Title: Channel Expressivity Measures

Speakers: Matthew Duschenes

Collection/Series: Perimeter Institute Graduate Students' Conference 2024

Date: September 12, 2024 - 12:00 PM

URL: https://pirsa.org/24090201

Abstract:

The dynamics of closed quantum systems undergoing unitary processes has been well studied, leading to notions of measures for the expressive power of parameterized quantum circuits, relative to the unique, maximally expressive, average behaviour of ensembles of unitaries. Such unitary expressivity measures have further been linked to concentration phenomena known as barren plateaus. However, existing quantum hardware are not isolated from their noisy environment, and such non-unitary dynamics must therefore be described by more general trace-preserving operations. To account for hardware noise, we propose several, non-unique measures of expressivity for quantum channels and study their properties, highlighting how average non-unitary channels differ from average unitary channels. In the limit of large composite system and environments, average noisy quantum channels are shown to be maximally globally depolarizing, with next-leading-order non-unital perturbative behaviour. Furthermore, we rigorously prove that highly-expressive parameterized quantum channels will suffer from barren plateaus, thus generalizing explanations of noise-induced phenomena. This work is based on forthcoming work with Diego Martin, Zoe Holmes, and Marco Cerezo, in affiliation with Los Alamos National Laboratory.

Channel Expressivity Measures

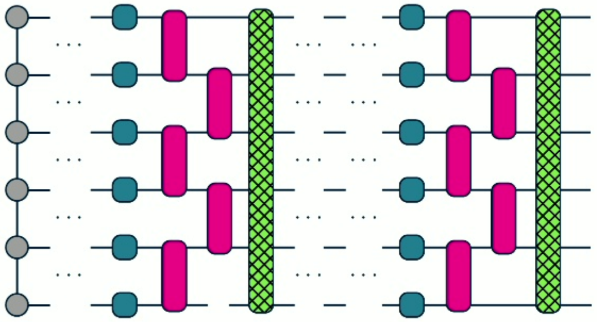
Matthew Duschenes*, Juan Carrasquilla, Raymond Laflamme,
Diego García-Martín, Martín Larocca, Zoë Holmes, Marco Cerezo

University of Waterloo, Institute for Quantum Computing, ETH Zurich, & Los Alamos National Laboratory

PI Graduate Student Conference

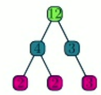
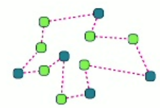
arXiv:2409.XXXXX

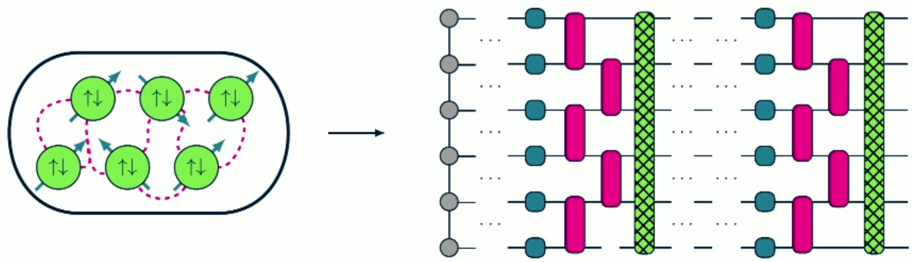
September 12-13, 2024



Quantum Tasks Of Interest

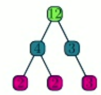
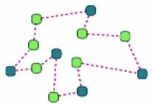
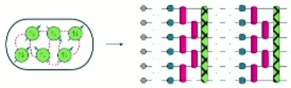
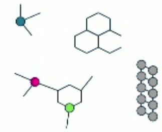
Ultimately, we want to do something *useful* with our quantum devices

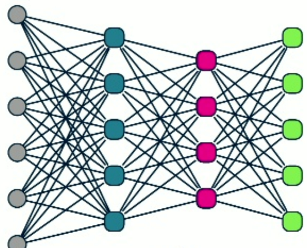
- *Quantum algorithms* i.e) Factoring numbers 
- *Optimization* problems i.e) Travelling Salesman Problem 
- *Compilation* tasks i.e) Form operators U given native gates $\{V\}$



Quantum Tasks Of Interest

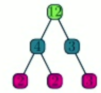
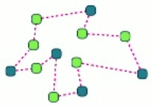
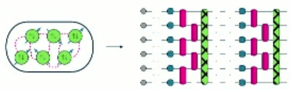
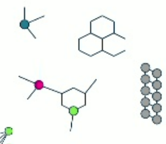
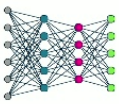
Ultimately, we want to do something *useful* with our quantum devices

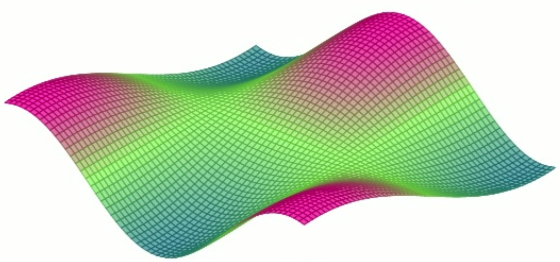
- *Quantum algorithms* i.e) Factoring numbers 
- *Optimization* problems i.e) Travelling Salesman Problem 
- *Compilation* tasks i.e) Form operators U given native gates $\{V\}$ 
- *Simulate* quantum systems i.e) Complicated molecules and chemical reactions 
- *Machine learning* functions i.e) Classification, Regression, Generative



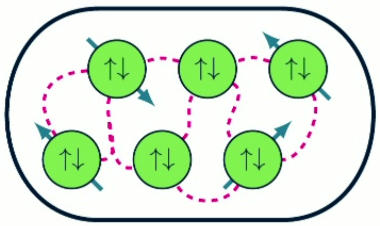
Quantum Tasks Of Interest

Ultimately, we want to do something *useful* with our quantum devices

- *Quantum algorithms* i.e) Factoring numbers 
- *Optimization* problems i.e) Travelling Salesman Problem 
- *Compilation* tasks i.e) Form operators U given native gates $\{V\}$ 
- *Simulate* quantum systems i.e) Complicated molecules and chemical reactions 
- *Machine learning* functions i.e) Classification, Regression, Generative 

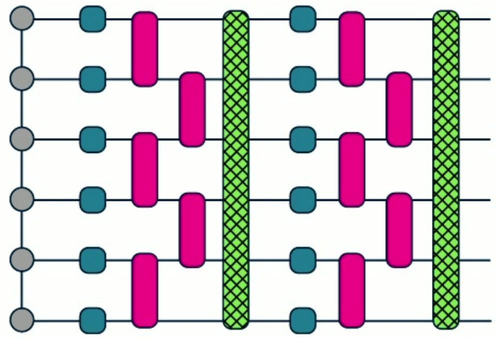


What Are Parameterized Quantum Systems?



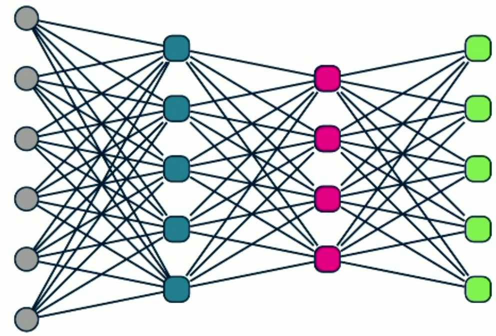
Quantum System

$H_\theta = \sum_\mu \theta_\mu G_\mu$



Quantum Circuit

$U_\theta = \prod_\mu U_\theta^\mu$

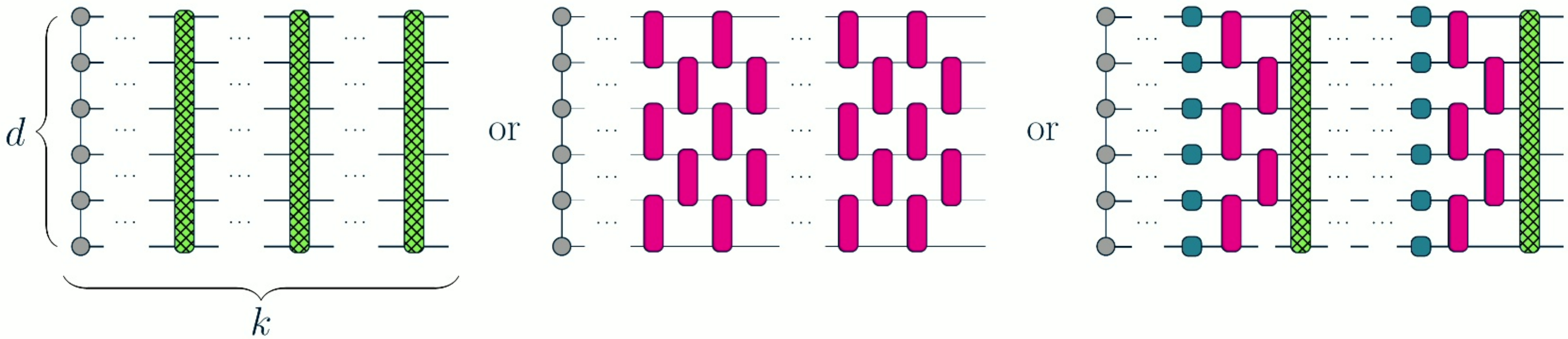


Classical Algorithm

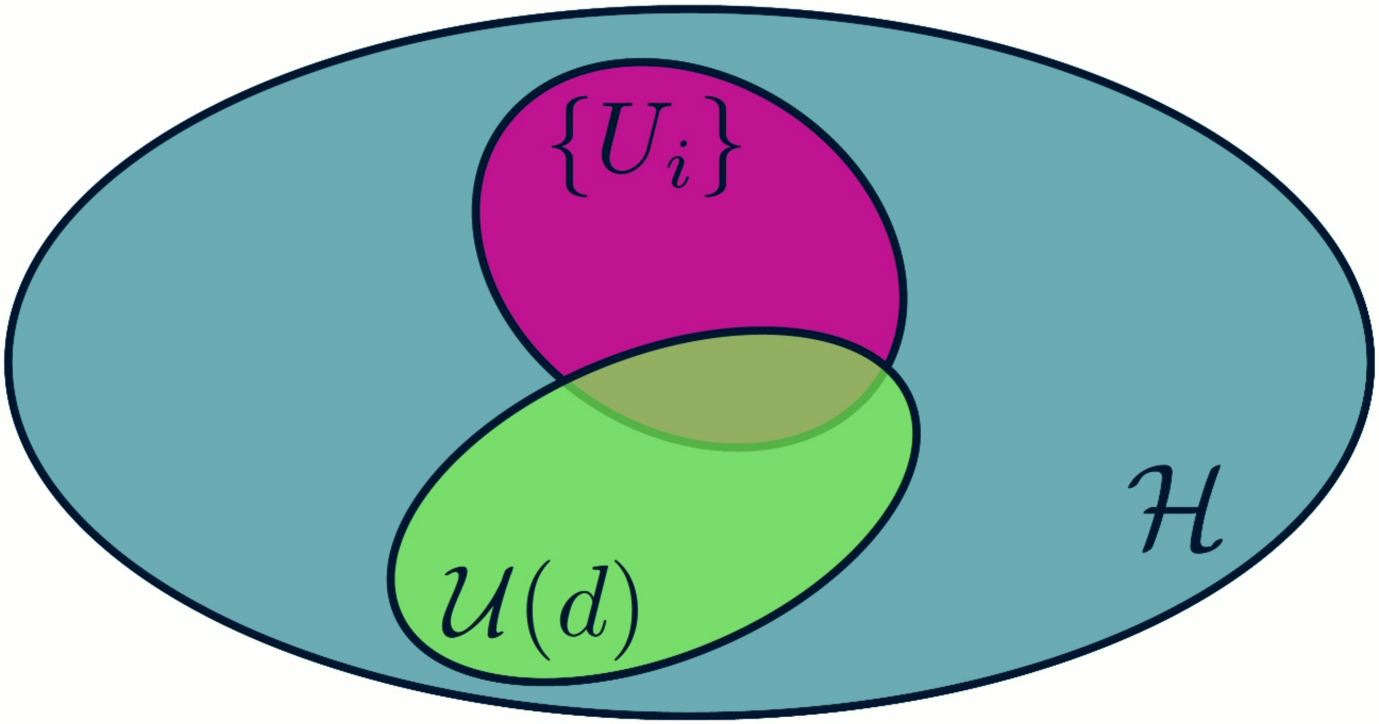
$f_\theta = \circ_\mu f_\theta^\mu$

Tasks of Interest: Unitary Compilation, State Preparation

How Expressive are our Ansatze?

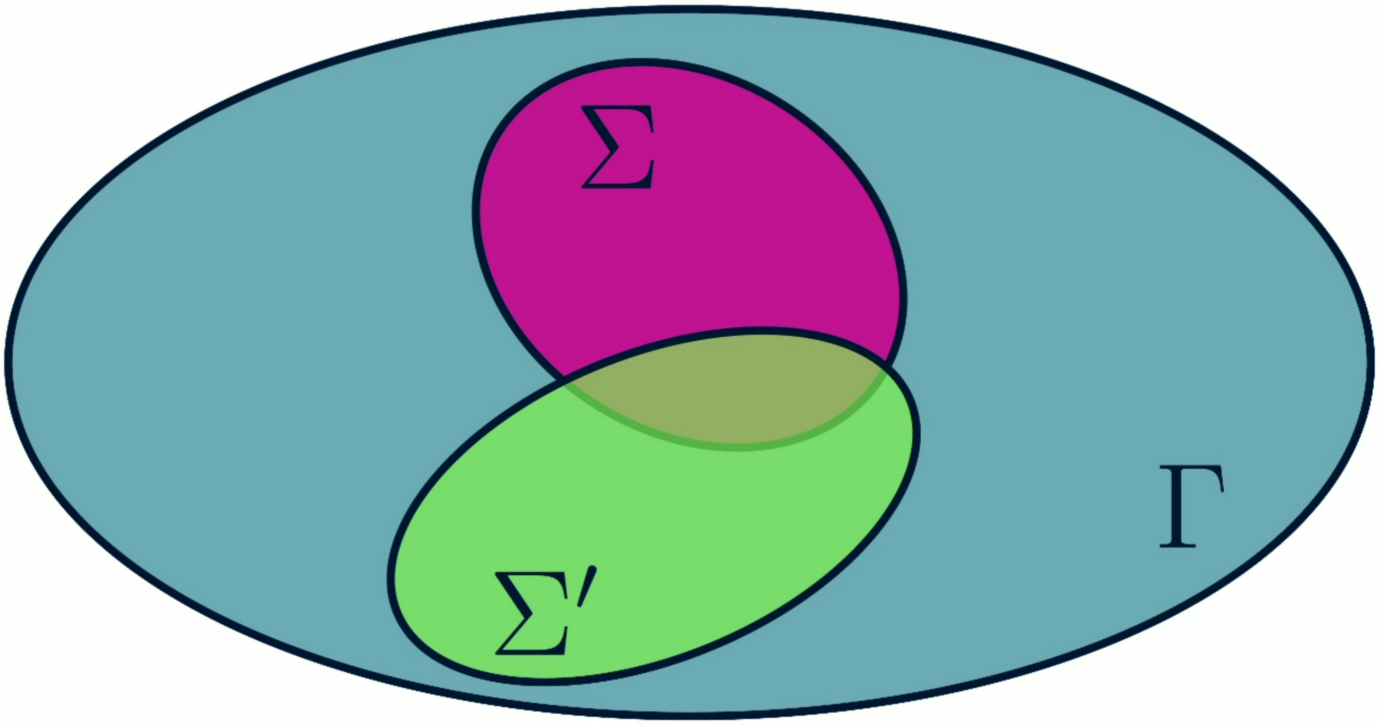


How Expressive are our Ansatze?

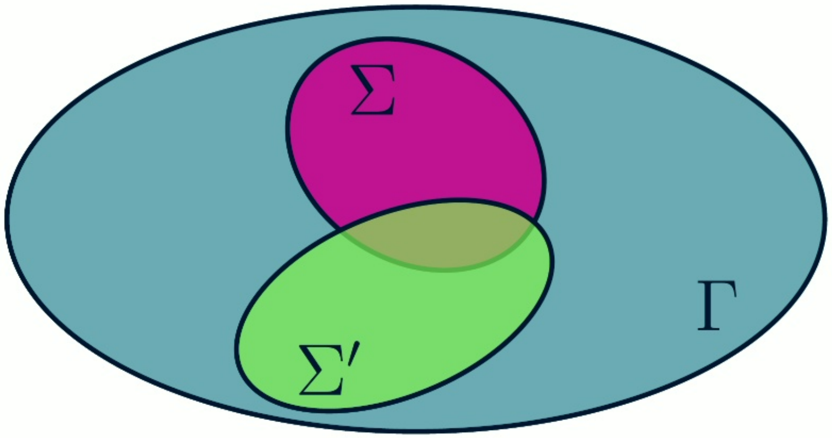


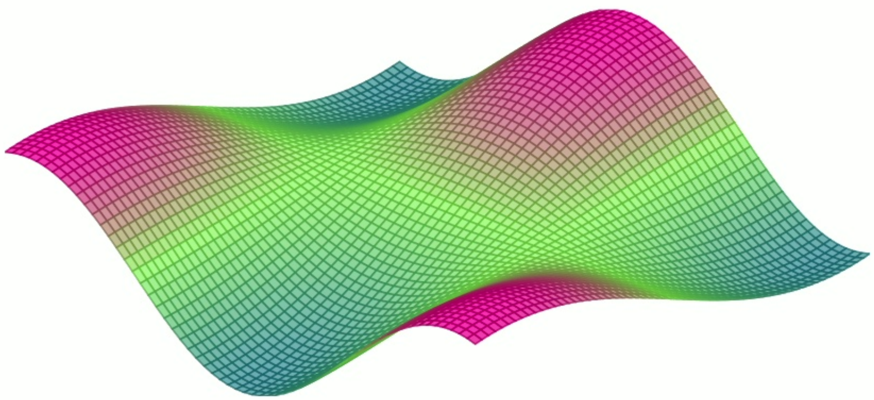
[1] Holmes, Z. *et al.*, *PRX Quantum* **3**, 010313 (2022)

How Expressive are our Ansatze?



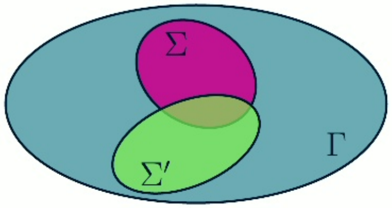
How Expressive are our Ansatze?





How Expressive are our Ansatze?

- Expressivity and trainability of *unitary ansatze* are well understood
- How does an ansatz compare to a *maximally expressive* reference ansatz?
- How do generalized expressivity measures for channels depend on:
  - Noise induced phenomena
  - Underlying (parameterized) unitary evolution
  - Coupling with the environment



Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

$$\mathcal{T}_{\Sigma}^{(t)} = \int_{\Sigma} d\Lambda \; \Lambda^{\otimes t} \tag{1}$$

- This allows us to define an *expressivity* measure between ensembles

$$\mathcal{E}_{\Sigma\Sigma'}^{(t)2} = \|\mathcal{T}_{\Sigma}^{(t)} - \mathcal{T}_{\Sigma'}^{(t)}\|^2 \;\sim\; \|\mathcal{T}_{\Sigma}^{(t)}\|^2 \;-\; \|\mathcal{T}_{\Sigma'}^{(t)}\|^2 \;+\; \cdots \tag{2}$$

- Twirls are crucially *trace-preserving*, with *ensemble-dependent* expansions

$$\mathcal{T}_{\Sigma}^{(t)}(\cdot) \;=\; \boxed{\underbrace{\frac{\mathrm{tr}\,(\cdot)}{d^t} I}_{\text{Depolarizing}}} \;+\; \boxed{\underbrace{\Delta_{\Sigma}^{(t)}(\cdot)}_{\text{Deviations}}} \tag{3}$$

Expressivity Measures

- Let an *ensemble* of channels $\Lambda \sim \Sigma$ over a space $\mathcal{H}$ of dimension d have an average behaviour defined by the t-order *twirl*

$$\mathcal{T}_{\Sigma}^{(t)} = \int_{\Sigma} d\Lambda \; \Lambda^{\otimes t} \tag{1}$$

- This allows us to define an *expressivity* measure between ensembles

$$\mathcal{E}_{\Sigma\Sigma'}^{(t)2} = \|\mathcal{T}_{\Sigma}^{(t)} - \mathcal{T}_{\Sigma'}^{(t)}\|^2 \;\sim\; \|\mathcal{T}_{\Sigma}^{(t)}\|^2 \;-\; \|\mathcal{T}_{\Sigma'}^{(t)}\|^2 \;+\; \cdots \tag{2}$$

- Twirls are crucially *trace-preserving*, with *ensemble-dependent* expansions

$$\mathcal{E}_{\Sigma}^{(t)} = \underbrace{\|\Delta_{\Sigma}^{(t)}\|}_{\text{Deviations}} \tag{4}$$

Reference Ensembles

Channels present several choices for reference ensembles

- *Haar* ~ Unitary Haar measure (uniformly random unitaries)

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}(\rho)} = \int_{\mathcal{U}(d_{\mathcal{H}})} dU\, U^{\otimes t}\, \rho^{\otimes t}\, U^{\otimes t\,\dagger} \tag{5}$$

- *cHaar* ~ Stinespring Unitary Haar measure (random channels) [2]

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}(\rho)} = \text{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(d_{\mathcal{H}} d_{\mathcal{E}})} dU\, U^{\otimes t}\, \rho^{\otimes t} \otimes\, \nu_{\mathcal{E}}^{\otimes t}\, U^{\otimes t\,\dagger}\right) \tag{6}$$

[2] Kukulski, R. *et al.*, *J. Math. Phys.* **62**, 062201 (2021)

Reference Ensembles

Channels present several choices for reference ensembles

- *Haar* $\sim$ Unitary Haar measure (uniformly random unitaries)

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}(\rho)} = \int_{\mathcal{U}(d_{\mathcal{H}})} dU\; U^{\otimes t}\; \rho^{\otimes t}\; U^{\otimes t\,\dagger} \tag{5}$$

- *cHaar* $\sim$ Stinespring Unitary Haar measure (random channels)

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}(\rho)} = \mathrm{tr}_{\mathcal{E}}\left(\int_{\mathcal{U}(d_{\mathcal{H}} d_{\mathcal{E}})} dU\; U^{\otimes t}\; \rho^{\otimes t} \otimes\; \nu_{\mathcal{E}}^{\otimes t}\; U^{\otimes t\,\dagger}\right) \tag{6}$$

- *Depolarizing* $\sim$ Maximally Depolarizing (single channel)

$$\boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}(\rho)} = \frac{\mathrm{tr}\left(\rho^{\otimes t}\right)}{d_{\mathcal{H}}^{t}}\; I^{\otimes t} \tag{7}$$

Behaviour of Random Quantum Channels

The t-order Haar, cHaar, Depolarize ensembles are related by

$$\lim_{d_{\mathcal{E}} \to 1} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\mathrm{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Haar}}}} \qquad \lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}} \to \infty} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\mathrm{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\mathrm{Depolarize}}}} \tag{8}$$

Behaviour of Random Quantum Channels

The t-order Haar, cHaar, Depolarize ensembles are related by

$$\lim_{d_{\mathcal{E}}\to 1} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Haar}}}} \qquad \lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}}\to\infty} \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{cHaar}}}} \to \boxed{\mathcal{T}^{(t)}_{\Sigma_{\text{Depolarize}}}} \tag{8}$$

The k-concatenated, t-order cHaar ensemble is *depolarizing* and *non-unital* [3]

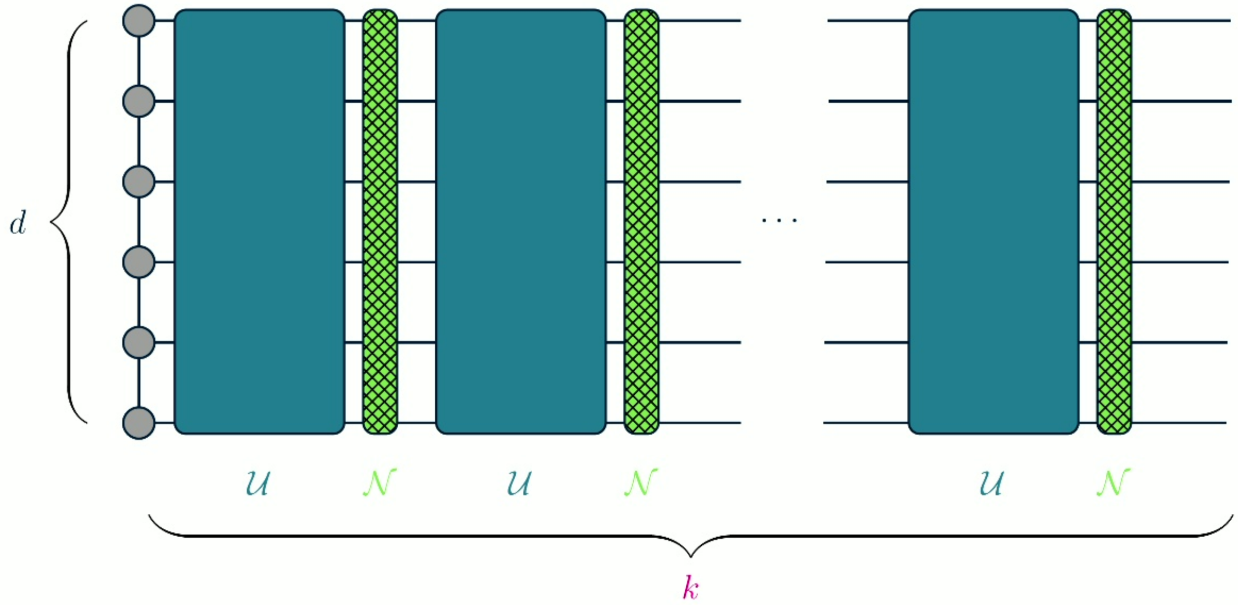
$$\lim_{\frac{d_{\mathcal{H}}}{d_{\mathcal{E}}}\to\infty} \mathcal{T}^{(t)k}_{\Sigma_{\text{cHaar}}}(\rho) = \boxed{\underbrace{\frac{\operatorname{tr}\left(\rho^{\otimes t}\right)}{d^t_{\mathcal{H}}} I^{\otimes t}}_{\text{Depolarize}}} + \boxed{\underbrace{O\left(\frac{1}{d^2_{\mathcal{H}} d_{\mathcal{E}}}\right) \sum_{P \neq I^{\otimes t}} P}_{\text{Non-Unital}}} \tag{9}$$

[3] Bai, J. *et al.*, *Quantum Information Processing* **23**, 1–18 (2024)

Relationships between Noise and Expressivity

Analytical *expressivities* for k layers of specific channel *ansatze*

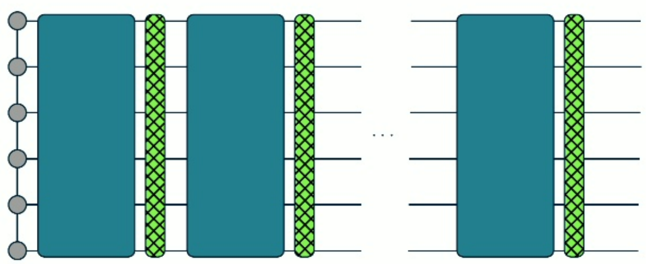
$$\Lambda^{(k)}_{\mathcal{U}\gamma}(\rho) = (\mathcal{N}_\gamma \circ \mathcal{U})^k(\rho) = \frac{\mathrm{tr}\,(\rho)}{d} I + \Delta^{(k)}_\gamma(\rho) \tag{10}$$



Relationships between Noise and Expressivity

Haar Random Unitaries + Fixed *Unital* Pauli Noise: *Increases* Expressivity

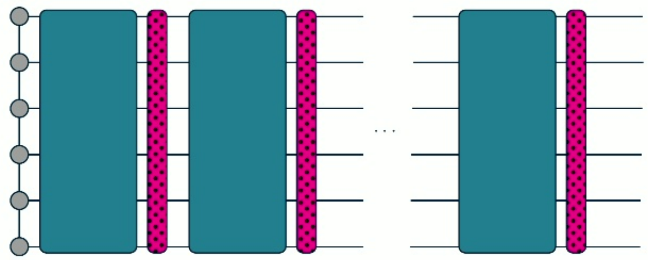
$$\mathcal{E}_{\mathcal{U}_{\gamma}}^{(t,k)} = O\left((1-\gamma)^{2k}\right) \tag{11}$$



Relationships between Noise and Expressivity

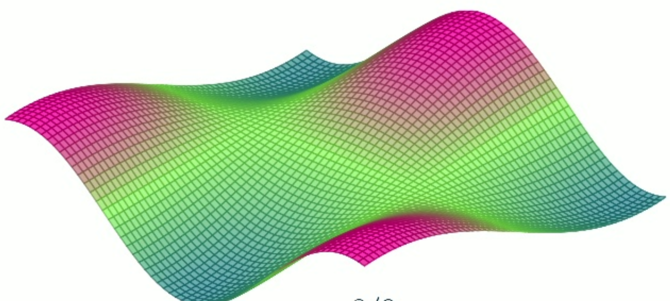
Haar Random Unitaries + Fixed *Non-Unital* Pauli Noise: *Decreases* Expressivity

$$\mathcal{E}^{(t,k)}_{\mathcal{U}\gamma\eta} = O\left(\eta\right) \tag{12}$$



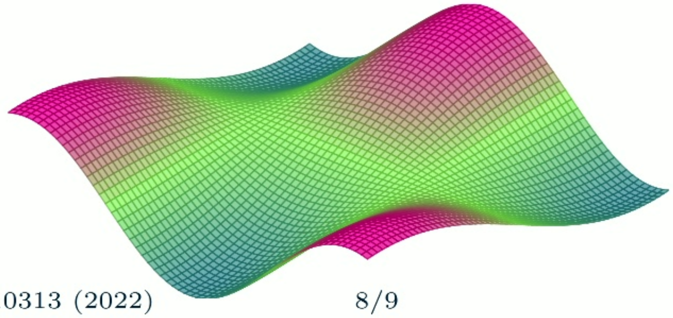
Expressivity versus Trainability

- Ensemble-dependent functions $\mathcal{F}$ may *concentrate* $p\left(|\mathcal{F}-\mu_{\mathcal{F}}| \geq \epsilon\right) \leq \sigma_{\mathcal{F}}^2/\epsilon^2$ (with *caveats* on ensembles, locality, norms, ...)



Expressivity versus Trainability

- Ensemble-dependent functions $\mathcal{F}$ may *concentrate* $p\left(|\mathcal{F}-\mu_{\mathcal{F}}| \geq \epsilon\right) \leq \sigma_{\mathcal{F}}^2/\epsilon^2$ (with *caveats* on ensembles, locality, norms, ...)
- *Objectives* and *gradients* $\mathcal{L} = \mathrm{tr}\left(O\Lambda(\rho)\right) \to \partial\mathcal{L}$ variances may decay [1]



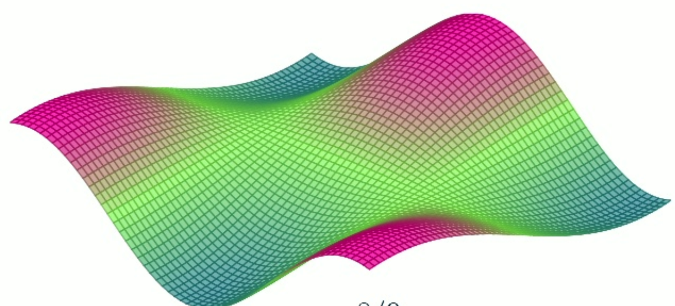
[1] Holmes, Z. *et al.*, *PRX Quantum* **3**, 010313 (2022)

Expressivity versus Trainability

- Ensemble-dependent functions $\mathcal{F}$ may *concentrate* $p\left(|\mathcal{F}-\mu_{\mathcal{F}}| \geq \epsilon\right) \leq \sigma_{\mathcal{F}}^2/\epsilon^2$ (with *caveats* on ensembles, locality, norms, ...)

- *Objectives* and *gradients* $\mathcal{L} = \mathrm{tr}\left(O\Lambda(\rho)\right) \rightarrow \partial\mathcal{L}$ variances may decay

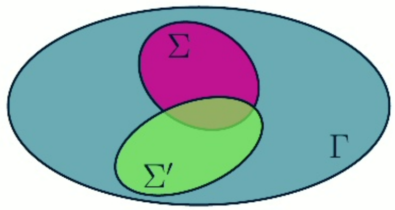
$$\sigma_{\mathcal{L}}^2, \sigma_{\partial\mathcal{L}}^2 \sim \underbrace{O\left(\frac{1}{\mathrm{poly}(d_{\mathcal{H}}, d_{\mathcal{E}})}\right) \|\rho\|_2^2 \, \|O\|_2^2}_{\text{Inherent}} + \underbrace{O\left(\frac{1}{\mathrm{poly}(d_{\mathcal{H}}, d_{\mathcal{E}})}\right) \|\rho\|_p^2 \, \|O\|_q^2 \, \mathcal{E}_{\Sigma\Sigma'}^{(2|p)}}_{\text{Expressivity}} \quad (13)$$



8/9

Operational Meaning of Channel Expressivity Measures

- *Noise induced* phenomena are actually channel *expressivity* phenomena!
- Channel expressivity is more subtly related to *usefulness* or *capability*
- Are there relationships between channel expressivity and their *simulability*? [4]



[4] Mele, A. *et al.*, *arXiv:2403.13927* (2024)