Title: Violation of Bell's inequality in continuous variable systems

Speakers: Gurpahul Singh

Collection/Series: Perimeter Institute Graduate Students' Conference 2024

Date: September 12, 2024 - 9:40 AM

URL: https://pirsa.org/24090195

Abstract:

Violations of Bell’s inequality have been studied for spin-1/2 systems in much detail. Turns out that one can show Bell violation for systems that are expressed in terms of continuous variables such as position and momentum. The most ubiquitous examples of such systems are Gaussian states, notably the two-mode squeezed vacuum state. I will talk about how one can quantify violations of local realism in such states. I will discuss the dependence of Bell violation on temperature as well as the result that entanglement is not a monotonic function of Bell's inequality.

Violation of Bell's Inequality
in
Continuous variable systems

By Gurpahul Singh

Supervisor – Eduardo Martín-Martínez
Collaborator – Kelly Wurtz

PERIMETER SCHOLARS INTERNATIONAL

12th September, 2024

Outline

- Literature review
- Original work

- Introduction and Motivation
- Gaussian Quantum Mechanics - a review
- Entanglement in a two-mode squeezed (TMS) thermal state
- Bell's inequality for continuous variable systems
- Bell operator for a TMS thermal state
- Conclusions and Outlook

Bell's Inequality

Spin-1/2 space

(discrete)

Bell (1964)

$$\boxed{|\langle\hat{B}\rangle| = |E(\hat{n}_1, \hat{n}_2) + E(\hat{n}_1, \hat{n}_2') + E(\hat{n}_1', \hat{n}_2) - E(\hat{n}_1', \hat{n}_2')| \leq 2}$$

$$E(\hat{n}_1, \hat{n}_2) = \langle(\hat{n}_1 \cdot \vec{\sigma}_1) \otimes (\hat{n}_2 \cdot \vec{\sigma}_2)\rangle$$

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

Bell's Inequality

Spin-1/2 space

(discrete)

Bell (1964)

$$\boxed{|\langle \hat{B} \rangle| = |E(\hat{n}_1, \hat{n}_2) + E(\hat{n}_1, \hat{n}_2') + E(\hat{n}_1', \hat{n}_2) - E(\hat{n}_1', \hat{n}_2')| \leq 2}$$

$$E(\hat{n}_1, \hat{n}_2) = \langle (\hat{n}_1 \cdot \vec{\sigma}_1) \otimes (\hat{n}_2 \cdot \vec{\sigma}_2) \rangle$$

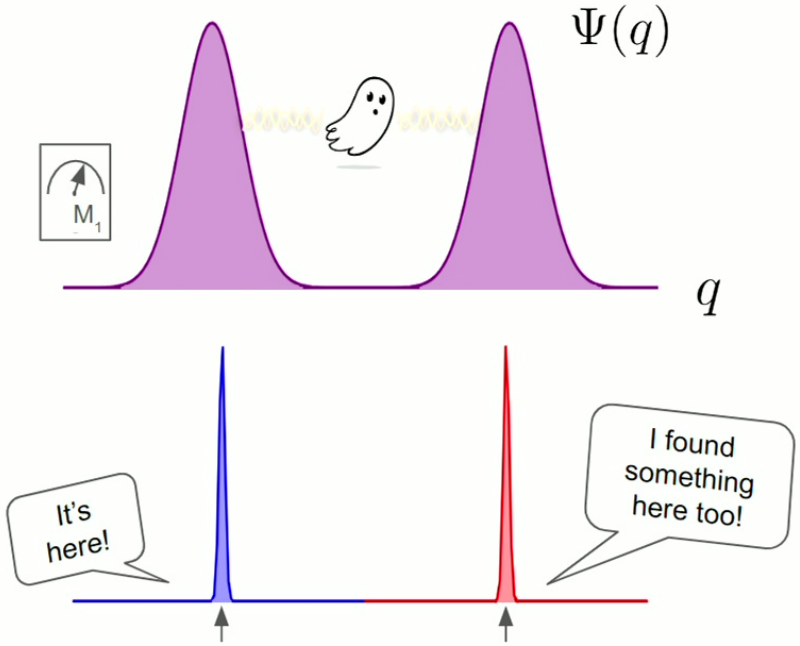
$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$\langle \hat{B} \rangle \leq 2 \implies$ Classical

$2 < \langle \hat{B} \rangle \leq 2\sqrt{2} \implies$ Quantum

4

- How do we compute Bell's inequality in the original EPR setting, i.e, for continuous variable systems?
- How do we extend the Bell's inequality to mixed states, particularly thermal states? What is the effect of temperature on Bell violation?
- What is the correlation of Bell's inequality violation with entanglement in such systems?



Einstein, Podolsky, Rosen (1935)

Chapter II

Gaussian Quantum Mechanics

(What kind of continuous variable (CV) systems will be used?)

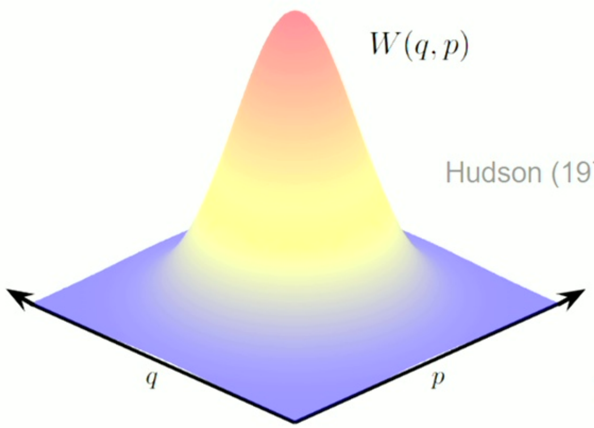
The Wigner quasi-probability distribution

$\hat{O} \in$ {Operators in Hilbert Space} $\xrightarrow[\text{Transform}]{\text{Wigner}}$ $O(q,p) \in$ {Functions in phase space}

$\hat{\rho}$ Density operator $\xrightarrow[\text{Transform}]{\text{Wigner}}$ $W(q,p)$ Wigner function

Wigner (1932)

Special CV system – Gaussian states



Hudson (1974)

- States that possess a Gaussian Wigner function. Being non-negative it represents true probability distribution.
- Fully characterized by their first and second moments. Thus, no need to deal with infinite dimensional Hilbert spaces!

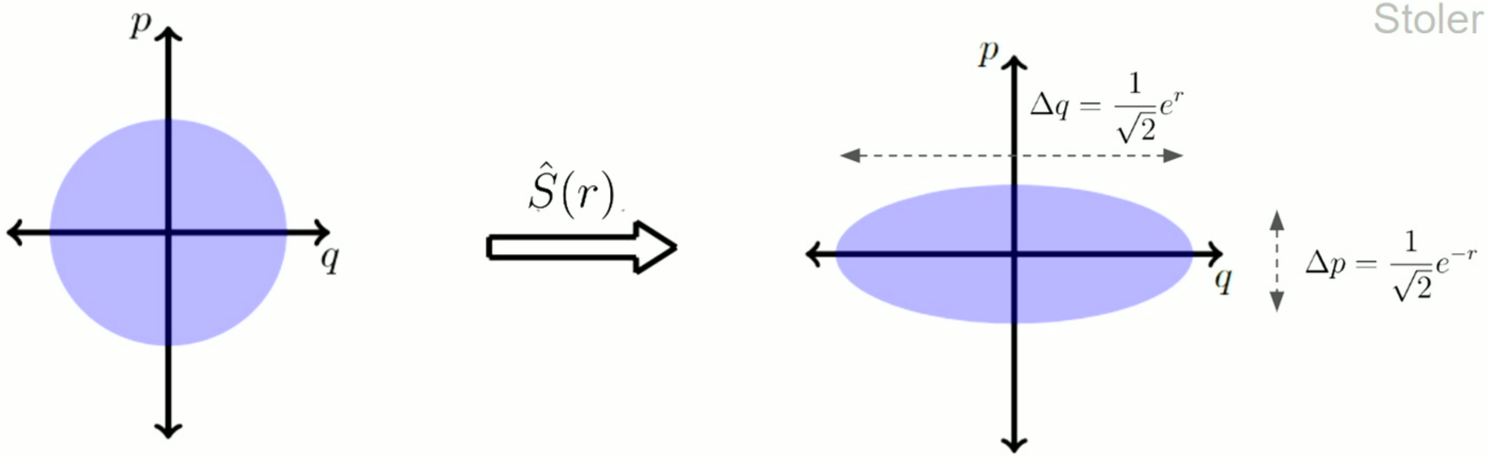
7

Special Gaussian state – Single-mode squeezed state

$$\hat{S}(r) = \exp[r(\hat{a}^{\dagger 2} - \hat{a}^2)] \quad \longrightarrow \quad \text{squeezing operator}$$

$r \rightarrow$ squeezing parameter

Stoler (1970)



Vacuum state $|0\rangle$

Squeezed vacuum state $\hat{S}(r)|0\rangle$

Two-mode squeezed vacuum (TMSV) state

- A pure bipartite entangled Gaussian state obtained by squeezing the vacuum state of two Harmonic oscillators.

$$\hat{S}(r) = \exp[r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})]$$

$$|\mathrm{TMSV}\rangle = \hat{S}(r)|0\rangle|0\rangle = \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \tanh^n(r)|n\rangle|n\rangle$$

- Entanglement increases as r increases. Lu (1974), Caves (1982).

But how do we quantify entanglement?

Entanglement in Gaussian states

$$\hat{\rho} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow{\text{Partial transpose}} \hat{\rho}^{T_2} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

Negativity is a measure of entanglement given by

$$\mathcal{N} = \sum_{i, \lambda_i < 0} |\lambda_i|$$

(unbounded from above)

where λ_i are the eigenvalues of $\hat{\rho}^{T_2}$. Logarithmic negativity:

$$E_N = \log_2(2\mathcal{N} + 1)$$

Życzkowski *et al* (1998),

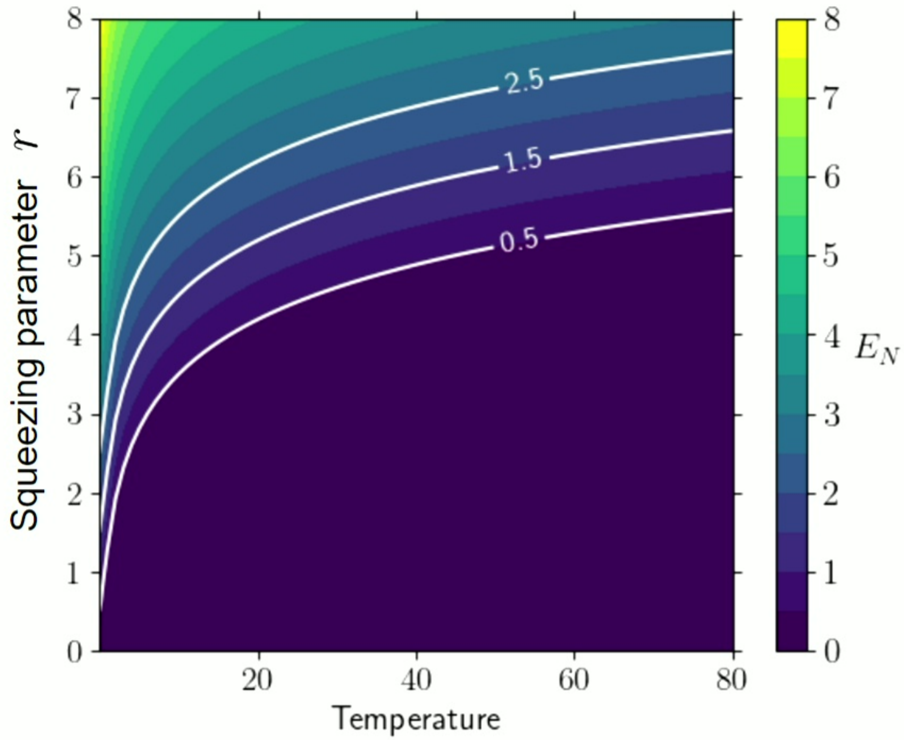
Logarithmic negativity can be used for mixed states unlike entanglement entropy.

Chapter III

Entanglement in a TMS thermal state

Log negativity for the TMS thermal state.

At T=0, for any r > 0, there is entanglement



$r \uparrow \Rightarrow E_N \uparrow$

$T \uparrow \Rightarrow E_N \downarrow$

Chapter IV

Bell's inequality for continuous variable systems

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$
$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$\sigma_y = -\mathrm{i}(|0\rangle\langle 1| - |1\rangle\langle 0|)$$

Pseudo-spin operators

Fock space representation

$$\hat{s}_z = \sum_{n=0}^{\infty} \{|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|\}$$

$$\hat{s}_x = \sum_{n=0}^{\infty} \{|2n+1\rangle\langle 2n| + |2n\rangle\langle 2n+1|\}$$

$$\hat{s}_y = -\mathrm{i}\sum_{n=0}^{\infty} \{|2n+1\rangle\langle 2n| - |2n\rangle\langle 2n+1|\}$$

Chen *et al* (2002)

$$|n\rangle \longrightarrow \boxed{\text{Change of basis}} \longrightarrow |q\rangle$$

Pseudo-spin operators

$$\hat{\Pi}_z = -\int_{-\infty}^{\infty} \mathrm{d}q\, |q\rangle\langle -q|$$

Position space representation

$$\hat{\Pi}_x = \int_0^{\infty} \mathrm{d}q\ \{|q\rangle\langle q| - |-q\rangle\langle -q|\}$$

Gour *et al* (2004)

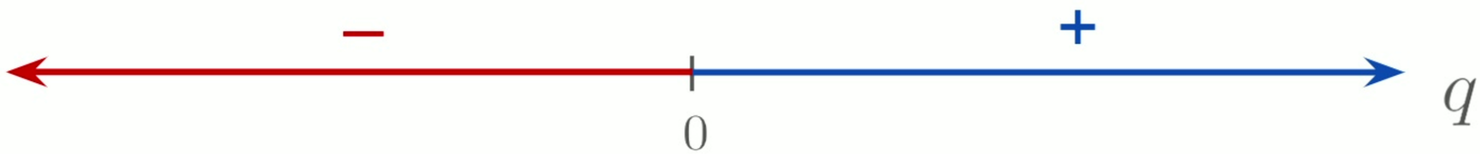
$$\hat{\Pi}_y = -\mathrm{i}\int_0^{\infty} \mathrm{d}q\ \{|q\rangle\langle -q| - |-q\rangle\langle q|\}$$

$$[\hat{\Pi}_i, \hat{\Pi}_j] = 2\mathrm{i}\epsilon_{ij}{}^k\hat{\Pi}_k \qquad \hat{\Pi}_i^2 = \mathbb{I}$$

Note: $\hat{\Pi}_z = \hat{s}_z \qquad \hat{\Pi}_{x,y} \neq \hat{s}_{x,y}$

Generalized pseudo-spin operators

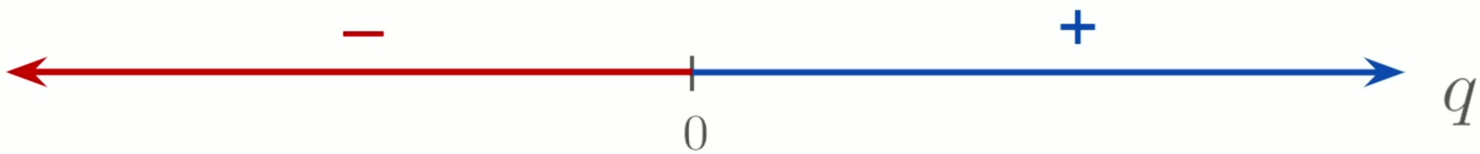
$$\hat{\Pi}_x = \int_0^\infty \mathrm{d}q \ \{|q\rangle\langle q| - |-q\rangle\langle -q|\} = \mathrm{sgn}(\hat{q})$$



Larsson (2003)

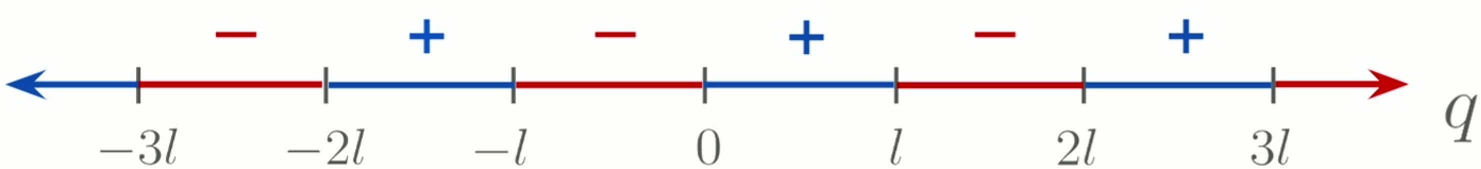
Generalized pseudo-spin operators

$$\hat{\Pi}_x = \int_0^\infty \mathrm{d}q \; \{|q\rangle\langle q| - |-q\rangle\langle -q|\} = \mathrm{sgn}(\hat{q})$$

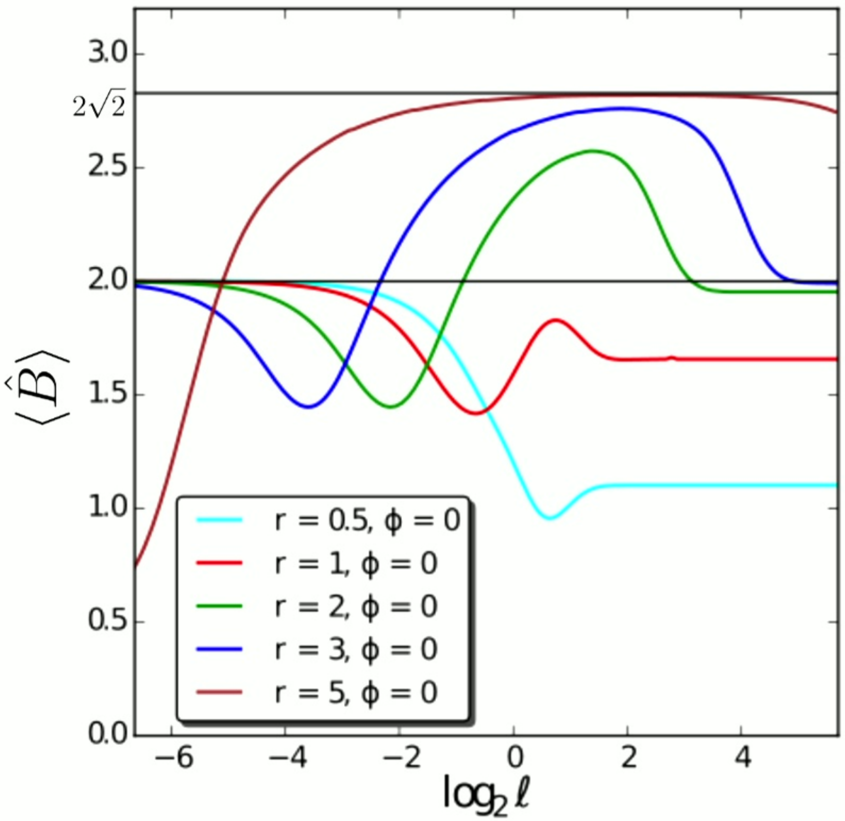


$$\hat{S}_z(l) = \sum_{-\infty}^{\infty} (-1)^n \int_{nl}^{(n+1)l} \mathrm{d}q \; |q\rangle\langle q|$$

Larsson (2003)



Bell violation for TMSV state with generalized pseudo-spin operators



$$\langle\hat{B}\rangle = |E(\hat{n}_1,\hat{n}_2) + E(\hat{n}_1,\hat{n}_2') + E(\hat{n}_1',\hat{n}_2) - E(\hat{n}_1',\hat{n}_2')|$$

$$E(\hat{n}_1,\hat{n}_2) = \langle(\hat{n}_1 \cdot \vec{S}^{(1)}(l)) \otimes (\hat{n}_2 \cdot \vec{S}^{(2)}(l))\rangle$$

Using the most optimized ‘angles’ for measurement

$$\langle\hat{B}\rangle = 2\sqrt{\left\langle \hat{S}_z^{(1)}(l)\hat{S}_z^{(2)}(l)\right\rangle^2 + \left\langle \hat{S}_x^{(1)}(l)\hat{S}_x^{(2)}(l)\right\rangle^2}$$

Martin and Vennin (2016)

Chapter V

Bell's inequality for a TMS thermal state

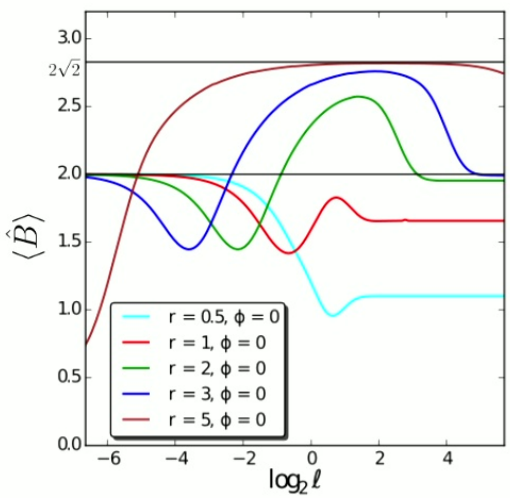
Bell operator for <u>mixed states</u> using generalized pseudo-spin operators

$$\hat{S}_x(l), \hat{S}_y(l), \hat{S}_z(l) \xrightarrow[\text{Transform}]{\text{Wigner}} S_x(l)(q,p), S_y(l)(q,p), S_z(l)(q,p)$$

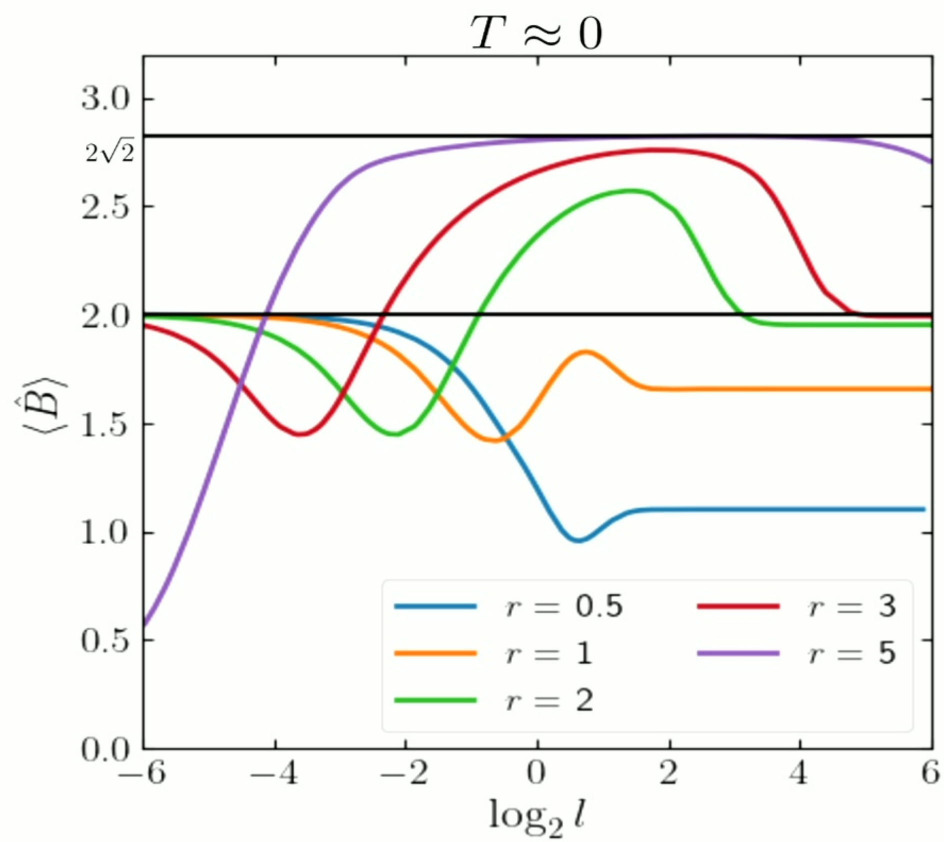
Thus, we performed the whole calculation of the correlation functions in phase space instead of Hilbert space. In this way, our protocol was generalized to mixed states.

Using this, we calculate the Bell operator for TMS thermal state.

$$\langle \hat{B} \rangle = 2\sqrt{\left\langle \hat{S}_z^{(1)}(l)\hat{S}_z^{(2)}(l) \right\rangle^2 + \left\langle \hat{S}_x^{(1)}(l)\hat{S}_x^{(2)}(l) \right\rangle^2}$$



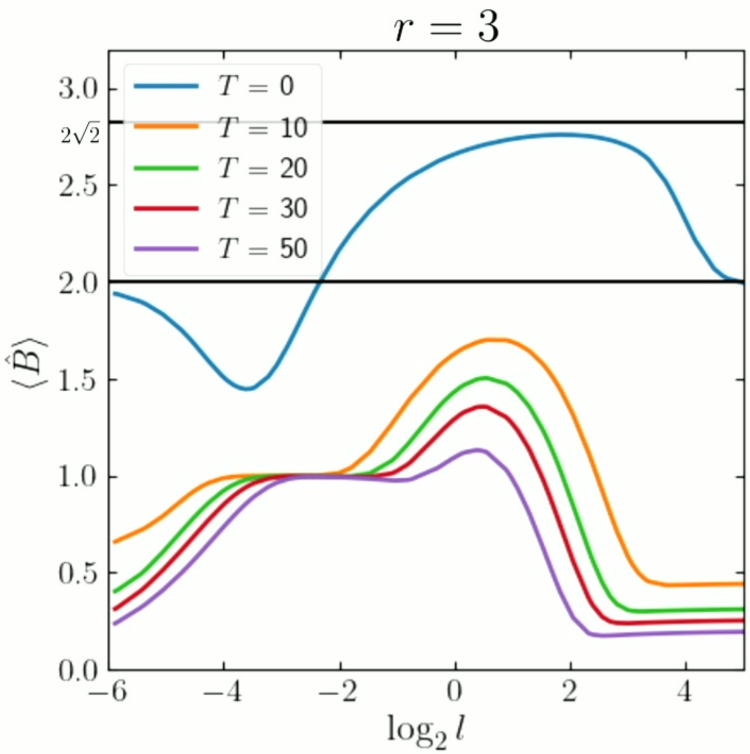
Martin and Vennin (2016)



Simulated by taking the limit $T \to 0$

20

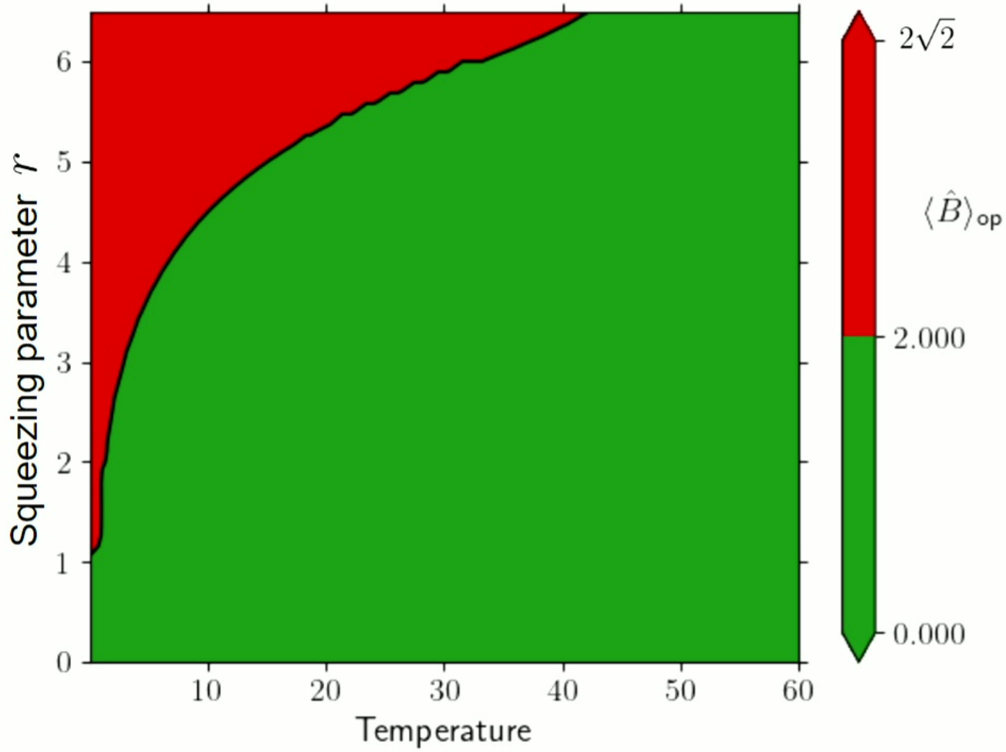
Dependence on Temperature



As T increases, the Bell value decreases drastically, moving out of the violation region.

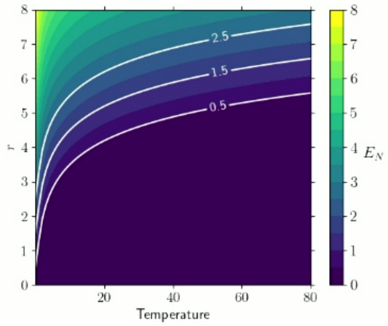
Optimized Bell operator as a function of r and T

$\langle \hat{B} \rangle_{\mathrm{op}}$ denotes the value of $\langle \hat{B} \rangle$ at the peak of the 'bump' (optimized bin-size)



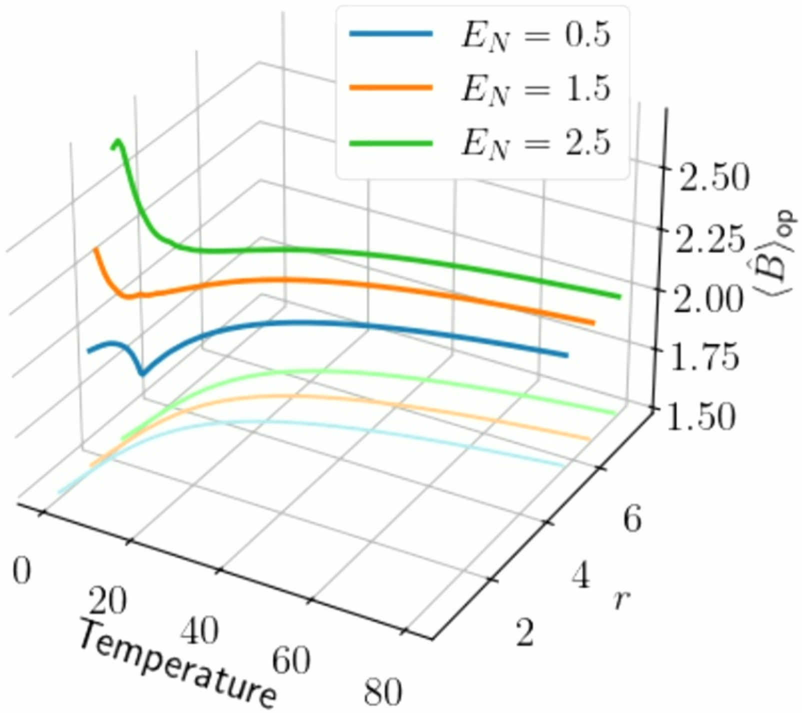
$r \uparrow \Rightarrow \langle \hat{B} \rangle_{\mathrm{op}} \uparrow$

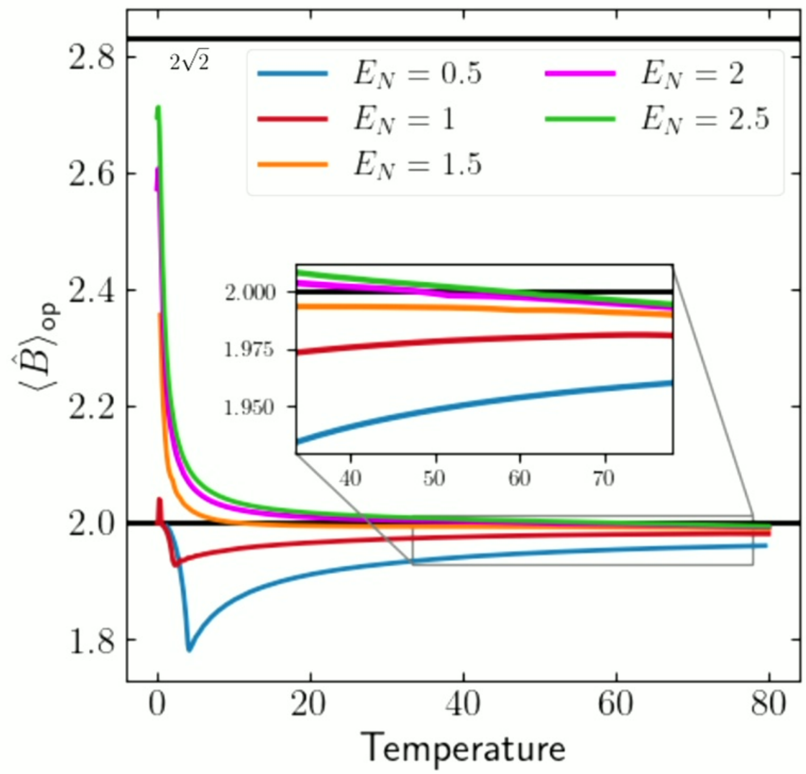
$T \uparrow \Rightarrow \langle \hat{B} \rangle_{\mathrm{op}} \downarrow$



Bell operator at equal entanglement

Bell violation is not a monotonic function of entanglement.





Conclusion

- We extended the pseudo-spin operator approach for calculating Bell's inequality to mixed states.

- Using the same, we computed the Bell violation for a TMS thermal state and analyzed its dependence on squeezing and temperature.

- We found that in this framework, the violation of Bell's inequality is not a monotonic function of entanglement.

Thank You!