Title: The spin-statistics theorem for TFTs

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Abstract: In quantum field theory (QFT) the spin-statistics theorem says that in a unitary QFT, a particle has half-integer spin if and only if it is a fermion. I show how to phrase this statement in the language of functorial field theories. More precisely, I explain when a functorial field theory "has fermions" and "has spinors" and when they are "related". I will then restrict to topological field theories (TFTs) and define unitary TFTs. There are counterexamples of the spin-statistics theorem for non-unitary TFTs. I will prove that every unitary TFT satisfies the spin-statistics theorem.

Spin-Statistics Theorem for TQFTs

- spin-statistics connection: integer spin $\Longleftrightarrow$ bosonic statistics
  - ——— theorem: every unitary QFT has a spin-statistics connection

$$Y^{n-1} \longmapsto \mathcal{H}(Y) \qquad \mathcal{H}(\emptyset) = \mathbb{C}$$

space (closed spin mfld) — states $\in$ sVect$_{\mathbb{C}}$ (graded by $(-1)^F$)

$$X^n : Y_1 \rightsquigarrow Y_2 \longmapsto Z(X) : \mathcal{H}(Y_1) \longrightarrow \mathcal{H}(Y_2)$$

compact spacetime (spin cobordism) — $(-1)^F$-preserving linear map

$$X : \emptyset \to \emptyset \longmapsto Z(X) \in \mathbb{C}$$

partition function

$$Z(Y_1 \sqcup Y_2) \simeq Z(Y_1) \otimes Z(Y_2)$$

$$\|$$

$$Z(Y_2 \sqcup Y_1) \simeq Z(Y_2) \otimes Z(Y_1)$$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_2 \otimes \mathcal{H}_1$$

statistics $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$

super vector space

$\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_2 \otimes \mathcal{H}_1$

$v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v$

Spin spin structures

$\rightarrow$ Def A "fermionic TQFT" of spacetime dimension n is a sym monoidal functor

$$(Bord_n^{spin}, \sqcup) \longrightarrow (sVect, \otimes)$$

$Y^{n-1} \longmapsto \mathcal{H}(Y) \qquad \mathcal{H}(\emptyset) = \mathbb{C}$

space (closed spin mfd) $\qquad$ states $\in \mathrm{sVect}_{\mathbb{C}}$ (graded by $(-1)^F$)

$X^n : Y_1 \rightsquigarrow Y_2 \longmapsto Z(X) : \mathcal{H}(Y_1) \longrightarrow \mathcal{H}(Y_2)$ $\quad (-1)^F$-preserving linear map

compact spacetime (spin cobordism)

$X : \emptyset \longrightarrow \emptyset \longmapsto Z(X) \in \mathbb{C}$ $\quad$ partition function

$$Z(Y_1 \sqcup Y_2) \simeq Z(Y_1) \otimes Z(Y_2)$$

$$(\mathrm{Bord}_n^{\mathrm{spin}}, \sqcup) \longrightarrow (\mathrm{sVect}, \otimes)$$

$$Cl_{+n} = \frac{\mathbb{R}[\gamma^1, \ldots, \gamma^n]}{\left(\gamma^i\gamma^j + \gamma^j\gamma^i = 0,\ (\gamma^i)^2 = 1\right)}$$
$i \neq j$

$\supseteq$ generated by norm 1 vectors $\sum a_i \gamma^i$

$\mathbb{Z}/2$ $\supseteq$ Spin

$\text{Pin}^+(n)$

$\mathbb{Z}/2 \leftarrow O(n) \leftarrow SO$

<u>Recall</u> $\text{Spin}(3)$ has irreps V_s labeled by $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$

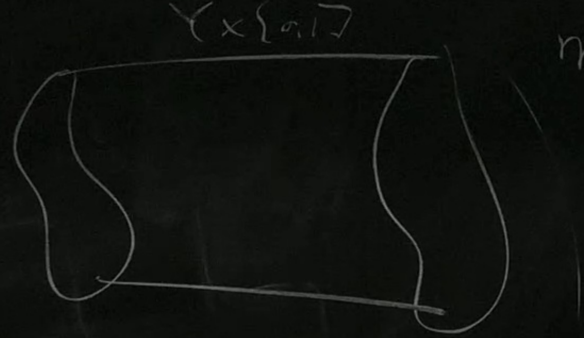
V_s descends to $SO(3) \Longleftrightarrow s \in \mathbb{Z}$

$\Rightarrow$ irrep of $\text{Spin}(n)$ has half-integer spin if $(-1)^{2s}$ acts as -1

$$Z(Y_1 \sqcup Y_2) \simeq Z(Y_1) \otimes Z(Y_2)$$
$$\|$$
$$Z(Y_2 \sqcup Y_1) \simeq Z(Y_2) \otimes Z(Y_1)$$

If Z is a fermionic TQFT,

get a $\mathbb{Z}/2$-rep $\quad Z\left((-1)^{2s}_Y\right) : \mathcal{H}(Y) \longrightarrow \mathcal{H}(Y)$



mapping cylinder of

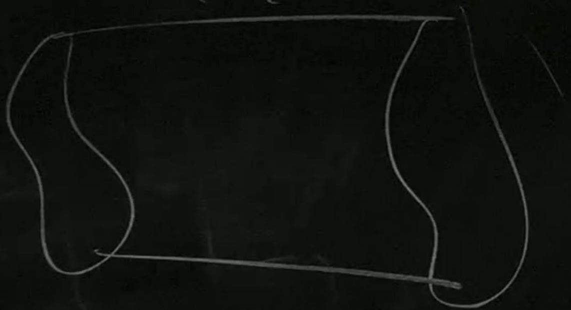
$(-1)^{2s}$. $\longleftarrow$ "$B\mathbb{Z}/2$-action on Bo

$Z(Y_2 \amalg Y_1) \cong Z(Y_2) \otimes Z(Y_1)$

a fermionic TQFT,

$\mathbb{Z}/2$-rep $\quad Z((-1)^{2S}_Y) : \mathcal{H}(Y) \longrightarrow \mathcal{H}(Y)$

$Y \times \{0,1\}$



mapping cylinder of

$(-1)^{2S}$. $\longleftarrow$ "$B\mathbb{Z}/2$-action on $\mathrm{Bord}_n^{\mathrm{Spin}}$"

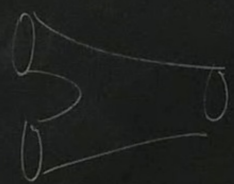
Def Z has spin-st connection if $\forall Y^{n-1}$ $Z((-1)^{2S}_Y) = (-1)^F_{\mathcal{H}(Y)}$

Ex dim 1 $Z(\bullet^+) = \mathbb{C}$ $Z(\overset{(-1)^{2S}}{\bullet\!\!-\!\!\!-\!\!\!|\!\!\!-\!\!\!-\!\!\bullet}) = -1$ $\mathrm{Bord}_1^{Spin} = \mathrm{Bord}_1^{SO\times\mathbb{Z}_2}$

does not have a spin-statistics connection

(Note if $g^2=1$ bosonic symmetry, then $Z(\overset{g}{\bullet\!\!-\!\!\!-\!\!\!|\!\!\!-\!\!\!-\!\!\bullet}) = -1$ has a spin-st)

Idea: incorporate unitarity by

$Z\Big($  $\Big) : \mathcal{H}(Y_1) \longrightarrow \mathcal{H}(Y_2)$

$Z\Big($  $\Big)^{\dagger} : \mathcal{H}(Y_2) \longrightarrow \mathcal{H}(Y_1)$

$\overset{?}{\parallel}$

$Z\Big($  $\Big)$

$\Rightarrow$ Def A Hermitian

Idea: incorporate unitarity by

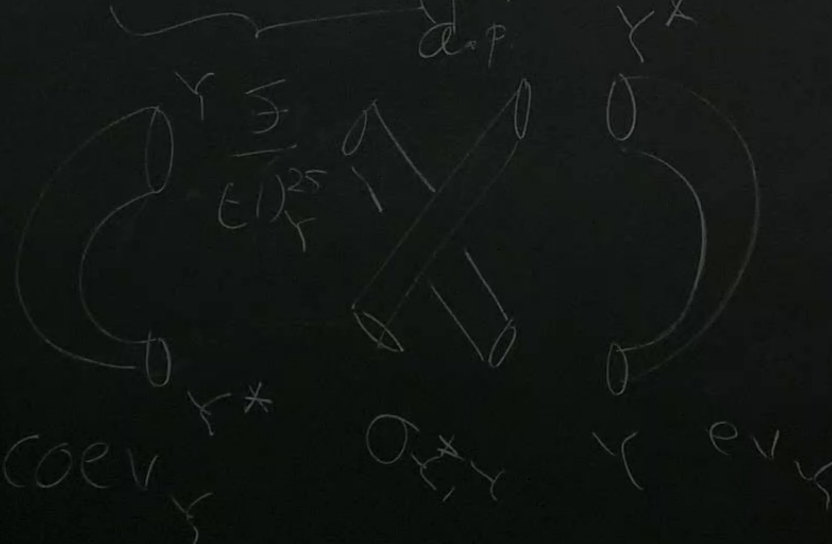
$$Z(\quad) : \mathcal{H}(Y_1) \longrightarrow \mathcal{H}(Y_2)$$

$$Z(\quad)^\dagger : \mathcal{H}(Y_2) \longrightarrow \mathcal{H}(Y_1)$$

$\overset{?}{=}$ ← unitary

$$Z(\quad)$$

$$Z(\underbrace{Y^{n-1}}_{\text{per a.p.}} \times S^1) = \mathrm{tr}_S(\mathrm{id}_{\mathcal{H}(Y)}) = \mathrm{tr}$$



$= \mathrm{tr}(S)$

$\mathrm{tr}_S(Z((-1)$

Lemma $\forall Y \quad Z($

$$\mathrm{tr}_S\left(Z((-1)^{2S}_Y) \right) = \mathrm{tr}_{\mathcal{H}_{ung}} \left((-1)^F_{\mathcal{H}(Y)} Z((-1)^{2S}_Y) \right)$$

$$= \mathrm{tr}(S)$$

<u>Lemma</u> $\forall Y \quad Z(Y \times S^1_{a.p.}) = \dim_{ung} \mathcal{H}(Y)$

$\Uparrow$

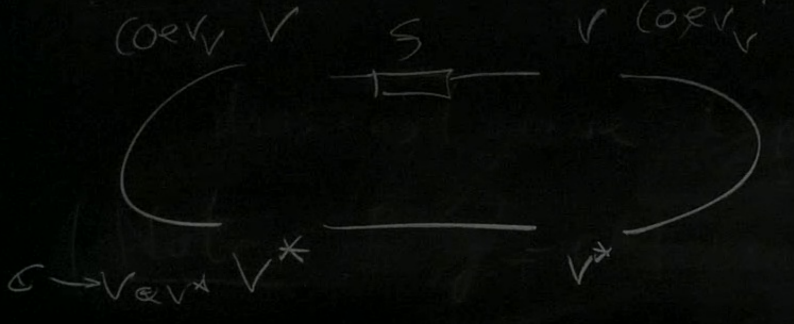
spin-statistics connection

Y

<u>Proof</u> $(-1)^F_{\mathcal{H}(Y)}$ & $Z((-1)^{2S}_Y)$ commute $\Rightarrow$ common eigenbasis

$sHilb^{fd}$ is a sym mon †-cat. $sHilb \xrightarrow{(-)^\dagger} sHilb^{op}$

with a dual functor



$| \; | \longmapsto$ "id_V"

$$tr^\dagger(S) = tr_{ungraded}(S)$$

$$coev_V^\dagger = ev_V \circ \left(id_{V^*} \otimes (-1)^{F_V} \right)$$

ion $\dagger$-cat. $\mathrm{sHilb} \longrightarrow \mathrm{sHilb}$ $\quad V \cong V$

ctor

$ev_V^\dagger$

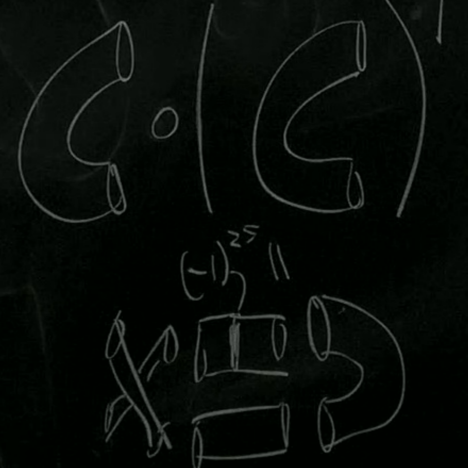
$$tr^\dagger(S) = tr_{\text{ungraded}}(S)$$

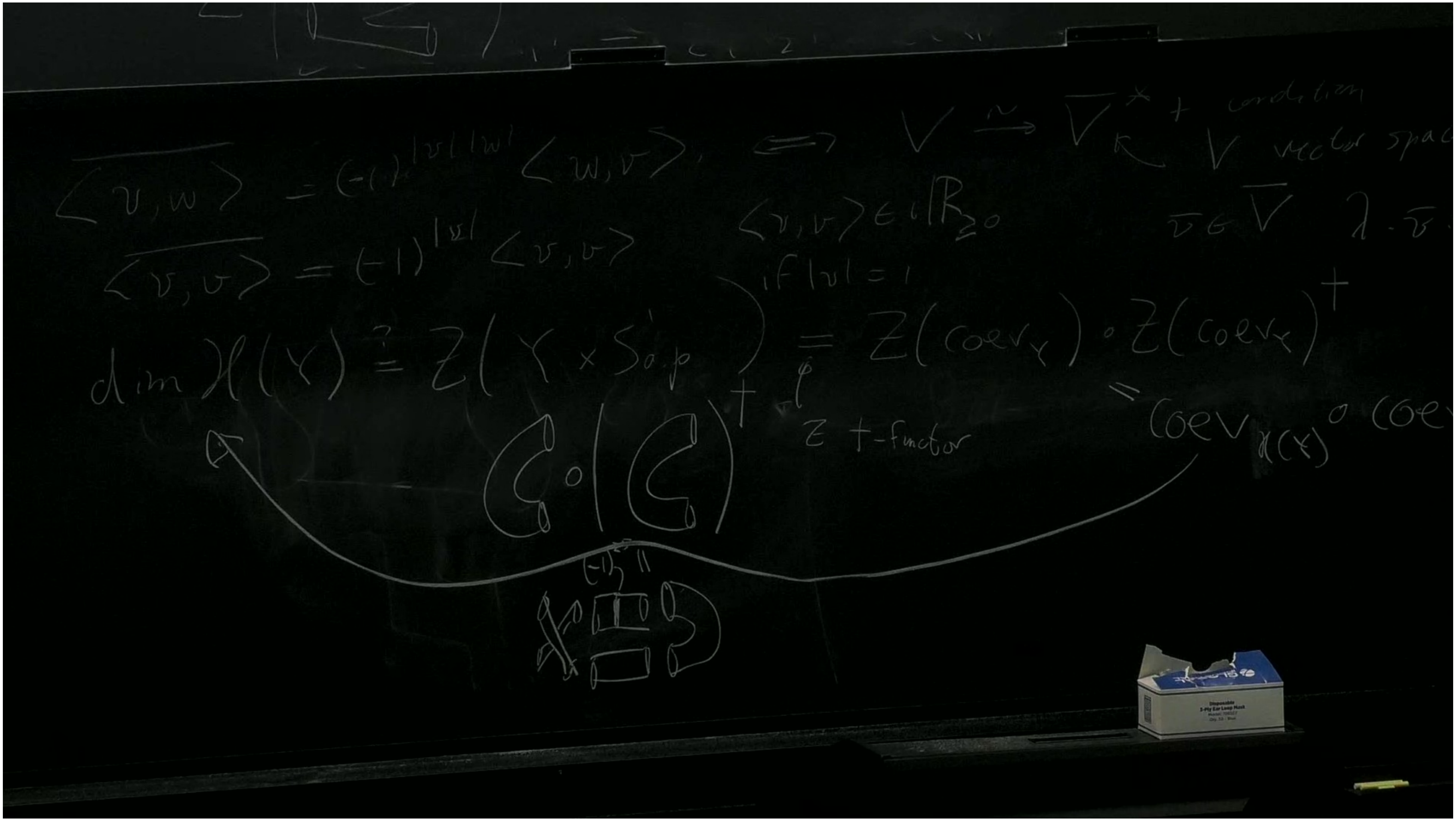
$$\boxed{coev_V^\dagger = ev_V \circ \left(id_{V^*} \otimes (-1)^F_V\right)}$$

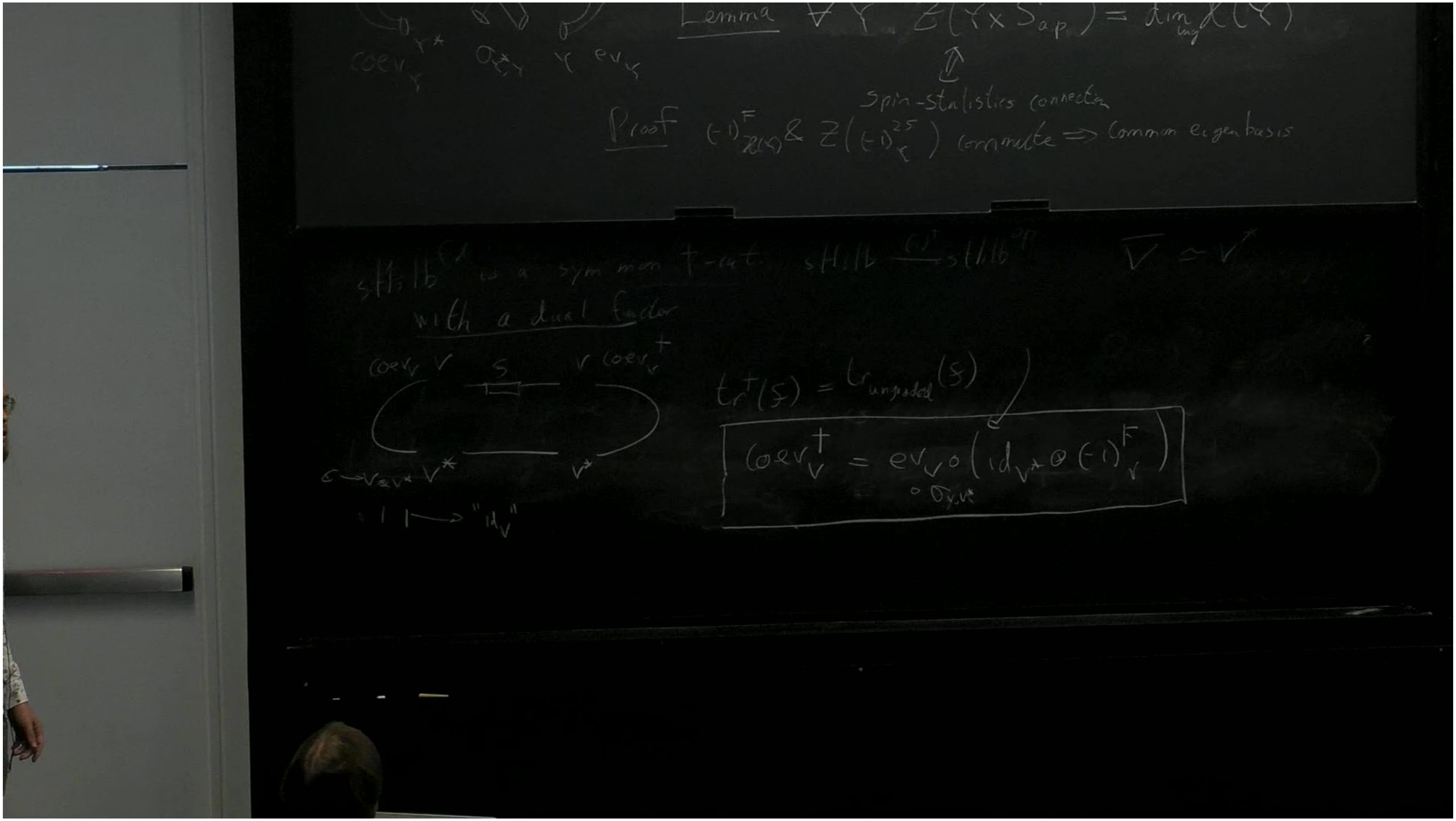
$$\overline{\langle v,w\rangle} = (-1)^{|v||w|}\langle w,v\rangle, \iff V \xrightarrow{\sim} \overline{V}^*_K$$

$$\overline{\langle v,v\rangle} = (-1)^{|v|}\langle v,v\rangle \qquad \langle v,v\rangle \in i\mathbb{R}_{\geq 0} \text{ if } |v|=1$$

$$\dim \mathcal{H}(Y) \stackrel{?}{=} Z(Y \times S^1_{o.p})$$




$\overline{\langle v,w\rangle} = (-1)^{|v||w|}\langle w,v\rangle$, $\iff$ $V \xrightarrow{\sim} \overline{V}^*$ + condition
V vector space
$\langle v,v\rangle \in i\mathbb{R}_{\geq 0}$
$\overline{v} \in \overline{V}$ $\lambda \cdot \overline{v}$
$\overline{\langle v,v\rangle} = (-1)^{|v|}\langle v,v\rangle$
if $|v|=1$
$\dim \mathcal{H}(Y) \overset{?}{=} Z(Y \times S^1_{a.p}) = Z(\mathrm{coev}_Y) \circ Z(\mathrm{coev}_Y)^\dagger$
$= \mathrm{coev}_{\mathcal{H}(Y)} \circ \mathrm{coev}$
Z $\dagger$-functor


Lemma
Spin-statistics connection
Proof
commute ⇒ Common eigenbasis
sHilb is a sym mon †-cat
with a dual functor
coev
tr unpgraded
ev