Title: Lecture - Classical Physics, PHYS 776

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COVARIANT PICTURE	N particles	scalar field
indep. var	$t \in \mathbb{R}$ time	$x^\mu \in M = \mathbb{R}^{1,3}$ sp.t.
config sp. (target sp.)	$Q = \mathbb{R}^{3 \times N}$	$Q = \mathbb{R}, \mathbb{C}$
history	$\vec{\gamma}_\alpha : \mathbb{R} \to Q$ $t \mapsto \vec{\gamma}_\alpha(t)$	$\varphi : M \to Q$ $x \mapsto \varphi(x)$
1st derivative	$\vec{v}_\alpha \sim \frac{d}{dt} \vec{\gamma}_\alpha$	$v_\mu \sim \frac{d}{dx^\mu} \varphi$

^{3} Sp.t.

$\mathcal{I}$, V vect sp.

Q_I

$\varphi^I(x)$

φ^I

$$S(\varphi) = \int_M d^4x \, \mathcal{L}(\varphi^I, \partial_\mu \varphi^I, x)$$

$$M \simeq \Sigma \times [t_0, t_1] \quad , \partial M = \Sigma_0 \cup \Sigma_1$$

$$\delta S(\varphi) \hat{=} 0 \quad w/ \quad \delta\varphi|_{\partial M} = 0$$

Noether thm Consider $\tilde{\delta}\varphi$ s.t.

$$\tilde{\delta}\mathcal{L} = \frac{d}{dx^\mu} R^\mu(\varphi, \partial\varphi, \partial^2\varphi, \ldots x)$$

Then

$$\nabla_\mu J^\mu \hat{=} 0 \quad , \quad J^\mu = \sum_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \tilde{\delta}\varphi^I - R^\mu$$

Sp.t. translation

Consider $a^\mu \in$

Consider two h

$\varphi(x)$

$\tilde{\varphi}(x) = \varphi$

Lagrangian $\mathcal{L}$

$$\tilde{\delta}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} a^\mu \partial$$

$$= a^\mu \Big(\frac{\partial}{\partial}$$

Sp. t. translations

Consider $a^\mu \in \mathbb{R}^{1,3}$ (fixed)

Consider two histories

$\varphi(x)$

$$\tilde{\varphi}(x) = \varphi(x + \epsilon a) \xrightarrow{\epsilon \to 0} \varphi(x) + \epsilon \underbrace{a^\mu \nabla_\mu \varphi}_{= \tilde{\delta}\varphi(x)} + \cdots$$

Lagrangian $\mathcal{L}$, when is the above a sym?

$$\tilde{\delta}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} a^\mu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (a^\nu \partial_\nu \varphi)$$

$$= a^\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu (\partial_\nu \varphi) \right) = a^\mu \frac{d}{dx^\mu} \mathcal{L} - a^\mu \frac{\partial}{\partial x^\mu} \mathcal{L}$$

$\frac{d}{dt}$ $\frac{d}{dx^\mu}\varphi$ Then

a 0+1d scalar field theory w/ $Q = V = \mathbb{R}^{3N}$

$\nabla_\mu J^\mu \hat{=} 0$,

Therefore if

$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0$ ($\mathcal{L}$ does <u>not</u> explicitly depend on x

$\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi, \cancel{x})$)

then $\tilde{\delta}\varphi = a^\mu \partial_\mu \varphi$ is a sym w/

$$\tilde{\delta}\mathcal{L} = \frac{d}{dx^\mu}\Big(\underbrace{a^\mu \mathcal{L}}_{\equiv R^\mu}\Big)$$

Σ_1

$\tilde{\delta}\varphi^F - R^\mu$

Sp. t. translations

Consider $a^\mu \in \mathbb{R}^{1,3}$ (fixed)

Consider two histories

$\varphi(x)$

$\tilde{\varphi}(x) = \varphi(x + \epsilon a) \xrightarrow{\epsilon \to 0} \varphi(x) + \epsilon \underbrace{a^\mu \nabla_\mu \varphi}_{=\tilde{\delta}\varphi(x)} + \cdots$

Lagrangian $\mathcal{L}$, when is the above a sym?

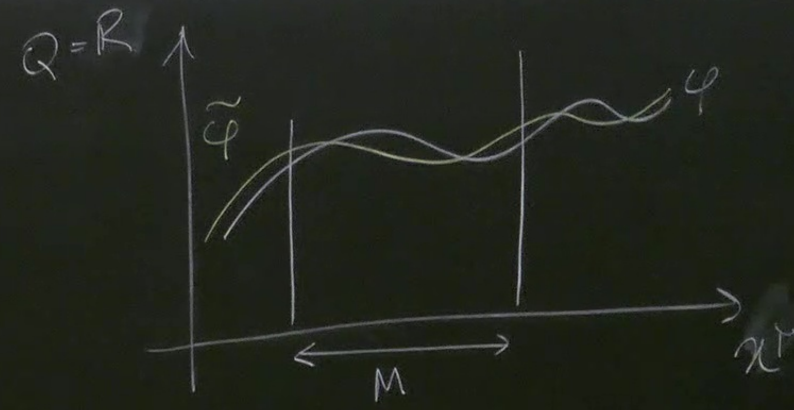
$$\tilde{\delta}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} a^\mu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (a^\nu \partial_\nu \varphi)$$

$$= a^\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu (\partial_\nu \varphi) \right) = a^\mu \frac{d}{dx^\mu} \mathcal{L} - a^\mu \frac{\partial}{\partial x^\mu} \mathcal{L}$$

$$F(q, v_\mu, a_{\mu\nu}, \dots, x)$$

$$\frac{d}{dx^\mu} F(\varphi(x), \partial\varphi(x), \partial^2\varphi(x), x)$$

$$:= \left(\frac{\partial F}{\partial q} \partial_\mu \varphi + \frac{\partial F}{\partial v_\nu} \partial_\mu(\partial_\nu \varphi) + \frac{\partial F}{\partial a_{\nu\rho}} \partial_\mu(\partial_\nu \partial_\rho \varphi) + \dots + \frac{\partial F}{\partial x^\mu} \right) \Bigg|_{(q, v, a) = (\varphi, \partial\varphi, \partial^2\varphi)}$$



3N

$\nabla_\mu J^\mu \hat{=} 0$, $J^\mu = \sum_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^I)} \delta\varphi^I - R^\mu$

$= a^\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \right.$

⤳ Noether current:

$$J^\mu = \boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}} (a^\nu \partial_\nu \varphi) - (a^\mu \mathcal{L})$$

$\pi^\mu \equiv$

$$= a^\nu \left(\pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \right)$$

"on shell"

Result: for all a^ν, $\nabla_\mu J^\mu \hat{=} 0$

$$\nabla_\mu \left(a^\nu (\pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \varphi \right) = a^\nu \nabla_\mu \left(\pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \right)$$

→ We conclude that

$$t^\mu{}_\nu := \pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}$$

is conserved on shell

$$\nabla_\mu t^\mu{}_\nu \hat{=} 0$$

CANONICAL STRESS ENERGY TENSOR

We conclude that

$$t^{\mu}{}_{\nu} := \pi^{\mu}\partial_{\nu}\varphi - \delta^{\mu}_{\nu}\mathcal{L}$$

is conserved on shell

$$\nabla_{\mu} t^{\mu}{}_{\nu} \triangleq 0$$

CANONICAL STRESS ENERGY TENSOR

$$\varphi) = a^{\nu}\nabla_{\mu}\left(\pi^{\mu}\partial_{\nu}\varphi - \delta^{\mu}_{\nu}\mathcal{L}\right)$$

a^{μ} const!

⇝ Noether charges

$$Q = \int_{\mathbb{R}^3} J^0 \, d^3x \quad \text{(in an inertial frame)}$$

$\forall a^{\mu}$ (const)

$$P_{\nu} = \int_{\mathbb{R}^3} t^{0}{}_{\nu} \, d^3x$$

(total) Energy-momentum (in that given frame!)

$\rightarrow$ We conclude that

$$t^\mu{}_\nu := \pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}$$

is conserved on shell

$$\nabla_\mu t^\mu{}_\nu \triangleq 0$$

CANONICAL STRESS ENERGY TENSOR

$$= a^\nu \nabla_\mu \left(\pi^\mu \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L} \right)$$

const!

$\rightsquigarrow$ Noether charges

$$Q = \int_{\mathbb{R}^3} J^0 \, d^3x \quad \text{(in an inertial frame)}$$

$\forall a^\mu$ (const)

$$P_\nu = \int_{\mathbb{R}^3} t^0{}_\nu \, d^3x$$

(total) Energy-momentum (in that given frame!)

<u>Rmk</u> ambiguity $J^\mu \longrightarrow J^\mu + \nabla_\nu \psi^{\nu\mu}$

effect $t^\mu{}_\nu$, but not P^μ!

Sometimes one can use ψ to fix issues

Ex

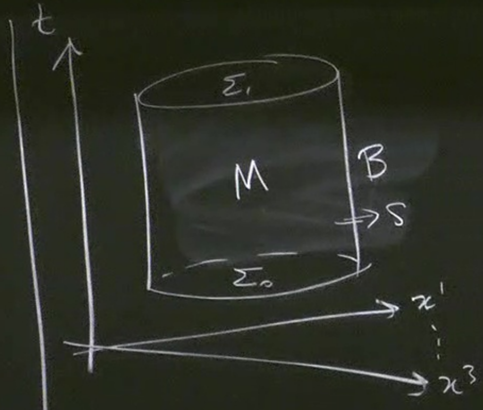
$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$$

$$\frac{\partial}{\partial x^\mu}\mathcal{L} = 0 \longrightarrow t^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}$$

$$= -\partial^\mu\varphi\,\partial_\nu\varphi + \delta^\mu_\nu\left(+\frac{1}{2}\partial^\rho\varphi\,\partial_\rho\varphi + \frac{1}{2}m^2\varphi^2 + V(\varphi)\right)$$

en. density $\quad \mathcal{E} = t^0{}_0 = \underbrace{+\dot\varphi^2 - \frac{1}{2}\dot\varphi^2}_{\frac{1}{2}\dot\varphi^2} + \frac{1}{2}m^2\varphi^2 + V(\varphi) \neq \pi_0$

lin. mom. density $\quad p_i = t^0{}_i = \dot\varphi\,\partial_i\varphi \neq \pi_i$



$$M = \Sigma \times [t_0, t_1]$$

$$\partial M = \Sigma_0 \cup \Sigma_1 \cup B$$

$$B = \partial\Sigma \times [t_0, t_1]$$

$$\underbrace{\int_B}_{\int_{t_0}^{t_1} \oint_{\partial\Sigma}}$$

e.g. $t^i{}_0$ = energy flow in direction x^i

$$0 \,\hat{=} \int_M D_\mu t^\mu{}_\nu = \int_{\Sigma_1} t^0{}_\nu - \int_{\Sigma_0} t^0{}_\nu + \int_{t_0}^{t_1} \oint_{\partial\Sigma} s_i t^i{}_\nu$$

$$P_\nu(t_1) - P_\nu(t_0) \,\hat{=}\, - \int_{t_0}^{t_1} \oint s_i t^i{}_\nu \rightsquigarrow$$

$t^i{}_\nu$ = energy-momentum current in direction x^i

Angular Momentum

$\varphi(x)$ $\qquad$ $\in$ Lorentz gr. $\qquad$ $\in$ Lorentz algebra

$$\tilde{\varphi}(x) = \varphi(\Lambda x) \xrightarrow{\Lambda \approx 1 + \epsilon\lambda} \varphi(x + \epsilon\lambda x) \approx \varphi(x) + \epsilon \underbrace{\lambda^\mu{}_\nu x^\nu \partial_\mu \varphi}_{\tilde{\delta}\varphi} + O(\epsilon^2)$$

Sym?

$$\tilde{\delta}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \lambda^\mu{}_\nu x^\nu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi)} \partial_\rho (\lambda^\mu{}_\nu x^\nu \partial_\mu \varphi) = \lambda^\mu{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \ldots \right.$$

$$= \left(\frac{d}{dx^\mu} - \frac{\partial}{\partial x^\mu} \right) (\lambda^\mu{}_\nu x^\nu \mathcal{L}) - \lambda^\mu{}_\nu \delta^\nu_\mu \mathcal{L} - \pi^\rho \lambda^\mu{}_\rho \partial_\mu \varphi$$

...lgebra.

$$\ldots = \varphi(x) + \epsilon \underbrace{\lambda^\mu{}_\nu x^\nu \partial_\mu \varphi}_{\tilde{\delta}\varphi} + O(\epsilon^2)$$

$$\ldots(\lambda^\mu{}_\nu x^\nu \partial_\mu \varphi) = \lambda^\mu{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi)} \partial_\rho \partial_\mu \varphi \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi)} \underbrace{\lambda^\mu{}_\nu \delta^\nu_\rho \partial_\mu \varphi}$$

$$\underbrace{\lambda^\mu{}_\nu \delta^\nu_\mu \mathcal{L}}_{\lambda^\mu{}_\mu \equiv 0 \text{ b.c. in } so(1,3)} + \underbrace{\pi^\nu \lambda^\mu{}_\nu \partial_\mu \varphi}$$

same as $\lambda^\mu{}_\nu t^\mu{}_\nu$

$\Rightarrow$ zero if $t^\mu{}_\nu$ is sym!

Angular Momentum

$\in$ Lorentz algebra

$\in$ Lorentz gr.

$\varphi(x)$

$\tilde{\varphi}(x) = \varphi(\Lambda x) \xrightarrow{\Lambda \simeq 1+\epsilon\lambda} \varphi(x + \epsilon\lambda x) \simeq \varphi(x) + \epsilon \underbrace{\lambda^\mu{}_\nu x^\nu \partial_\mu \varphi}_{\tilde{\delta}\varphi} + O(\epsilon^2)$

Sym?

$\tilde{\delta}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \lambda^\mu{}_\nu x^\nu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi)} \partial_\rho (\lambda^\mu{}_\nu x^\nu \partial_\mu \varphi) = \lambda^\mu{}_\nu x^\nu \left(\frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi)} \partial_\rho \ldots \right.$

$+ \underbrace{\pi^\nu \lambda^\mu{}_\nu \partial_\mu \varphi}$

if $\partial_\mu \mathcal{L} = 0$

$= \frac{d}{dx^\mu} \left(\lambda^\mu{}_\nu x^\nu \mathcal{L} \right) - \underbrace{\lambda^\mu{}_\nu \delta^\nu_\mu \mathcal{L}}$

$\lambda^\mu{}_\mu \equiv 0$ b.c. in $so(1,3)$

$\Longrightarrow$ if $\frac{\partial}{\partial x^\mu} \mathcal{L} = 0$ & $t^\mu{}_\nu$ is sym $\Longrightarrow$ $\tilde{\delta}\mathcal{L} = \frac{d}{dx^\mu}(\lambda^\mu{}_\nu x^\nu \mathcal{L})$

∂x^μ ... $\delta \mathcal{L} = \frac{\partial}{\partial x^\mu}$

$$J^\mu = \pi^\mu \lambda^\rho{}_\nu x^\nu \partial_\rho \varphi - \lambda^\mu{}_\nu x^\nu \mathcal{L}$$

$$= \lambda^\rho{}_\nu \left(x^\nu \pi^\mu \partial_\rho \varphi - \delta^\mu_\rho x^\nu \mathcal{L} \right)$$

$$= \lambda^\rho{}_\nu x^\nu t^\mu{}_\rho$$

$[t^{\mu\nu} \text{ sym}]$

$$= \underbrace{\lambda_{\rho\nu}}_{\text{skew}} x^{[\nu} t^{\rho]\mu} \rightsquigarrow \nabla_\mu J^\mu \hat{=} 0 \underset{\forall \lambda \text{ const}}{\rightsquigarrow} \nabla_\mu M^{\mu\nu\rho} \hat{=} 0, \quad M^{\mu\nu\rho} =$$

$\lambda^\rho{}_\nu\, x^\nu\, \partial_\rho \varphi - \lambda^\mu{}_\nu\, x^\nu L$

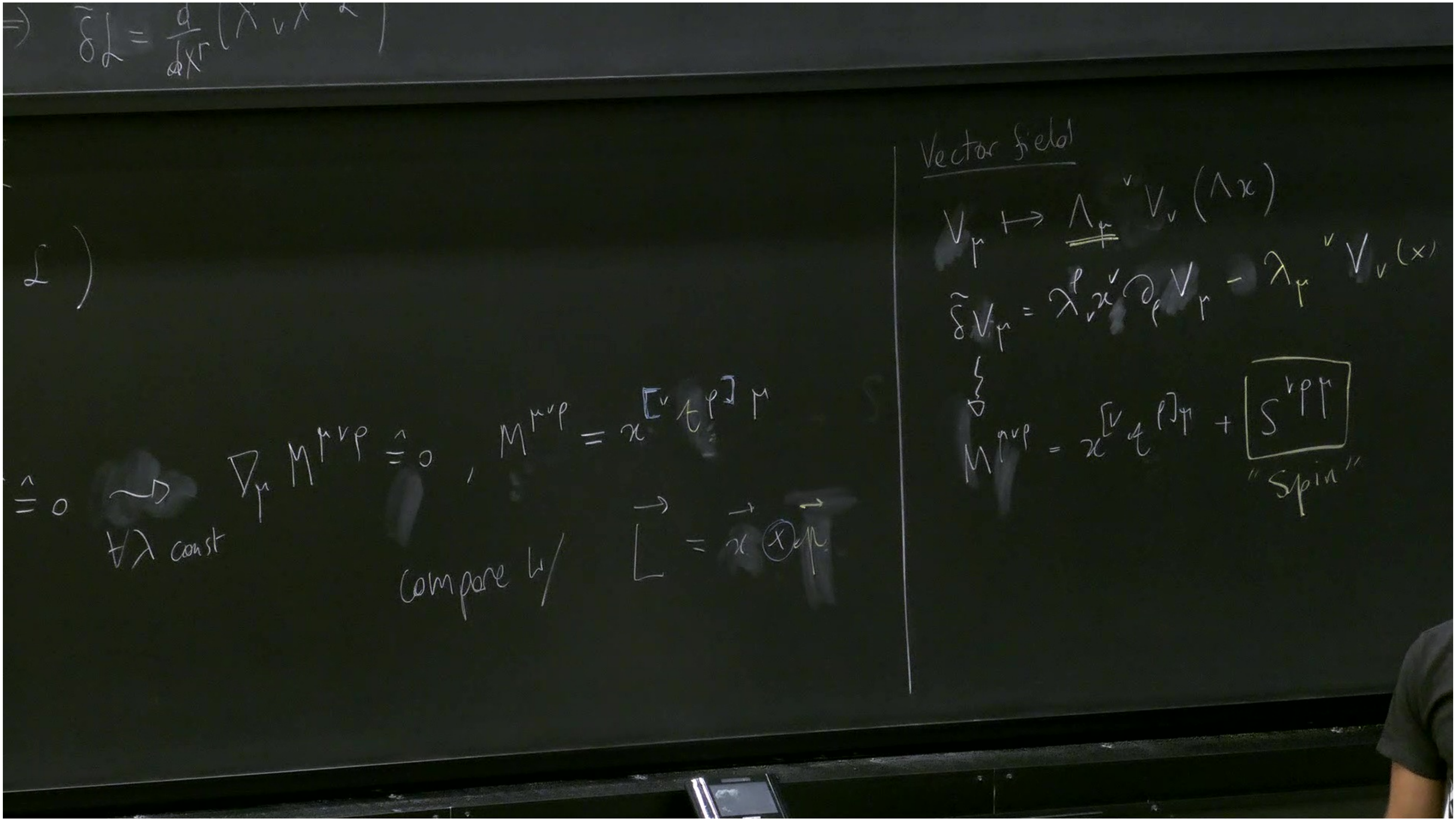
$\pi^\mu \lambda^\rho{}_\nu x^\nu \partial_\rho \varphi$

$\lambda^\rho{}_\nu \left(x^\nu \pi^\mu \partial_\rho \varphi - \delta^\mu_\rho\, x^\nu \mathcal{L} \right)$

$\lambda^\rho{}_\nu\, x^\nu\, t^\mu{}_\rho$

$\underbrace{\lambda_{\rho\nu}}_{\text{skew}}\, x^{[\nu} t^{\rho]\mu} \rightsquigarrow \nabla_\mu J^\mu \hat{=} 0 \underset{\forall \lambda \text{ const}}{\rightsquigarrow} \nabla_\mu M^{\mu\nu\rho} \hat{=} 0, \quad M^{\mu\nu\rho} = x^{[\nu} t^{\rho]\mu}$

compare w/ $\vec{L} = \vec{x} \times \vec{p}$


Vector field
$V_\mu \mapsto \Lambda_\mu{}^\nu V_\nu(\Lambda x)$
$\tilde{\delta} V_\mu = \lambda^\rho{}_\nu x^\nu \partial_\rho V_\mu - \lambda_\mu{}^\nu V_\nu(x)$
$M^{\mu\nu\rho} = x^{[\nu} t^{\rho]\mu} + \boxed{S^{\nu\rho\mu}}$
"Spin"
$\nabla_\mu M^{\mu\nu\rho} \hat{=} 0$, $M^{\mu\nu\rho} = x^{[\nu} t^{\rho]\mu}$
$\forall \lambda$ const
compare w/ $\vec{L} = \vec{x} \times \vec{p}$
$\tilde{\delta} \mathcal{L} = \frac{d}{dx^\mu}(\lambda^\nu{}_\nu x \ldots)$