Title: Categories of line defects and cohomological Hall algebras

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Series: Mathematical physics

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Abstract: BPS line defects in 4d N=2 supersymmetric QFT are described by a monoidal category with a list of desired properties. For example, the Grothendieck group of this category is supposed to coincide with quantization of functions on Coulomb branch of the theory compactified on a circle. Based on an observation, that at a given vacuum the spectrum of PBS particles can be quipped with an algebra structure â€“ cohomological Hall algebra of the corresponding BPS quiver â€“ we propose a category generated by certain bimodules over this algebra that possesses expected properties of the category of lines. Based on a joint work with Davide Gaiotto and Wei Li.Â

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D. Gaiotto Wei Li

$\frac{1}{2}$ BPS in 4d $N=2$ SUSY

CoHA

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$\frac{1}{2}$ BPS lines in 4d $N=2$ SUSY

CoHA

$\boxed{1.}$ What do we want?

– $\mathbb{C}$-linear abelian Cat

– $Q^2=0$ $\quad \partial_{x_1}, \partial_t, \partial_{\bar{z}} = \partial_{x_2} - i\partial_{x_3}$

all exact

HT twist

A

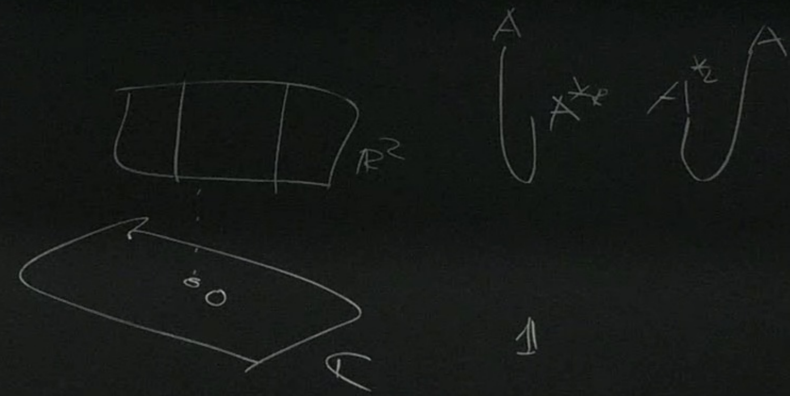
$f \in Hom(A, B)$

B

HT twist

- 2d top $\rightsquigarrow$ monoidal, rigid

$A \otimes B \neq B \otimes A$



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D. Gaiotto Wei Li

$\frac{1}{2}$ BPS lines in 4d $N=2$ SUSY

GoHA

$\boxed{1.}$ What do we want? Line(T)

- $\mathbb{C}$-linear abelian Cat
- $Q^2=0$ $\quad \partial_{x_1}, \partial_t, \partial_{\bar{z}} = \partial_{x_2} - i\partial_{x_3}$ all exact

HT twist

A, $f \in \mathrm{Hom}(A,B)$, B

- 2d top $\rightsquigarrow$ monoidal, rigid

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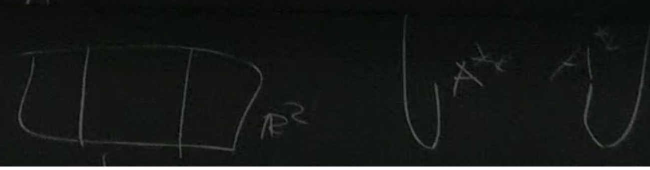

HT twist

- 2d top $\rightsquigarrow$ monoidal, rigid

$A \otimes B \neq B \otimes A$

- holomorphic $\Rightarrow$ $\varphi(z)$, $A^0 \otimes B^z \simeq B^z \otimes A^0$

$r_{A,B} : A \otimes B \to B \otimes A$

$z^{\#}\varphi(z) = r_{A,B} + O(z)$

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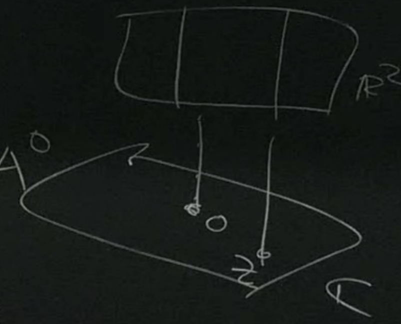
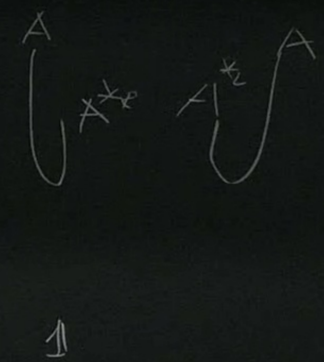




HT twist

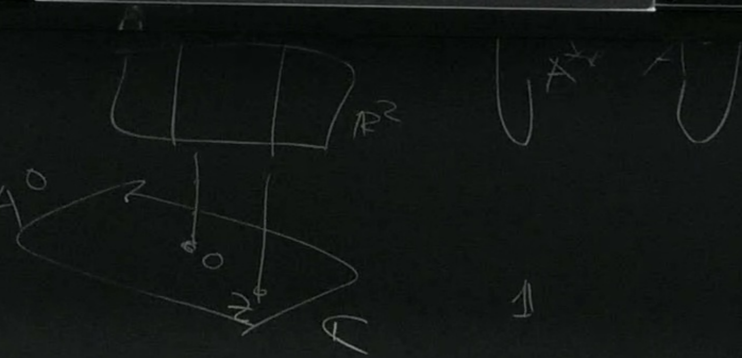
— 2d top ⟶ monoidal, rigid

$A \otimes B \neq B \otimes A$

— holomorphic $\Rightarrow \varphi(z) : A^{0} \otimes B^{z} \simeq B^{z} \otimes A^{0}$

$r_{A,B} : A \otimes B \longrightarrow B \otimes A$

$z^{\#} \varphi(z) = r_{A,B} + O(z)$

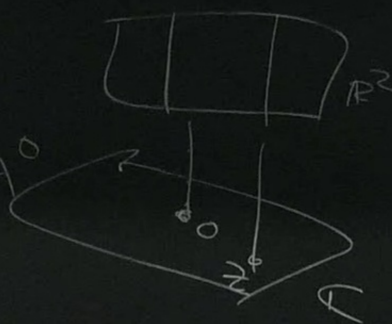


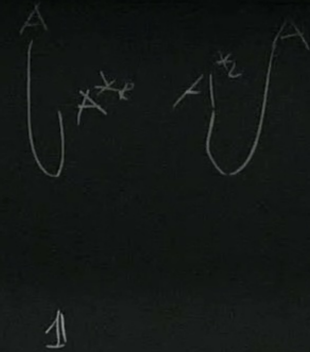
— look at $\mathbb{R}^3 \times S^1$ go to IR $K_0(\mathrm{Line}) \simeq \mathbb{C}[M_C]$

...noidal, rigid

$A \otimes B \neq B \otimes A$

$\varphi(z) : A^0 \otimes B^z \simeq B^z \otimes A^0$





$A \otimes B \to B \otimes A$

$z^{\#}\varphi(z) = r_{A,B} + O(z)$

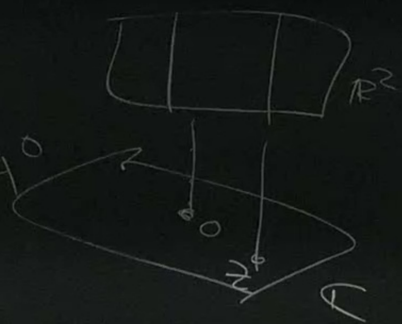
$\mathbb{R}^3 \times S^1$ go to IR $K_0(\mathrm{Lme}) \simeq \mathbb{C}[M_C]$ ← this is a cluster alg.

Ex $AD(A_1, A_2)$

...noidal, rigid

$A \otimes B \neq B \otimes A$

$\varphi(z) : A^{0} \otimes B^{z} \simeq B^{z} \otimes A^{0}$



$A^{*z} \quad A^{z}$

$\mathbb{1}$

$: A \otimes B \longrightarrow B \otimes A$

$z^{\#}\varphi(z) = r_{A,B} + O(z)$

$\mathbb{R}^3 \times S^1$ go to $\mathbb{R}$ $K_0(\mathrm{Lmo}) \simeq \mathbb{C}[M_C]$ ← this is a cluster alg.

$\rightsquigarrow$ Baseground in $\mathbb{C}_q^{2} \rightsquigarrow \mathbb{C}_q[M]$

HT twist

— 2d top ⟶ monoidal, rigid

$A \otimes B \not\simeq B \otimes A$

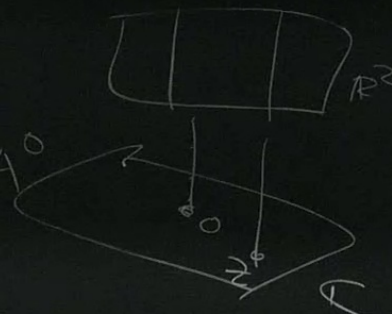
— holomorphic $\Leftrightarrow \varphi(z): \overset{0}{A} \otimes \overset{z}{B} \simeq \overset{z}{B} \otimes \overset{0}{A}$

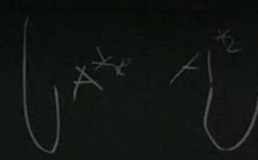
$r_{A,B} : A \otimes B \to B \otimes A$

$z^{\#} \varphi(z) = r_{A,B} + O(z)$

— look at $\mathbb{R}^3 \times S^1$ goto $\mathbb{R}$ $K_0(\mathrm{Line}) \simeq \mathbb{C}[M_C]^{c}$ ⟵ this i...

⟲ background in $\mathbb{C}_q^2 \rightsquigarrow \mathbb{C}_q[M$...





$\mathcal{N}=2$ $U(1)^r$ w/o matter

$\Gamma \cong (\mathbb{Z}^{2r}, \langle , \rangle)$ Line^{IR} semisimple Γ graded $\simeq \mathrm{Vect}^{\Gamma}$ $K_0(\mathrm{Line}^{IR}) = \mathbb{C}[\Gamma]$

$$\rightsquigarrow \mathbb{T}_q[\Gamma, \langle\rangle]$$

$$(X_\gamma, q)_{\gamma \in \Gamma}$$

$$X_\gamma X_{\gamma'} = q^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}$$

$o \to o \leftarrow o$

$$\mathbb{C}_q[\mathcal{M}_c] \hookrightarrow \mathbb{T}_q[\Gamma, \langle , \rangle] \qquad X_\gamma$$

$$\searrow \mathbb{T}_q[\Gamma, \langle\rangle] \qquad \Big\downarrow \mu_i \qquad (1 - X_{\gamma_i})^{\langle \gamma_i, \gamma \rangle} X_\gamma$$

Choose a basis in Γ

$\{\gamma_i\} = Q_0$

$r_{ij} = \langle \gamma_i, \gamma_j \rangle$

Note: No 1, 2-cycles in Q

$o \to o \leftarrow o$

$$\mathbb{C}_q[\mathcal{M}_C] \hookrightarrow \mathbb{T}_q[\Gamma, \langle , \rangle] \qquad X_\gamma \overset{\mu_i}{\longmapsto} \left(1 + X_{\gamma_i}\right)^{\langle \gamma, \gamma_i \rangle} X_\gamma$$

$$\mathbb{C}_q[\mathcal{M}_C] \longrightarrow \mathbb{T}_q[\Gamma, \langle , \rangle]$$

$$\Phi_q(X_{\gamma_i})^{-1} X_\gamma \Phi_q(X_{\gamma_i}) = \left(1 + X_{\gamma_i}\right)^{\langle \gamma, \gamma_i \rangle} X_\gamma$$

$$\Phi_q(X) = \prod_{k \geq 0} (1 - q^{2k+1} X)$$

$\mathbb{T}_q[\Gamma, \langle , \rangle]$

$(X_\gamma, q)_{\gamma \in \Gamma}$

$$X_\gamma X_{\gamma'} = q^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}$$

Choose a basis in Γ

$\{\gamma_i\} = Q_0$

$r_{ij} = \langle \gamma_i, \gamma_j \rangle$

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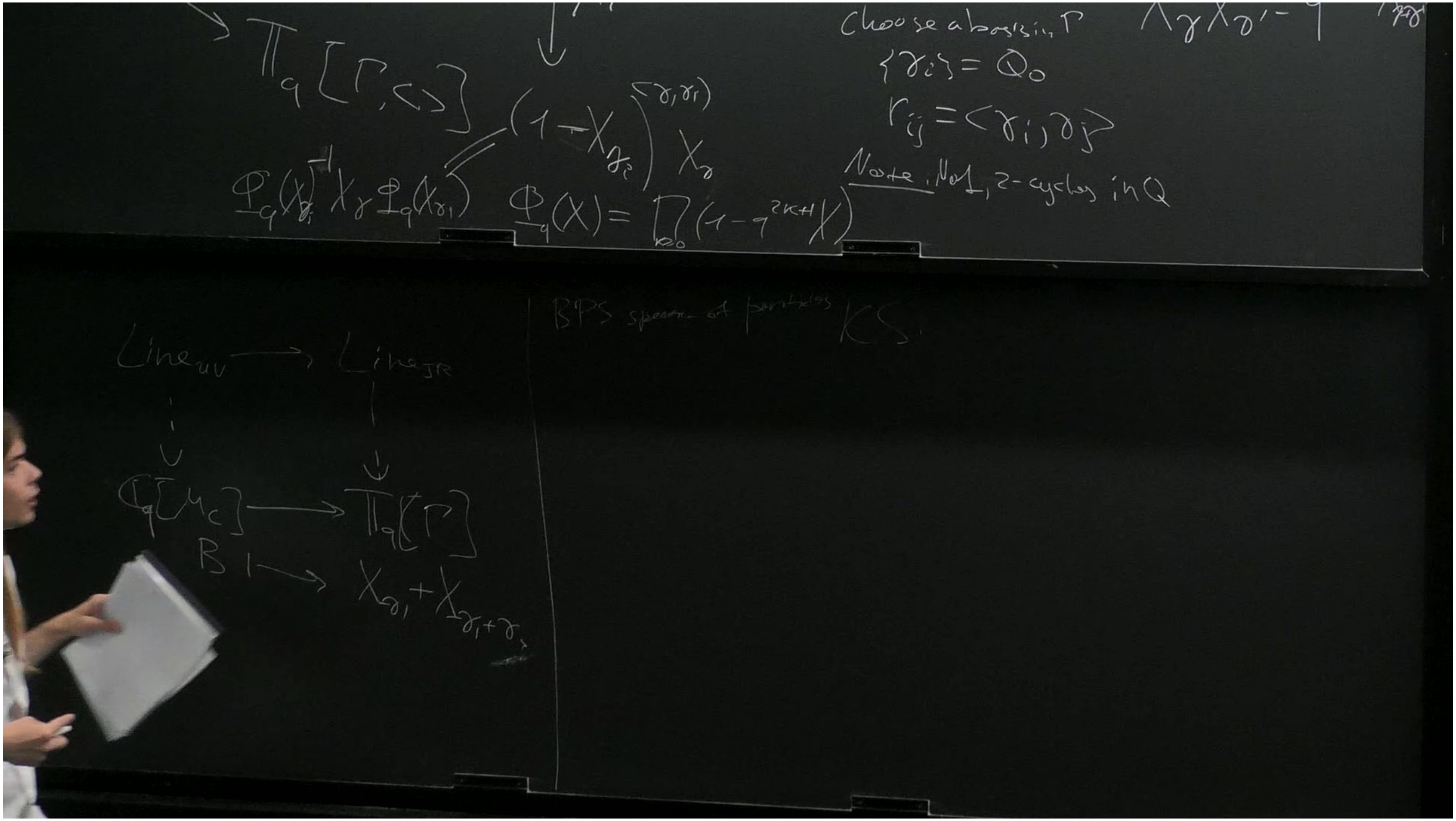
$$\mathbb{C}_q[\mathcal{M}_C] \longrightarrow \mathbb{T}_q[\Gamma]$$

$$B \longmapsto X_{\gamma_1} + X_{\gamma_1 + \gamma_2}$$


Choose a basis in Γ
{γi} = Q0
rij = <γi, γj>
Note: No 1, 2-cycles in Q
KS

$\gamma\, \Phi_q(X_{\gamma_1})$ $\qquad \Phi_q(X) = \prod_{k\geq 0}(1-q^{2k+1}X)$

Note: No 1, 2-cycles in Q

BPS spectrum of particles:

$K\S$ automorphic ... S' as S $\qquad S = \prod_{\gamma \in \mathrm{BPS\,spectrum}} \Phi(X_\gamma)^{\Omega_\gamma}$

Fix Q. Q_f, $f \in \Gamma^\vee$

$\overset{i}{\circ} \rightarrow \circ \leftarrow \overset{j}{\circ}$

$f_i > 0$ $\qquad f_j < 0$

$\dim f_n = 1$

$A_Q = H^*\left(\coprod_\gamma R_\gamma / G_\gamma\right)$

$S = \mathrm{tr}_{A_Q}(-q)^{\gamma} X_\gamma$

Def: $\hat{B}_f = H^*\left(\coprod_\gamma R_\gamma \bigoplus_{i \in Q_0} \mathrm{Hom}(\mathbb{C}^{\max(0, f_i)}, \mathbb{C}^{\gamma_i}) \,/\, G_\gamma\right)$

In $\dim(0 \ldots 0, 1)$ at f_n

$H^*[pt] = \mathbb{C}\cdot\beta$

$\mathbb{T}_q[\Gamma]$

$X_{\gamma_1} + X_{\gamma_1+\gamma_2}$

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$e^i_n = X^n$ in $(0 \ldots 0\,1\,0 \ldots 0)$

$S \mid e^i(z) = \sum_{n \geq 0} \frac{e^i_n}{z^{n+1}}$

$\mathcal{M}_Q \times \mathcal{M}_Q \qquad \mathcal{M}^f_Q$

$e^i(z)\, e^j(w) = (z-w)^{\langle \gamma_i, \gamma_j \rangle}\, e^j(w)\, e^i(z)$

Prop: $e^i_{n+f_i}\, b_f = \pm\, b_f\, e^i_n$

— look at $\mathbb{R}^3 \times S^1$ go to IR $K_0(L\mathcal{M}_Q) \simeq \mathbb{C}[\mathcal{M}_C]$ ← this is

⇝ background in $\mathbb{C}_q^{\ldots}$

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$e^i_n = X^n$

$$e^i(z) = \sum_{n\geq 0} \frac{e^i_n}{z^{n+1}}$$

$$e^i(z)e^j(w) = (z-w)^{\langle\gamma_i,\gamma_j\rangle} e^j(w)e^i(z)$$

$\mathcal{M}_Q \times \mathcal{M}^f_Q \quad \mathcal{M}^f_Q$

<u>Prop</u>: $e^i_{n+f_i} b_f = \pm b_f e^i_n$ $\quad \langle\gamma_f,\gamma_i\rangle = f_i$

<u>Def</u>: $\mathcal{C}_Q$ abelian envelope of $\{B_f\}_{f\in\Gamma^\vee}$

$\Phi_q(X_{\gamma_i})^{-1} X_\gamma \Phi_q(X_{\gamma_1})$ $\Phi_q(X) = \prod_{k\geq 0}(1-q^{2k+1}X)$

Note: No 1, 2-cycles in Q

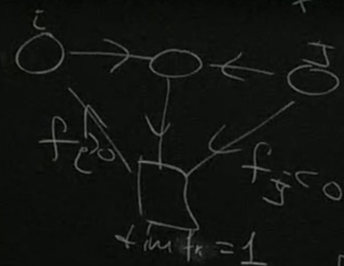
$\mathcal{M}_C] \longrightarrow \mathbb{T}_q[\Gamma]$

$B \longmapsto X_{\gamma_1} + X_{\gamma_1+\gamma_2}$

BPS spectrum of particles:

X & automorphic an[illegible]

Fix Q: Q_f, $f \in \Gamma^\vee$



In dim (0 ... 0, 1 at f_i)

$S = \prod_{\gamma \in BPS\,spectrum} \Phi(X_\gamma)^{\Omega(\gamma)}$

$A_Q = H^*\left(\coprod_\gamma R_\gamma / G_\gamma\right)$

$S = \mathrm{tr}_{A_Q}\left(-q^* X_\gamma\right)$

Def: $\hat{B}_f = H^*\left(\coprod_\gamma R_\gamma \oplus_{i \in Q_0} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^{\gamma_i})\right) / G_\gamma$

$H^*[pt] = \mathbb{C} \cdot b_f$

$\mathcal{M}^f_Q$

Def: $\mathcal{C}_Q$ abelian envelope of $\{B_f\}_{f\in\Gamma^\vee}$

- B_f is free as right/left A_Q-module

  $B_f = V_f^\pi \otimes_{\mathbb{C}} A_Q = A_{\geq\Theta} \otimes_{\mathbb{C}} V^\Theta \otimes_{\mathbb{C}} A_{<\Theta}$

  gradim $V_f^\pi \in \mathbb{Z}_q[M_C]$

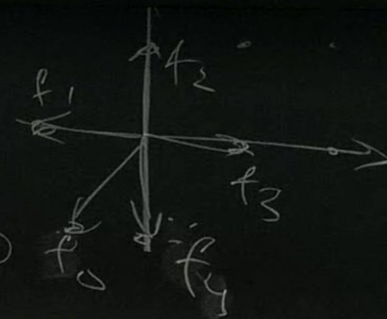
- $\mathcal{C}_Q$ is monoidal $\otimes_{A_Q}$

Ex: $Q: \underset{1}{\circ} \to \underset{2}{\circ}$

envelope of $\{B_f\}_{f\in\Gamma}$

$A_{\geq\theta} \otimes_{\mathbb{C}} V^{\theta} \otimes_{\mathbb{C}} A_{<\theta}$

$B_{\{1,1\}} = B_{f_3} \otimes B_{f_1}$



$0 \to q B_i \to B_{i+1} \otimes B_{i-1} \to A \to 0$

$0 \to A \to B_{i+1} \otimes B_{i+1} \to q B_i \to 0$

$\bigotimes_{A_\phi}$

$\mathcal{C}_Q$ abelian envelope of $\{B_f\}_{f \in \Gamma^\vee}$

is free as right/left A_Q-module

$= V_f^\pi \otimes_{\mathbb{C}} A_Q = A_{\geq \Theta} \otimes_{\mathbb{C}} V^\Theta \otimes_{\mathbb{C}} A_{< \Theta}$

$V_f^\pi \in A_Q[M_c]$

is monoidal $\otimes_{A_Q}$ $\quad B_i \otimes B^\omega_{i+2} \xrightarrow{\sim} B^\omega_{i+2} \otimes B_i$

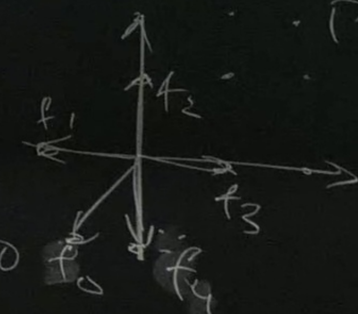
Ex: Q: $\underset{1}{\circ} \to \underset{2}{\circ}$

$B_{\{1,1\}} = B_{f_3} \otimes B_{f_1}$

$0 \to q B_i \to B_{i+1} \otimes B_{i-1} \xrightarrow{ev} A \to 0$

$0 \to A \xrightarrow{coev} B_{i+1} \otimes B_{i+1} \to q B_i \to 0$

$\Gamma = \mathbb{Z}^2 \quad \langle \partial_1, \partial_2 \rangle = 1$



Prop: $e^i_{n+f_i}\, \ell_f = \pm \ell_f\, e^i_n$ | $\langle \gamma_f, \gamma_i \rangle = f_i$ $\quad w \in \mathbb{C}$

Def: $\mathcal{C}_Q$ abelian envelope of $\{B_f\}_{f \in \Gamma^\vee}$

- B_f is free as right/left A_Q-module

$B_f = V_f^\pi \otimes_{\mathbb{C}} A_Q = A_{\geq \theta} \otimes_{\mathbb{C}} V^\theta \otimes_{\mathbb{C}} A_{<\theta}$

grading $V_f^\pi \in \mathbb{C}_q[M]$

- $\mathcal{C}_Q$ is monoidal $\otimes_{A_Q}$ $\quad B_i \otimes B^\omega_{i+2} \xrightarrow{\sim} B^\omega_{i+2} \otimes B_i$

$e^i(z)\, \ell_f = z^{f_i}\, \ell\, e^i(z)$

Ex: Q: $\underset{1}{\circ} \to \underset{2}{\circ}$

$B_0 = X_{-\gamma_1}$, $B_1 = X_{\gamma_2}$

$B_{\{1,1\}} = B_{f_3} \otimes B_{f_1}$

$0 \to q B_i \to B_{i+1} \otimes B_{i-1} \xrightarrow{ev} A_Q \to 0$

$0 \to A_Q \xrightarrow{coev} B_{i-1} \otimes B_{i+1} \to q B_i \to 0$

$\Gamma = \mathbb{Z}^2$ $\quad \langle \gamma_1, \gamma_2 \rangle = 1$