

**Title:** Large charge sector in the theory of a complex scalar field with quartic self-interaction

**Speakers:** Sergio Sanjurjo

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**Abstract:**

The theory  $S = \int \text{d}^{4-\epsilon}x \left( \frac{1}{2} |\partial \phi|^2 - \frac{m^2}{2} |\phi|^2 - \frac{g}{16} |\phi|^4 \right)$  exhibits a global  $U(1)$  symmetry, and the operators  $\phi^n$  ( $\bar{\phi}^n$ ) have charge  $n$  ( $-n$ ) with respect to this symmetry. By rescaling the fields and the coupling constant, it is possible to work in a double limit  $n \rightarrow \infty$ ,  $g \rightarrow 0$  with  $\lambda = gn$  kept constant. In this way, it is possible to compute 2-point functions of the form  $\langle \phi^n(x) \bar{\phi}^n(0) \rangle$  in the large  $n$  limit, either diagrammatically by a resummation of the leading contribution at all orders in  $g$ , or using semiclassical methods through the saddle point approximation. This second approach is particularly powerful because it can also be applied to the theory on a curved background. This allows obtaining the form of the 2-point function for an arbitrary metric, and by functionally differentiating with respect to it, it is also possible to obtain, in the flat theory, the 3-point function  $\langle T^{ij}(z) \phi^n(x) \bar{\phi}^n(0) \rangle$  in which an energy-momentum tensor has been inserted. This allows for a non-trivial check of the conformal symmetry of this sector of the theory by verifying the Ward identities that this 3-point function should satisfy.

# Large charge sector in the theory of a complex scalar with quartic self-interaction

Sergio Sanjurjo

Supervisor: Diego Rodríguez Gómez

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## $|\phi|^4$ theory and global $U(1)$ symmetry

We are going to study the theory with action

$$S = \int d^{4-\epsilon}x \left( \partial\phi\partial\bar{\phi} - \frac{g}{4}(\phi\bar{\phi})^2 \right)$$

The renormalization group flow shows that there exists non trivial  $g = g_{WF}(\epsilon)$  which makes the theory conformal: the Wilson-Fisher fixed point.

### Global $U(1)$ symmetry

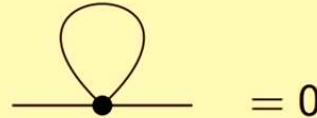
The action is invariant under a transformation

$$\phi(x) \rightarrow e^{i\alpha}\phi(x), \quad \bar{\phi}(x) \rightarrow e^{-i\alpha}\bar{\phi}(x)$$

The operator  $\phi^n(x)$  has charge  $n$  under this  $U(1)$ . Similarly,  $\bar{\phi}^n(x)$  has charge  $-n$ .

## Computing $\langle \phi^n(x) \bar{\phi}^n(0) \rangle$ diagrammatically

To compute  $\langle \phi^n(x) \bar{\phi}^n(0) \rangle$  with  $n \gg 1$  we note that


$$= 0$$

and that there is only one dominant diagram at each order in perturbation theory, which allows a resummation of the full result into an exponential (we denote  $\lambda = gn$ )

$$\langle \phi^n(x) \bar{\phi}^n(0) \rangle = \langle \phi^n(x) \bar{\phi}^n(0) \rangle_0 e^{-i \frac{n\lambda}{4} \mathcal{K}}$$

where

$$\mathcal{K} = \frac{1}{G(0, x)^2} \int d^4 z G(0, z)^2 G(z, x)^2$$

and the free theory 2-point function is

$$\langle \phi^n(x) \bar{\phi}^n(0) \rangle_0 = n! G(0, x)^n$$

## $\langle \phi^n(x) \bar{\phi}^n(0) \rangle$ anomalous dimension and double limit

$\mathcal{K}$  diverges and after regularization with an energy scale  $\mu$  we obtain

$$\mathcal{K} = \frac{-i}{8\pi^2} \log x^2 \mu^2 + o(1)$$

### Large charge 2-point function

$$\langle \phi^n(x) \bar{\phi}^n(0) \rangle = \frac{1}{(4\pi)^{2n} x^{2\Delta_{\phi^n}}}, \quad \Delta_{\phi^n} = n \left( 1 + \frac{\lambda}{32\pi^2} \right)$$

### Double limit

The appropriate limit to keep control over the correction to the dimension is

$$n \rightarrow \infty, \quad g \rightarrow 0, \quad \lambda = gn \text{ constant}$$

## Theory in a curved background

We now consider the same theory in a euclidean setting with a minimal coupling to a metric  $g$ . The action is

$$S^g = \int \sqrt{g} \left[ g^{ij} \partial_i \bar{\phi} \partial_j \phi + \xi R \bar{\phi} \phi + \frac{g}{4} (\bar{\phi} \phi)^2 \right]$$

Using the path integral we can write

$$\langle \phi^n(x) \bar{\phi}^n(0) \rangle^g = Z^{-1} \int \phi^n(x) \bar{\phi}^n(0) e^{-S^g}$$

and this way we can easily insert the energy-momentum tensor

$$\begin{aligned} -2 \frac{\delta}{\delta g_{ij}(z)} \left( Z^{-1} \int e^{-S^g} \right) \Big|_{g=\delta} &= Z^{-1} \int T^{ij}(z) e^{-S^\delta} \\ \langle T^{ij}(z) \phi^n(x) \bar{\phi}^n(0) \rangle &= -2 \frac{\delta}{\delta g_{ij}(z)} \left( \langle \phi^n(x) \bar{\phi}^n(0) \rangle^g \right) \Big|_{g=\delta} \end{aligned}$$

## Computing $\langle \phi^n(x) \bar{\phi}^n(0) \rangle^g$ with a saddle point

The result is

$$\langle \phi^n(x) \bar{\phi}^n(0) \rangle^g = \langle \phi^n(x) \bar{\phi}^n(0) \rangle_0^g e^{-\frac{n\lambda}{4} \mathcal{K}^g}$$

where the free theory 2-point function is

$$\langle \phi^n(x) \bar{\phi}^n(0) \rangle_0^g = n! G^g(0, x)^n$$

We denoted

$$\mathcal{K}^g = \frac{1}{G^g(0, x)^2} \int \sqrt{g} (G_0^g)^2 (G_x^g)^2$$

with the Green function being a solution to

$$(\partial_g^2 - \xi R) G^g(x, y) = -\frac{\delta(x - y)}{\sqrt{g(x)}}$$

## Computing $\langle T^{ij}(z)\phi^n(x)\bar{\phi}^n(0)\rangle$ with a functional derivative

Given that by definition

$$\langle \phi(x)\bar{\phi}(0)\rangle_0^g = G^g(0, x)$$

we have

$$\langle T^{ij}(z)\phi(x)\bar{\phi}(0)\rangle_0 = -2 \left. \frac{\delta G^g(0, x)}{\delta g_{ij}(z)} \right|_{g=\delta}$$

We also denote

$$D^{ij}(z, x, 0) = -2 \left. \frac{\delta \mathcal{K}^g(0, x)}{\delta g_{ij}(z)} \right|_{g=\delta}$$

Large charge 3-point function with a  $T^{ij}(z)$  insertion

$$\frac{\langle T^{ij}(z)\phi^n(x)\bar{\phi}^n(0)\rangle}{\langle \phi^n(x)\bar{\phi}^n(0)\rangle} = n \left( \frac{\langle T^{ij}(z)\phi(x)\bar{\phi}(0)\rangle_0}{\langle \phi(x)\bar{\phi}(0)\rangle_0} - \frac{\lambda}{4} D^{ij}(z, x, 0) \right)$$

## Checking the Ward identities

Given that we have a 3-point function with an energy-momentum tensor we can make a non-trivial check of the conformal invariance of the theory using the Ward identities.

### Ward identities

Translational symmetry implies

$$\partial_i^z \langle T^{ij}(z) \phi^n(x_1) \bar{\phi}^n(x_2) \rangle = [\delta(z - x_1) \partial_{x_1}^j + \delta(z - x_2) \partial_{x_2}^j] \langle \phi^n(x_1) \bar{\phi}^n(x_2) \rangle$$

Scale invariance implies

$$\delta_{ij} \langle T^{ij}(z) \phi^n(x_1) \bar{\phi}^n(x_2) \rangle = \Delta_{\phi^n} [\delta(z - x_1) + \delta(z - x_2)] \langle \phi^n(x_1) \bar{\phi}^n(x_2) \rangle$$

The computed 3-point function does not satisfy the Ward identity associated to scale invariance. Instead of the anomalous dimension  $\Delta_{\phi^n}$  there is only a factor  $n$ : quantum corrections are missing.

## Adding counterterms to get contact terms

A reasonable solution is to add by hand the term

$$\frac{\delta^{ij}}{d} \frac{n\lambda}{32\pi^2} [\delta(z - x_1) + \delta(z - x_2)] \langle \phi^n(x_1) \bar{\phi}^n(x_2) \rangle$$

to the computed 3-point function.

### Generalizing the source terms

Contact terms appear in the 3-point function of operators  $\mathcal{O}_i$  of dimensions  $d$ ,  $\Delta$  and  $\Delta$  when computed via functional derivatives of sources after adding the following term to the action

$$S_J = \sum_{i=1}^3 \int J^i \mathcal{O}_i - c \int J^1 (J^2 \mathcal{O}_2 + J^3 \mathcal{O}_3)$$

This means that the “wrong” result we obtained is not a true conformal anomaly: it can be reabsorbed in counterterms.

Thank you!

