepsilon}_{N_e} + \underbrace{\frac{1}{e^{\beta\mu} - 1}}_{N_0}$$

Mathematical Identity:

$\mu < 0$

$$\int_0^{\infty} \frac{x^{\nu-1} dx}{e^{\beta\mu x} - 1} = \Gamma(\nu) \left(e^{\beta\mu} + \frac{e^{2\beta\mu}}{2^\nu} + \frac{e^{3\beta\mu}}{3^\nu} + \dots \right)$$

zeta-function: $\zeta(\nu) = 1 + \frac{1}{2^\nu} + \frac{1}{3^\nu} + \dots$

$$N_e = \frac{\pi V}{h^3} (2m)^{3/2} \int_0^{\infty} \frac{\epsilon^{1/2}}{e^{-\beta \mu} e^{\beta \epsilon} - 1} d\epsilon = \frac{\pi V}{h^3} \frac{(2m)^{3/2}}{\beta^{3/2}} \int_0^{\infty} \frac{x^{1/2} dx}{e^{-\beta \mu} e^{\beta x} - 1}$$

$x = \beta \epsilon, dx = \beta d\epsilon$

(3D)

$$v = m/v$$

$$N_e \leq \frac{\pi V}{h^3} (2m k T)^{3/2} \frac{\pi^{3/2}}{2} \zeta(3/2)$$

$\uparrow \frac{e^{\beta \mu}}{2}$

$$N_e = \frac{\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{\epsilon^{1/2}}{e^{\beta m \epsilon} e^{-1}} d\epsilon = \frac{\pi V}{h^3} \beta^{3/2} (2m)^{3/2} \int_0^\infty \frac{x^{1/2} dx}{e^{-\beta m x} e^{-1}}$$

$x = \beta \epsilon, dx = \beta d\epsilon$

3D

$$U = 3/2$$

$$N_e \leq \frac{\pi V}{h^3} (2mkT)^{3/2} \Gamma(3/2) \zeta(3/2)$$

The number of particles in the $\epsilon=0$ state

$$\text{is } N - N_{e,max} = N - \frac{V}{2h^3} (2mkT\pi)^{3/2} \zeta(3/2)$$

$$N_e = \frac{\pi V}{h^3} (2m)^{3/2} \int_{u_{\min}}^{\infty} \frac{e^{-\beta u}}{e^{-\beta u} - 1} du \quad x = \beta u, \quad dx = \beta du$$

$$= \frac{\pi V}{h^3} (2m)^{3/2} \int_0^{\infty} \frac{x^{3/2} dx}{e^{-\beta u} - 1}$$

(3D)

$u = mv^2$

$$N_e \leq \frac{\pi V}{h^3} (2mkT)^{3/2} \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right)$$

The number of particles in the $\epsilon=0$ state

$$N_0 = N - N_{e, \max} = N - \frac{V}{2h^3} (2mkT\pi)^{3/2} \zeta\left(\frac{3}{2}\right)$$

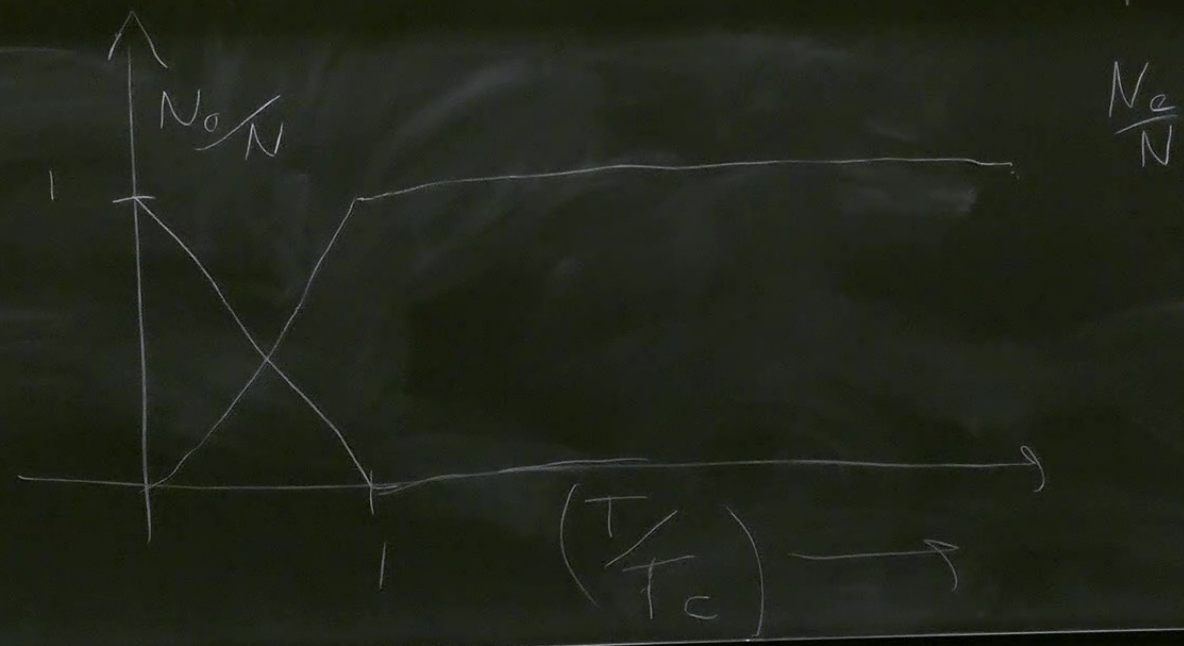
At what $T = T_c$ do we get BEC?

$$N = N_{e, \max} = \frac{V}{2h^3} (2mkT_c \pi)^{3/2} \zeta\left(\frac{3}{2}\right)$$

$$T_c = \left(\frac{N 2h^3}{\zeta\left(\frac{3}{2}\right)V} \right)^{2/3} \frac{1}{2mk \pi}$$

$$N_0 \quad N \quad N_{0,max} \quad N = \frac{2}{h^3} (2mkT\pi)^{3/2} \left(\frac{2}{\lambda}\right)$$

Need $T < T_c$ for a condensed phase



$$p_{\text{app}} = \frac{p_{\text{max}}}{p}$$

Two dimensions

$$N_e = \frac{A}{h^2} 2m\pi \int \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1} = \frac{A}{h^2} 2m\pi kT \int \frac{x^0 dx}{e^{\beta\varepsilon - x} - 1}$$

$\nu = 1$, $\zeta(1)$ diverges

N_e is unbounded.

$T < T_c = 0$

No cond phase

$$p_{\text{clp}} = \frac{p_{\text{max}}}{p}$$

no dimensions

$$N_e = \frac{A}{h^2} 2m\pi \int \frac{dE}{e^{\beta(E-\mu)} - 1} = \frac{A}{h^2} 2m\pi kT \int \frac{x^0 dx}{e^{\beta E - x} - 1}$$

$\nu=1$, $\zeta(1)$ diverges

N_e is unbounded.

$$T < T_c = 0$$

No condensed phase in 2D.

Thermodynamic Approach to Phase Transitions

Up till now, we assumed non-interacting particles
— products of single-particle states

ans
tides

$$Z_N = (Z_1)^N$$

Thermodynamic Approach to Phase Transitions

Up till now, we assumed non-interacting particles

- products of single-particle states

Now we allow interactions

- harder to solve
- discontinuities and singularities in thermodynamic functions and observables
- phase transitions

ions
particles

We begin with a thermodynamic model,
the van der Waals gas defined
through this equation of state:

$$\left(P + \frac{a}{v^2}\right)(v - b) = kT$$

↑
 $\frac{V}{N}$

ables

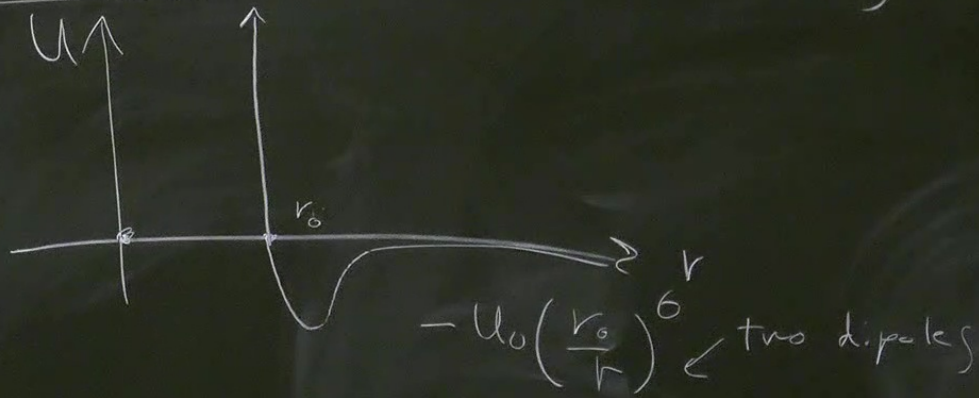
the van der Waals gas defined
through this equation of state:

$$\left(P + \frac{a}{v^2}\right)(v - b) = kT$$

David Tong $\frac{V}{N}$ Statistical Physics, 2.5
Pathria, 10.1-10.3

- phase transitions

First-ord improvement to ideal gas law

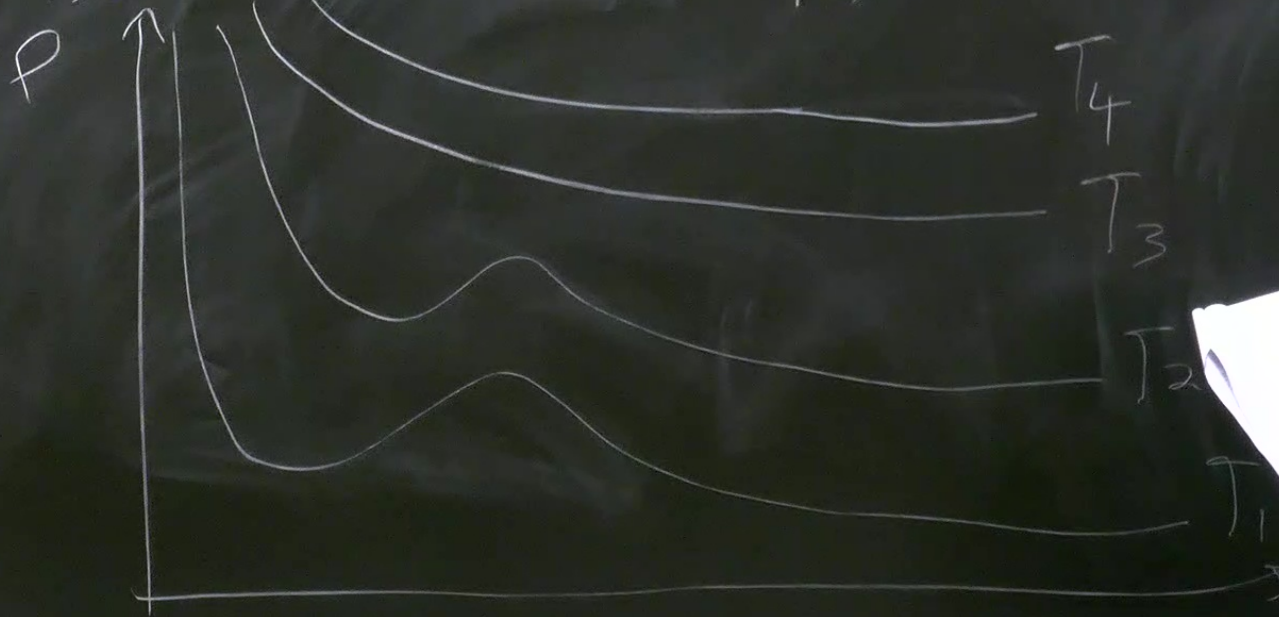


Pathria, 10.1-10.3

Empirically:

- when $a=0, b=0$, $P_v = kT$ ideal gas
- a increases $P \rightarrow$ interactions between particles
Should increase pressure
- b reduces $v \rightarrow$ particles now have some
volume - reduces overall
volume available

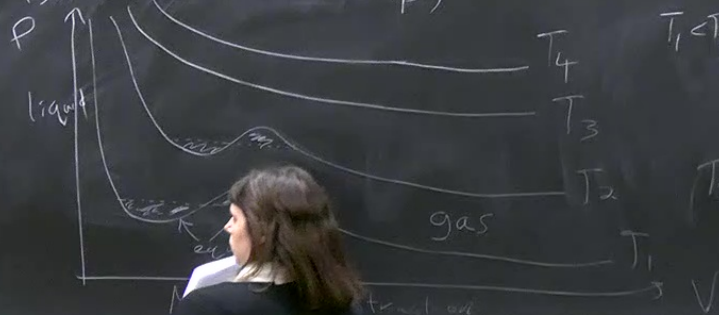
Let's look at plots for the equation of state:
Isotherms (constant temp)



$$T_1 < T_2 < T_3 < T_4$$

For h

Let's look at plots for the equation of state:
 Isotherms (constant temp)



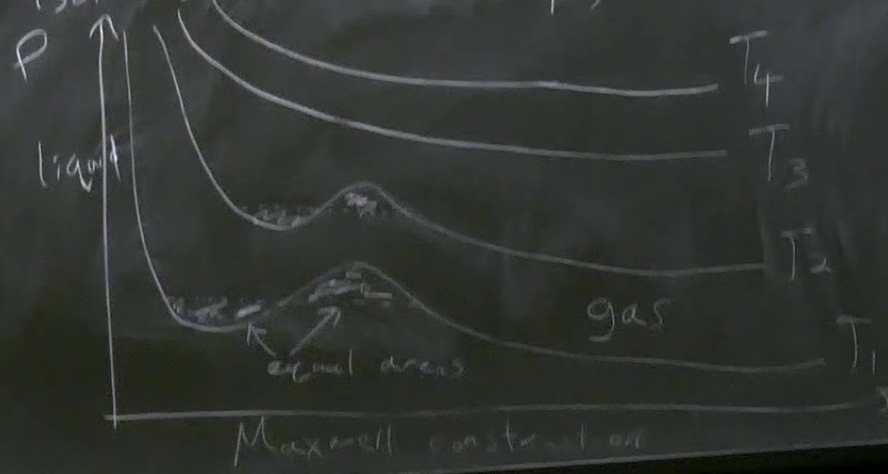
$$T_1 < T_2 < T_3 < T_4$$

For high T , pressure decreases as volume increases, physical $\frac{\partial P}{\partial V} < 0$

T_2 and T_1 , unphysical regions

Let's look at plots for the equation of state:

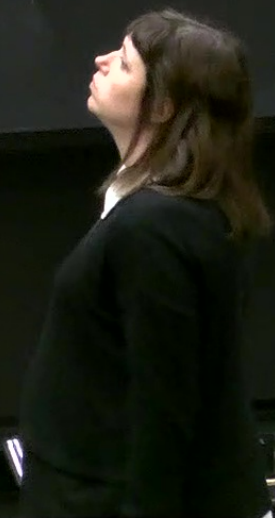
Isotherms (constant temp)



$$T_1 < T_2 < T_3 < T_4$$

For high T , pressure decreases as volume increases physical $\frac{\partial P}{\partial V} < 0$

T_2 and T_3 unphysical regions



Maxwell construction

V

