Title: Large charge sector in the theory of a complex scalar field with quartic self-interaction

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Collection/Series: Perimeter Institute Graduate Students' Conference 2024

Date: September 13, 2024 - 2:35 PM

URL: https://pirsa.org/24090200

Abstract:

The theory $S = \int\text{d}^{4-\epsilon}x\left(\frac{1}{2}|\partial\phi|^2 - \frac{m^2}{2}|\phi|^2-\frac{g}{16}|\phi|^4\right)$ exhibits a global $U(1)$ symmetry, and the operators ϕ^n ($\bar\phi^n$) have charge n ($-n$) with respect to this symmetry. By rescaling the fields and the coupling constant, it is possible to work in a double limit $n\to\infty$, $g\to 0$ with $\lambda = gn$ kept constant. In this way, it is possible to compute 2-point functions of the form $\langle \phi^n(x) \bar\phi^n(0) \rangle$ in the large n limit, either diagrammatically by a resummation of the leading contribution at all orders in g, or using semiclassical methods through the saddle point approximation. This second approach is particularly powerful because it can also be applied to the theory on a curved background. This allows obtaining the form of the 2-point function for an arbitrary metric, and by functionally differentiating with respect to it, it is also possible to obtain, in the flat theory, the 3-point function $\langle T^{ij}(z) \phi^n(x) \bar \phi^n(0) \rangle$ in which an energy-momentum tensor has been inserted. This allows for a non-trivial check of the conformal symmetry of this sector of the theory by verifying the Ward identities that this 3-point function should satisfy.

Large charge sector in the theory of a complex scalar with quartic self-interaction

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September 12th 2024

$|\phi|^4$ theory and global $U(1)$ symmetry

We are going to study the theory with action

$$S = \int d^{4-\epsilon}x \left(\partial\phi\partial\bar{\phi} - \frac{g}{4}(\phi\bar{\phi})^2 \right)$$

The renormalization group flow shows that there exists non trivial $g = g_{WF}(\epsilon)$ which makes the theory conformal: the Wilson-Fisher fixed point.

Global $U(1)$ symmetry

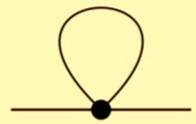
The action is invariant under a transformation

$$\phi(x) \to e^{i\alpha}\phi(x), \quad \bar{\phi}(x) \to e^{-i\alpha}\bar{\phi}(x)$$

The operator $\phi^n(x)$ has charge n under this $U(1)$. Similarly, $\bar{\phi}^n(x)$ has charge $-n$.

Computing $\langle \phi^n(x)\bar{\phi}^n(0)\rangle$ diagrammatically

To compute $\langle \phi^n(x)\bar{\phi}^n(0)\rangle$ with $n \gg 1$ we note that



$$= 0$$

and that there is only one dominant diagram at each order in perturbation theory, which allows a resummation of the full result into an exponential (we denote $\lambda = gn$)

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle = \langle \phi^n(x)\bar{\phi}^n(0)\rangle_0 \, e^{-i\frac{n\lambda}{4}\mathcal{K}}$$

where

$$\mathcal{K} = \frac{1}{G(0,x)^2}\int \mathrm{d}^4 z \, G(0,z)^2 G(z,x)^2$$

and the free theory 2-point function is

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle_0 = n!\, G(0,x)^n$$

$\langle \phi^n(x)\bar{\phi}^n(0)\rangle$ anomalous dimension and double limit

$\mathcal{K}$ diverges and after regularization with an energy scale μ we obtain

$$\mathcal{K} = \frac{-i}{8\pi^2}\log x^2\mu^2 + o(1)$$

Large charge 2-point function

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle = \frac{1}{(4\pi)^{2n}x^{2\Delta_{\phi^n}}}, \quad \Delta_{\phi^n} = n\left(1 + \frac{\lambda}{32\pi^2}\right)$$

Double limit

The appropriate limit to keep control over the correction to the dimension is

$$n \to \infty, \quad g \to 0, \quad \lambda = gn \;\; \text{constant}$$

Theory in a curved background

We now consider the same theory in a euclidean setting with a minimal coupling to a metric g. The action is

$$S^g = \int \sqrt{g} \left[g^{ij}\partial_i\bar{\phi}\partial_j\phi + \xi R\bar{\phi}\phi + \frac{g}{4}(\bar{\phi}\phi)^2 \right]$$

Using the path integral we can write

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle^g = Z^{-1} \int \phi^n(x)\bar{\phi}^n(0) e^{-S^g}$$

and this way we can easily insert the energy-momentum tensor

$$-2\frac{\delta}{\delta g_{ij}(z)} \left(Z^{-1} \int e^{-S^g} \right)\Big|_{g=\delta} = Z^{-1} \int T^{ij}(z) e^{-S^\delta}$$

$$\langle T^{ij}(z)\phi^n(x)\bar{\phi}^n(0)\rangle = -2\frac{\delta}{\delta g_{ij}(z)} \left(\langle \phi^n(x)\bar{\phi}^n(0)\rangle^g \right)\Big|_{g=\delta}$$

Computing $\langle \phi^n(x)\bar{\phi}^n(0)\rangle^g$ with a saddle point

The result is

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle^g = \langle \phi^n(x)\bar{\phi}^n(0)\rangle^g_0 \, e^{-\frac{n\lambda}{4}\mathcal{K}^g}$$

where the free theory 2-point function is

$$\langle \phi^n(x)\bar{\phi}^n(0)\rangle^g_0 = n! \, G^g(0,x)^n$$

We denoted

$$\mathcal{K}^g = \frac{1}{G^g(0,x)^2}\int \sqrt{g}(G^g_0)^2(G^g_x)^2$$

with the Green function being a solution to

$$(\partial_g^2 - \xi R)G^g(x,y) = -\frac{\delta(x-y)}{\sqrt{g(x)}}$$

Computing $\langle T^{ij}(z)\phi^n(x)\bar{\phi}^n(0)\rangle$ with a functional derivative

Given that by definition

$$\langle\phi(x)\bar{\phi}(0)\rangle_0^g = G^g(0,x)$$

we have

$$\langle T^{ij}(z)\phi(x)\bar{\phi}(0)\rangle_0 = -2\,\frac{\delta\, G^g(0,x)}{\delta\, g_{ij}(z)}\bigg|_{g=\delta}$$

We also denote

$$D^{ij}(z,x,0) = -2\,\frac{\delta\,\mathcal{K}^g(0,x)}{\delta\, g_{ij}(z)}\bigg|_{g=\delta}$$

Large charge 3-point function with a $T^{ij}(z)$ insertion

$$\frac{\langle T^{ij}(z)\phi^n(x)\bar{\phi}^n(0)\rangle}{\langle\phi^n(x)\bar{\phi}^n(0)\rangle} = n\left(\frac{\langle T^{ij}(z)\phi(x)\bar{\phi}(0)\rangle_0}{\langle\phi(x)\bar{\phi}(0)\rangle_0} - \frac{\lambda}{4}D^{ij}(z,x,0)\right)$$

Checking the Ward identities

Given that we have a 3-point function with an energy-momentum tensor we can make a non-trivial check of the conformal invariance of the theory using the Ward identities.

Ward identities

Translational symmetry implies

$$\partial_i^z \langle T^{ij}(z)\phi^n(x_1)\bar{\phi}^n(x_2)\rangle = [\delta(z-x_1)\partial^j_{x_1} + \delta(z-x_2)\partial^j_{x_2}]\langle \phi^n(x_1)\bar{\phi}^n(x_2)\rangle$$

Scale invariance implies

$$\delta_{ij}\langle T^{ij}(z)\phi^n(x_1)\bar{\phi}^n(x_2)\rangle = \Delta_{\phi^n}[\delta(z-x_1)+\delta(z-x_2)]\langle \phi^n(x_1)\bar{\phi}^n(x_2)\rangle$$

The computed 3-point function does not satisfy the Ward identity associated to scale invariance. Instead of the anomalous dimension Δ_{ϕ^n} there is only a factor n: quantum corrections are missing.

Adding counterterms to get contact terms

A reasonable solution is to add by hand the term

$$\frac{\delta^{ij}}{d}\frac{n\lambda}{32\pi^2}\left[\delta(z-x_1)+\delta(z-x_2)\right]\langle\phi^n(x_1)\bar{\phi}^n(x_2)\rangle$$

to the computed 3-point function.

Generalizing the source terms

Contact terms appear in the 3-point function of operators $\mathcal{O}_i$ of dimensions d, Δ and Δ when computed via functional derivatives of sources after adding the following term to the action

$$S_J=\sum_{i=1}^{3}\int J^i\mathcal{O}_i-C\int J^1\left(J^2\mathcal{O}_2+J^3\mathcal{O}_3\right)$$

This means that the "wrong" result we obtained is not a true conformal anomaly: it can be reabsorbed in counterterms.

Thank you!