Title: Paraconsistency of relativistic nonsignalling, and some other features of causal spectral toposes

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Paraconsistency of relativistic nonsignalling, and some other features of causal spectral toposes

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von Neumann'32, Birkhoff–von Neumann'36, Husimi'37

- A partially ordered set $(L, \leq)$ is a bounded lattice iff
  - $\exists$ a supremum/join $x \vee y \in L$ $\forall x, y \in L$,
  - $\exists$ an infimum/meet $x \wedge y \in L$ $\forall x, y \in L$,
  - $\exists$ a greatest element $1 \in L$,
  - $\exists$ a smallest element $0 \in L$.
- A bounded lattice is complete iff all of its subsets have meets and joins.
- A bounded lattice is orthocomplemented iff $\forall x \in L$ $\exists x^{\perp} \in L$ s.t. $x^{\perp\perp} = x$, $x^{\perp} \vee x = 1$, $x^{\perp} \wedge x = 0$, if $x \leq y$ then $x^{\perp} \geq y^{\perp}$ $\forall y \in L$.
- A boolean algebra is an orthocomplemented lattice $(L, ^{\perp})$ satisfying $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $\forall x, y, z \in L$.
- An orthocomplemented lattice $(L, ^{\perp})$ is orthomodular iff $x \vee y = ((x \vee y) \wedge y^{\perp}) \vee y$ $\forall x, y \in L$.
- Example: A set of all projections on a Hilbert space, or in any W*-algebra, is an orthomodular lattice: $0 := 0$, $1 := \mathbb{I}$, $P \leq Q := \mathrm{ran}(P) \subseteq \mathrm{ran}(Q)$, $P^{\perp} := \mathbb{I} - P$, $\mathrm{ran}(P \vee Q) := \mathrm{ran}(P) \cap \mathrm{ran}(Q)$, $P \wedge Q := (P^{\perp} \vee Q^{\perp})^{\perp}$.

II. Bi-Heyting algebras

Skolem'1919, Zarycki'27, Heyting'30, Birkhoff'40, McKinsey–Tarski'46, Klemke'71, Rauszer'71'74, Lawvere'76,'86,'89,'90

- A bounded latice $(L, \leq, \wedge, \vee, 0, 1)$ is called:
  - Heyting iff $\forall x, y \in L\ \exists!\ x \to y \in L\ \forall z \in L \quad z \leq x \to y :\iff z \wedge x \leq y$;
  - co-Heyting iff $\forall x, y \in L\ \exists!\ x \prec y \in L\ \forall z \in L \quad z \geq x \prec y :\iff z \vee y \geq x$;
  - bi-Heyting iff it is Heyting and co-Heyting.
- Defining $\neg x := x \to 0$ and $\lrcorner x := 1 \prec x$, we get:
  - $\neg x \vee x \leq 1$, i.e. $\neg$ does not satisfy *tertium non datur* (law of excluded middle),
  - $\lrcorner x \wedge x \geq 0$, i.e. $\lrcorner$ does not satisfy *ex falso quodlibet* (law of noncontradiction).
- In general, logics invalidating the law of noncontradiction are called paraconsistent.
- Zarycki–Lawvere boundary operator, $\partial(\cdot) := (\cdot) \wedge \lrcorner(\cdot)$, satisfies Leibniz rule: $\partial(x \wedge y) = (\partial x \wedge y) \vee (y \wedge \partial x)$.

III. Spectral presheaf and outer daseinisation

Isham–Butterfield'98, de Groote'05, Döring–Isham'08, Cannon'13, Cannon–Döring'18

- Stone'36 duality:
  - Every boolean algebra B determines a Stone space S_B := a set of nonzero boolean homomorphisms $B \to \{0, 1\}$, equipped with a suitable (totally disconnected compact Hausdorff) topology.
  - Given any t.d.c.H. topological space S, the set of all closed-and-open subsets of S forms a boolean algebra B_S.
  - $S_{B_S} = S$, $B_{S_B} = B$.
- Let L be a complete orthomodular lattice.
- Let $\mathbf{B}(L)$:= a category with {objects := boolean subalgebras of L; morphisms := inclusions}.
- A spectral presheaf := a contravariant functor $\Sigma_L : \mathbf{B}(L) \to \mathbf{Set}$, s.t. $\{B \mapsto S_B; (B_1 \hookrightarrow B_2) \mapsto \text{ restriction: } (S_{B_2} \to S_{B_1})\}$.
- Consider: $$\delta_B(x) := \underbrace{\left\{ s \in S_B \,:\, s\left(\underbrace{\bigwedge\{y \in B \,:\, y \geq x\}}_{\text{best approx. of } x \text{ in } B} \right) = 1 \right\}}_{\text{elements of } S_B \text{ for which the best approx. of } x \text{ holds}} \forall B \in \mathrm{Ob}(\mathbf{B}(L)).$$
- An outer daseinisation of $x \in L$:= a contravariant functor $\delta(x) : \mathbf{B}(L) \to \mathbf{Set}$, s.t. $\{B \mapsto \delta_B(x) \subseteq S_B; (B_1 \hookrightarrow B_2) \mapsto \text{ restriction: } (S_{B_2} \to S_{B_1})\}$.

III. Spectral presheaf and outer daseinisation

Döring–Isham'08, Döring'16, Cannon'13, Cannon–Döring'18, Eva'15'16, Döring–Eva–Ozawa'21, RPK'24

- $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$:= set of subfunctors F of Σ_L, s.t. $F(B)$ is a closed-and-open set $\forall B \in \mathrm{Ob}(\mathbf{B}(L))$.
- $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$ is a complete bi-Heyting algebra, when equipped with: $x \leq y :\iff x_B \subseteq y_B\ \forall B \in \mathrm{Ob}(\mathbf{B}(L))$, $(p \wedge q)_B := \mathrm{int}(p_B \cap q_B)$, $(p \vee q)_B := \mathrm{cl}(p_B \cup q_B)$.
- $\delta : L \to \mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$ is an injective, $(0, 1, \vee)$-preserving map.
- By an adjoint functor theorem for posets, there exists a surjective, $(0, 1, \wedge)$-preserving map $\varepsilon : \mathrm{Sub}_{\mathrm{clop}}(\Sigma_L) \to L$, s.t. $\delta(x) \leq y \iff x \leq \varepsilon(y)$, i.e. $\delta \dashv \varepsilon$ is a Galois connection.
- $\multimap(\cdot) := \delta((\varepsilon(\cdot))^{\perp})$ is a proper paraconsistent negation on $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$, i.e. $x \wedge \multimap x \geq 0$ and $(x \wedge \multimap x = 0$ iff $x \in \{0, 1\})$.
- Eva'15'16 (claim, no proof): $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \multimap)$, equipped with implication $x \Rightarrow y := \multimap x \vee y$, satisfies the axioms and rules of inference of the relevant paraconsistent logic **DL** (of Routley'77).
- RPK'24:
    1) Two rules of inference of **DL** (affixing and *modus ponens*) are not provable, so $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \multimap, \Rightarrow)$ is a model of a weaker relevant paraconsistent logic, $\mathbf{DL}_0$.
    2) Some logical rules that are generally different in the relevant paraconsistent systems, become equivalent in $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \multimap, \Rightarrow)$. In particular, $(x \Rightarrow \multimap x) \Rightarrow \multimap x$ iff $x \vee \multimap x = 1$, so $\mathbf{DL}_0 = \mathbf{DK}_0 := \{$**DK** of Routley–Meyer'76 minus *modus ponens* minus affixing$\}$.

V. Causal logic

Cegła–Jadczyk'77'79, Cegła–Florek'79'81'05'06, Cegła'89, Casini'02'03, Nobili'06, Cegła–Florek–Jancewicz'17

- Let (M, g) := arbitrary (≥ 2)-dimensional lorentzian space-time.
- For $S \subseteq M$, let $S^{\perp} :=$ {all $x \in M$ not connected with S by a time-like curve}.
- The set $L_{(M,g)}$ of subsets $S \subseteq M$, s.t. $S = S^{\perp\perp}$, equipped with $S_1 \leq S_2 := S_1 \subseteq S_2$, $S_1 \wedge S_2 := S_1 \cap S_2$, $S_1 \vee S_2 := (S_1 \cap S_2)^{\perp\perp}$, is a complete, atomic, orthomodular lattice.
- $L_{(M,g)}$ does not satisfy so-called covering law, so it cannot be represented as a lattice of projections in a W*-algebra.
- (Maximal) boolean subalgebras of $L_{(M,g)}$ = (space-like) achronal surfaces of (M, g).
- If $S^{\perp} :=$ {all $x \in M$ not connected with S by a time-like or null-like curve}, then $L_{(M,g)}$ is orthocomplemented but not orthomodular.
- Lattices defined by discretised space-times are also orthocomplemented but not orthomodular.
- Nobili'06: «(...) it is difficult to define unambiguously (...) the boundaries of causal completions».

V. Causal logic

RPK'19/'24

Observation:

- The Cannon–Döring–Eva construction works as well without assuming orthomodularity of L.
- The only thing that gets lost with such a weakening is characterisation of L by Σ_L up to an isomorphism.

Results:

1) Causal nonsignalling, encoded by $^{\perp}$, becomes represented by a Galois construction $\delta \dashv \varepsilon$, as a proper paraconsistent negation $\multimap$, satisfying the rules and axioms of $\mathbf{DL_0}$ logic.
2) Since $\delta \dashv \varepsilon$ is monotone, the strengthening (resp., weakening) of nonsignalling corresponds to strengthening (resp., weakening) of paraconsistent negation.
3) The Zarycki–Lawvere boundary operator encodes the properties of the causal boundary of the causally complete sets.
4) Since $\varepsilon \circ \delta = \mathrm{id}_L$, we can study the relationship between L and $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$ as a relationship between two resource theories in the sense of del Rio–Krämer–Renner'15: if Ω is a monoid of (some) endomorphisms of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L)$, and Θ is a monoid of (some) endomorphisms of L, then (L, Θ) represents a restricted agent within an underlying resource theory $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_L), \Omega)$ iff there exists a submonoid $\Xi \subseteq \Omega$ s.t. $\Theta = \{\varepsilon \circ f \circ \delta : f \in \Xi\}$.

VI. Causal logic in a spectral presheaf

Haag–Schroer'62, Haag–Kastler'64,..., Haag'92/'96

- Given a W*-algebra $\mathcal{N}$, and its sub-W*-algebra $\mathcal{A}$, a commutant (of $\mathcal{A}$ in $\mathcal{N}$) $:= \mathcal{A}^\bullet$ $:= \{x \in \mathcal{N} : xy = yx \; \forall y \in \mathcal{A}\}$.
- Minimal setting for algebraic q.f.t.:
  1) a functor $\mathfrak{A}$ from the category of subsets of space-time with embeddings as morphisms to the category of sub-W*-algebras of a W*-algebra with embeddings as morphisms,
  2) if $S_1 \subseteq S_2$ then $\mathfrak{A}(S_1) \subseteq \mathfrak{A}(S_2)$,
  3) if $S = \bigcup_j S_j$ then $\mathfrak{A}(S) = \bigwedge_j \mathfrak{A}(S_j) := (\bigcup_j \mathfrak{A}(S_j))^{\bullet\bullet}$,
  4) if $S_1 \subseteq S_2^\perp$ then $\mathfrak{A}(S_1) \subseteq (\mathfrak{A}(S_2))^\bullet$ (=: causality).
- Haag–Schroer duality property $:= (\mathfrak{A}(S^\perp) = (\mathfrak{A}(S))^\bullet)$.
- Haag'92/'96 postulate:
  1) consider $(L_{(M,g)}, {}^\perp)$, where (M, g) is a Minkowski space-time, and ${}^\perp$ is a time-like nonsignalling;
  2) consider an orthomodular lattice $(L_\mathcal{N}, {}^\bullet)$ of sub-W*-algebras of a W*-algebra $\mathcal{N}$;
  3) an algebraic q.f.t., for the vacuum sector of the theory, is given by the orthomodular lattice homomorphism $(L_{(M,g)}, {}^\perp) \to (L_\mathcal{N}, {}^\bullet)$.
- In general, Haag's postulate is too strong:
  1) Haag–Schroer duality does not hold in several models;
  2) $\wedge$-preservation is usually not required and not verified.

VIII. Vacuum algebraic q.f.t.: Beyond Haag's postulate

RPK'24

- Consider the following categorical reformulation, and weakening, of Haag's postulate: a vacuum pre-a.q.f.t. := an injective, $(0,1,\vee)$-preserving functor $\mathfrak{N}^\flat : (L_{(M,g)}, {}^\perp) \to (L_{\mathcal{N}}, {}^\bullet)$, satisfying $\mathfrak{N}^\flat((\cdot)^\perp) \leq (\mathfrak{N}^\flat(\cdot))^\bullet$.
- By the adjoint functor theorem, $\mathfrak{N}^\flat$ has a surjective, $(0,1,\wedge)$-preserving adjoint $\mathfrak{N}^\sharp : (L_{\mathcal{N}}, {}^\bullet) \to (L_{(M,g)}, {}^\perp)$, so $\mathfrak{N}^\flat \dashv \mathfrak{N}^\sharp$ is a monotone Galois connection.
- We will say that $\mathfrak{N}^\flat$ satisfies the Haag–Schroer duality iff $\mathfrak{N}^\flat((\cdot)^\perp) = (\mathfrak{N}^\flat(\cdot))^\bullet$.

VIII. Vacuum algebraic q.f.t.: Beyond Haag's postulate

RPK'24

- Combining earlier constructions, we define a spectral presheaf vacuum pre-a.q.f.t. as an injective, $(0, 1, \vee)$-preserving functor $\mathfrak{M}^\flat : (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(M,g)}}), \multimap) \to (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}), \dashv)$, s.t. $\mathfrak{M}^\flat(\multimap(\cdot)) \leq \dashv(\mathfrak{M}^\flat(\cdot))$ and the following diagram commutes

$$\begin{array}{ccc} (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{(M,g)}}), \multimap) & \xrightarrow{\mathfrak{M}^\flat} & (\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}), \dashv) \\ \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\ (L_{(M,g)}, {}^\perp) & \xrightarrow{\mathfrak{N}^\flat} & (L_{\mathcal{N}}, {}^\bullet). \end{array} \tag{1}$$

- We will say that $\mathfrak{M}^\flat$ satisfies the Haag–Schroer duality iff $\mathfrak{M}^\flat(\multimap(\cdot)) = \dashv(\mathfrak{M}^\flat(\cdot))$.
- This opens a way to study the relationship between the Zarycki–Lawvere causal boundaries and the Zarycki–Lawvere boundaries of sub-W*-algebras.
- If $\mathfrak{M}^\sharp \circ \mathfrak{M}^\flat = \mathrm{id}$, then $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma_{L_{\mathcal{N}}}), \neg, \urcorner, \dashv)$ is not only an algebraic model of a bi-intuitionist–relevant logic, but also a state space for construction of the resource theories, exhibiting all three other lattices in (1) as restricted theories of agents.