Title: Relativistic Concepts in Point-Free Spaces

Speakers: Nesta van der Schaaf

Series: Quantum Foundations, Quantum Information

Date: September 19, 2024 - 4:35 PM

URL: https://pirsa.org/24090136

Relativistic concepts in point-free spaces

Causalworlds 2024

Nesta van der Schaaf
joint with
Chris Heunen
Prakash Panangaden

19 Sept. 2024

{arXiv:2406.15406}

The University of Edinburgh
School of Informatics

Overview

Ordered Locales

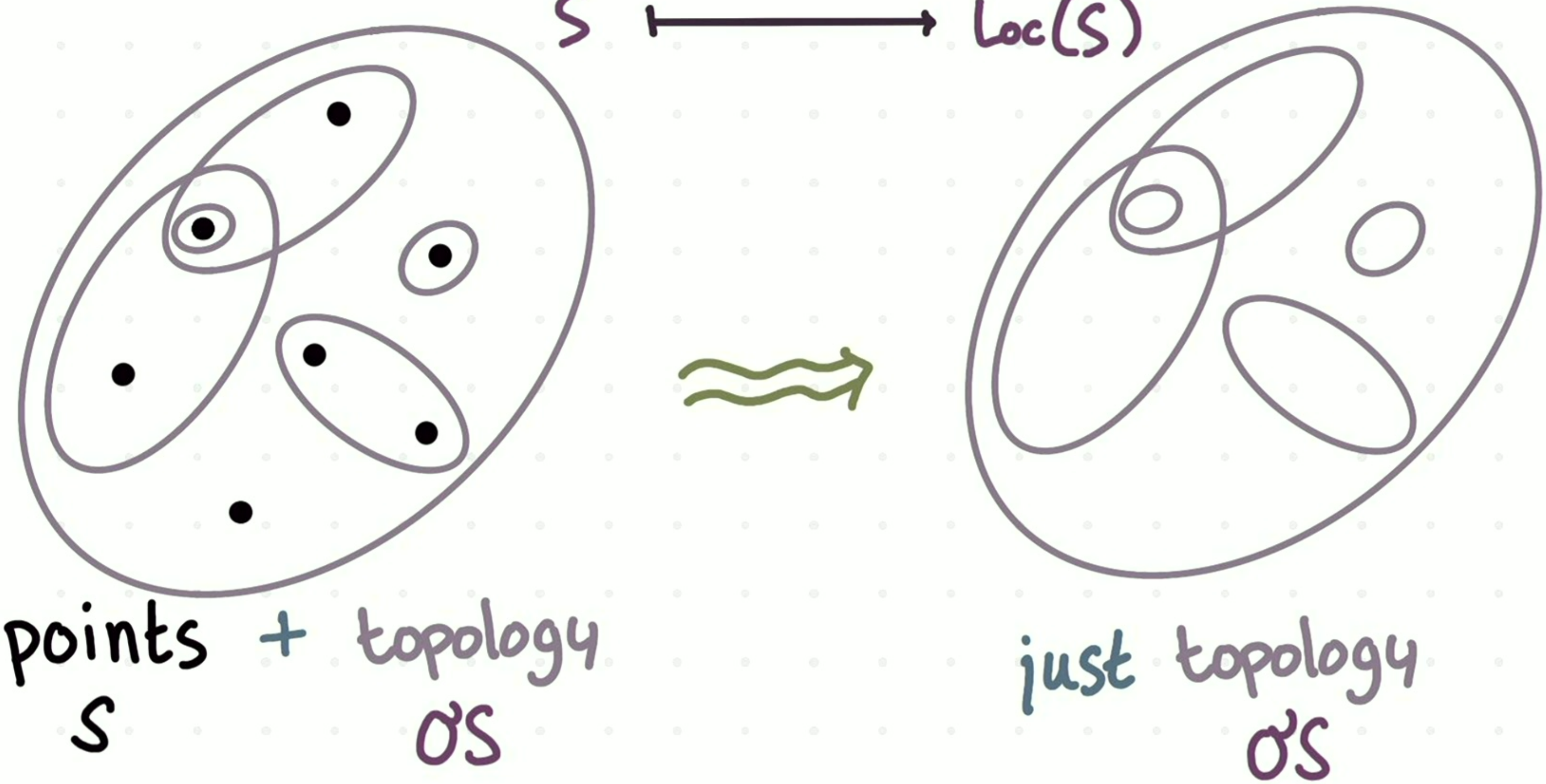
Causal Coverage

Domain of dependence

Causal Boundaries

Locales

$$\begin{array}{ccc} \mathbf{Top} & \longrightarrow & \mathbf{Loc} \\ S & \longmapsto & \mathrm{Loc}(S) \end{array}$$



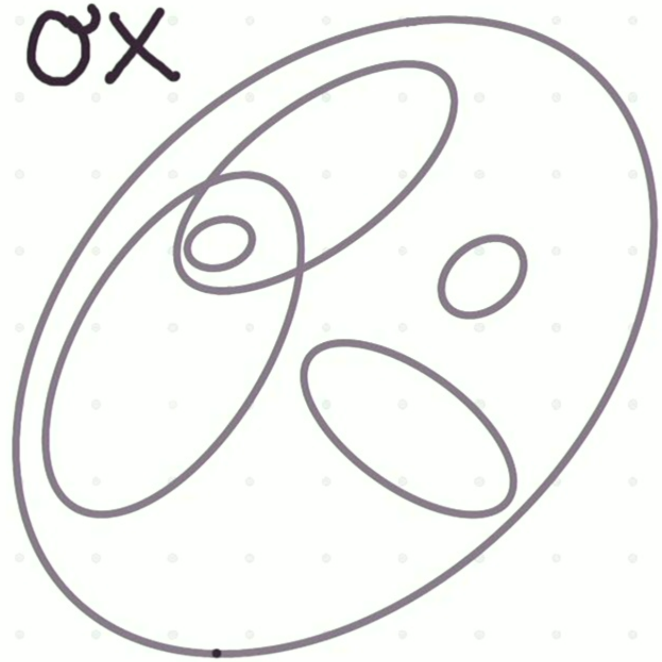
Locales

inclusion $\sqsubseteq$

intersection (meet) $\wedge$

union (join) $\vee$

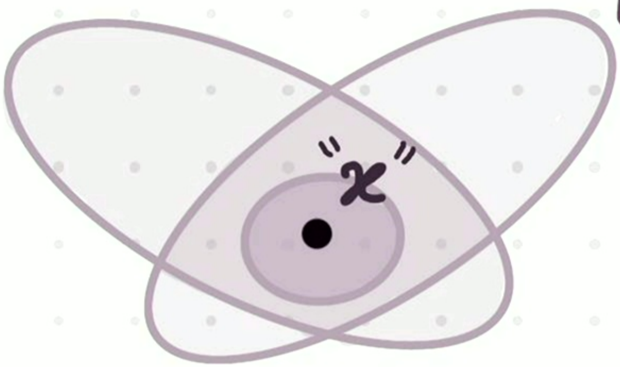
$\mathcal{O}X$



just topology

Locales

compl. prime filters:



prime opens:

$P \neq X$,

$U \wedge V = P$

$\Longrightarrow P = U$ or V

Idea

causal order

$x \leq y$

region causality

$U \trianglelefteq V$

Ordered Locales

ORD. LOC.: a locale X with preorder $\trianglelefteq$ on $\mathcal{O}X$:
+ axioms

{arXiv:2303.03813} JPAA vol. 228 (7)

THEOREM

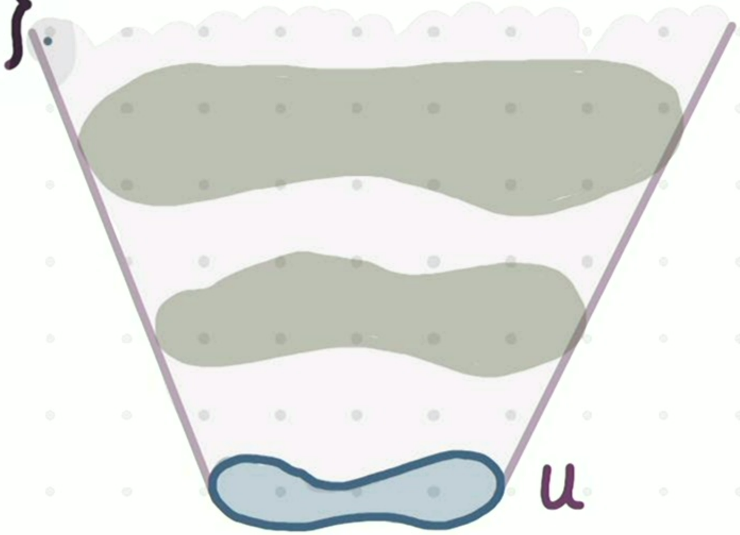
$$\begin{tikzcd} \mathbf{OrdTop}^{\bullet}_{OC} \arrow[r, shift left=2, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc}^{\bullet} \arrow[l, shift left=2, "\mathrm{pt}"] \end{tikzcd}$$

COROLLARY

$$\left\{ \begin{array}{c} \text{sober } T_0\text{-ordered} \\ \text{spaces with OC} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{spatial ordered} \\ \text{locales with } \bullet \end{array} \right\}$$

Ordered Locales

CONES $\uparrow u := \bigvee\{w \in \mathcal{O}X : u \trianglelefteq w\}$



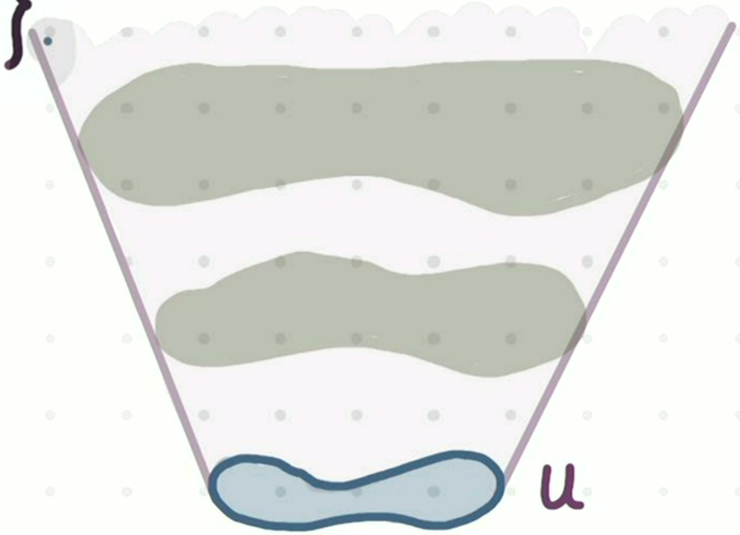
Ordered Locales

CONES

$$\Uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}$$

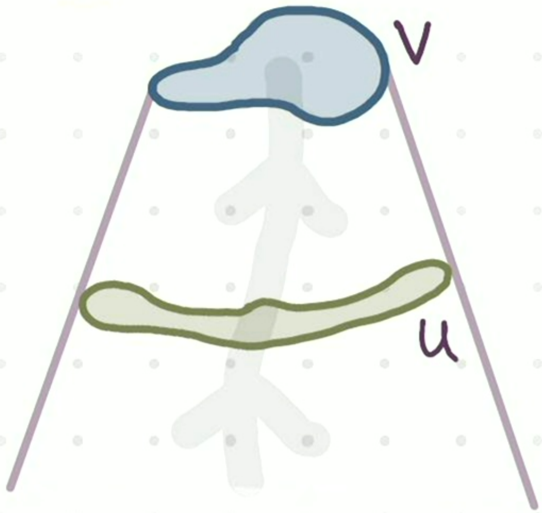
In spacetime :

$$\Uparrow U = I^+(U) = J^+(U)$$
$$\Downarrow U = I^-(U) = J^-(U)$$



Causal Coverage

[Christensen, Crane 05]



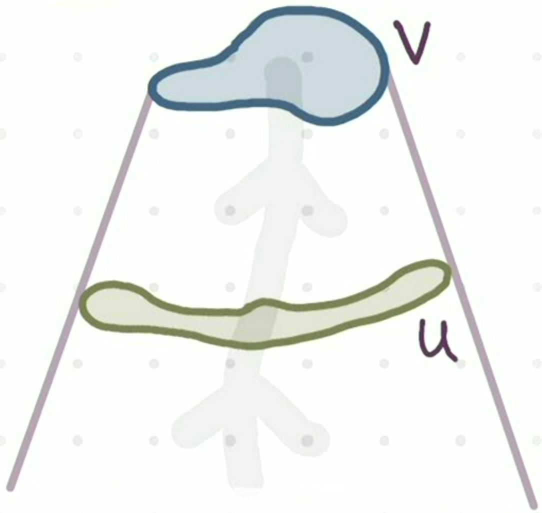
IDEA: all info. reaching V
must pass through U:

$$U \in Cov^{-}(V)$$

Coverages:
- sites [Johnstone82]
- formal topology [Sambin]

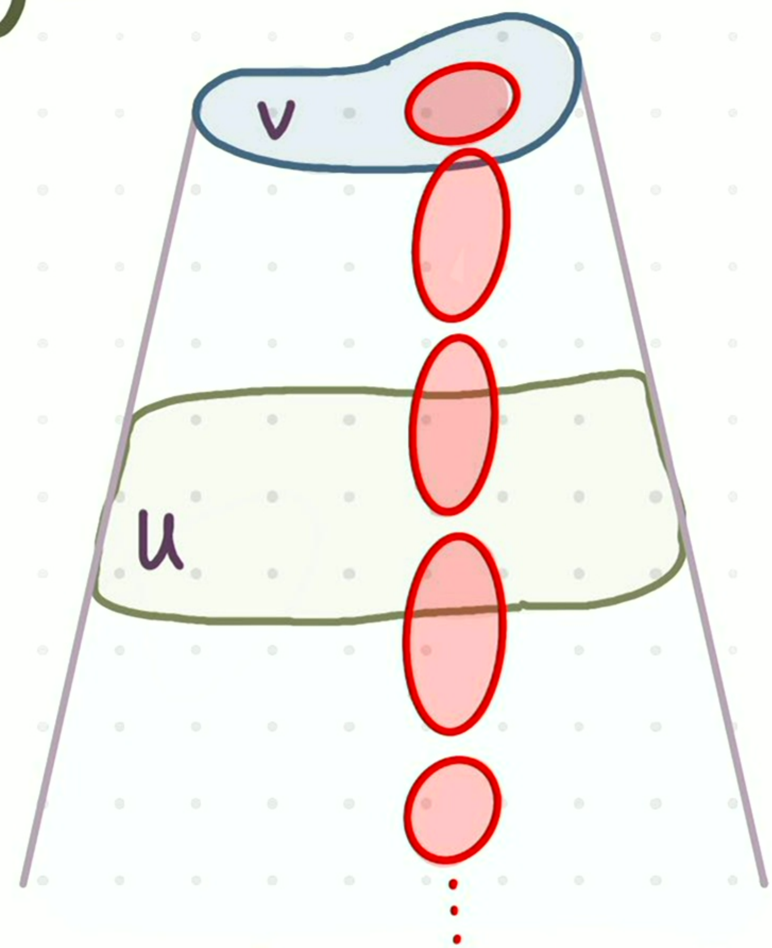
Causal Coverage

[Christensen, Crane 05]



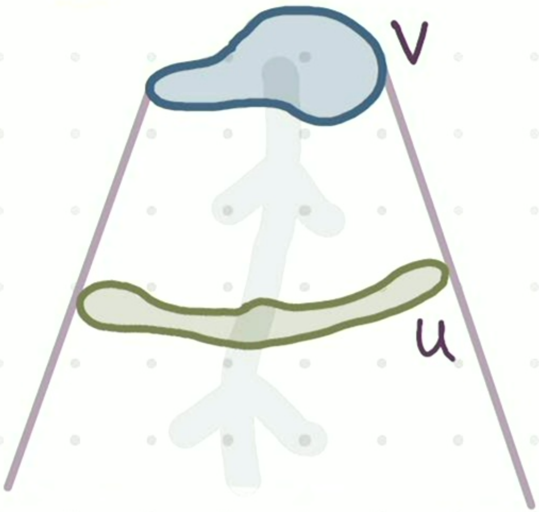
IDEA all info. reaching V **must** pass through U:

$$U \in Cov^{-}(V)$$



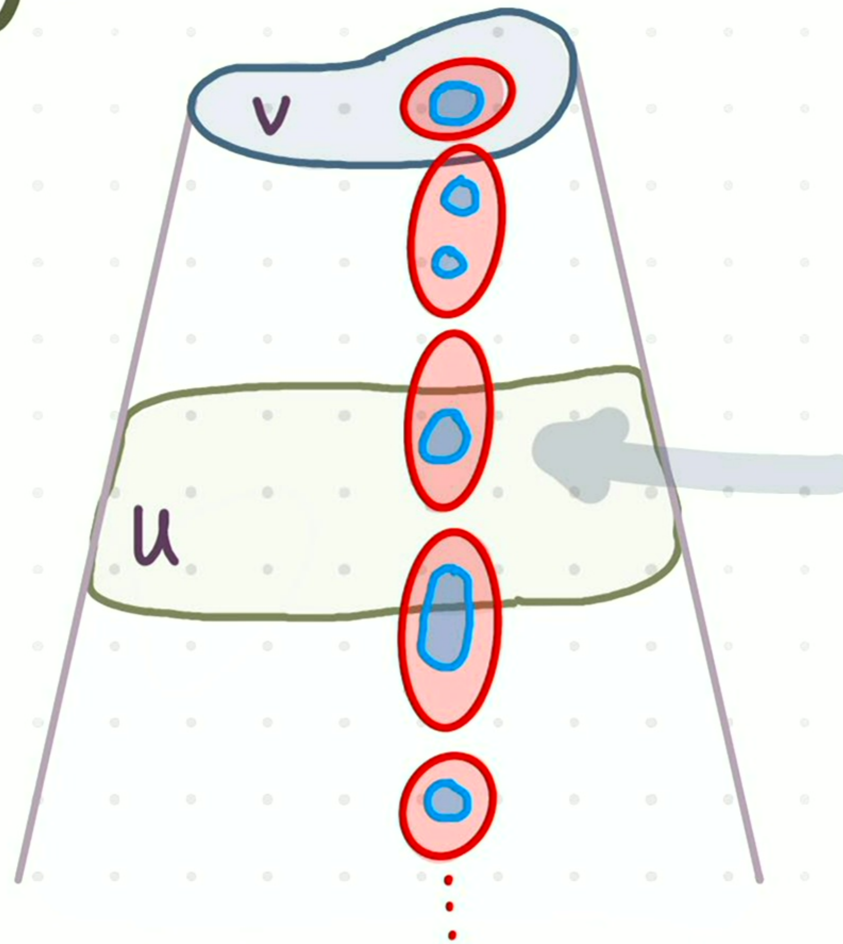
Causal Coverage

[Christensen, Crane 05]



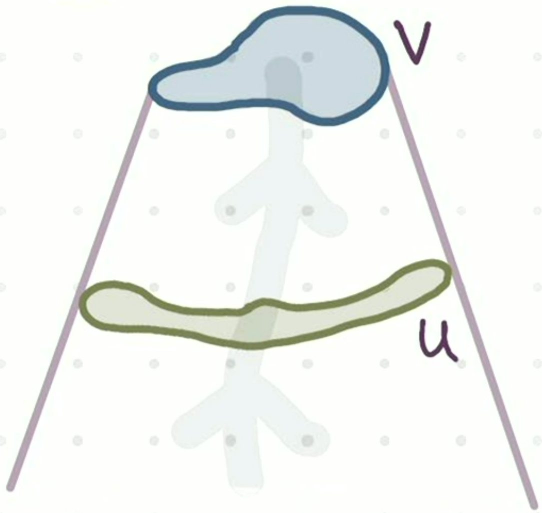
IDEA all info. reaching V **must** pass through U:

$$U \in \mathrm{Cov}^{-}(V)$$



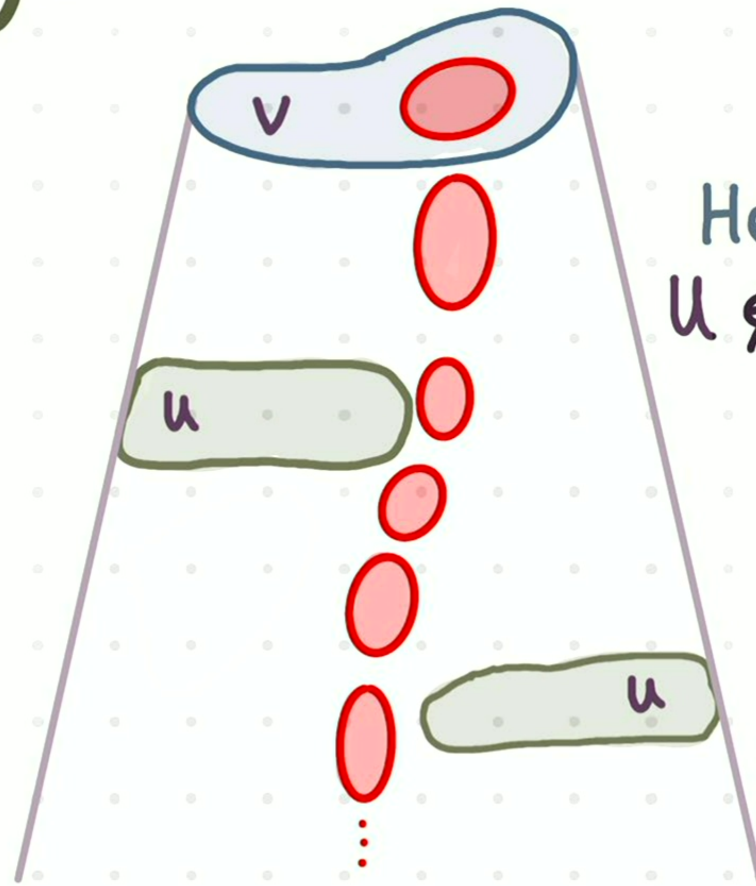
Causal Coverage

[Christensen, Crane 05]



IDEA: all info. reaching V **must** pass through U:

$$U \in \mathrm{Cov}^{-}(V)$$



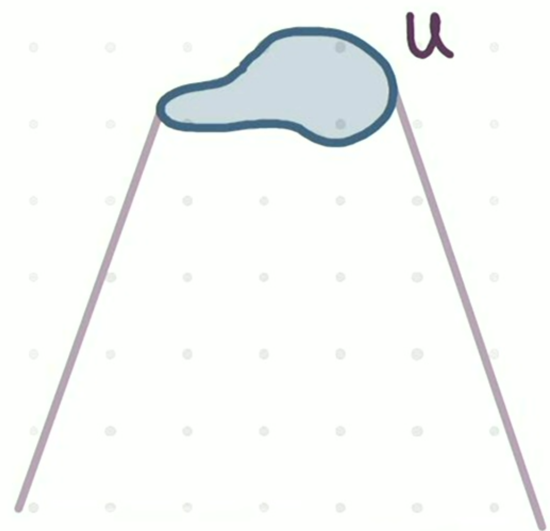
Here:

$$U \notin \mathrm{Cov}^{-}(V)$$

Causal Coverage

LEMMA

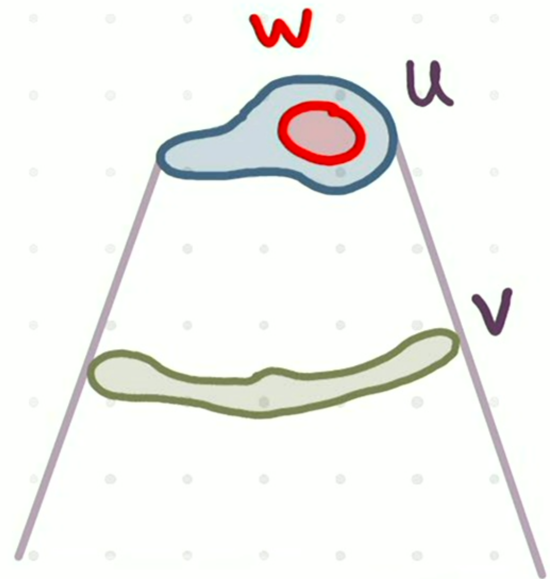
- $U \in Cov^-(U)$
- $\downarrow U \in Cov^-(U)$



Causal Coverage

LEMMA

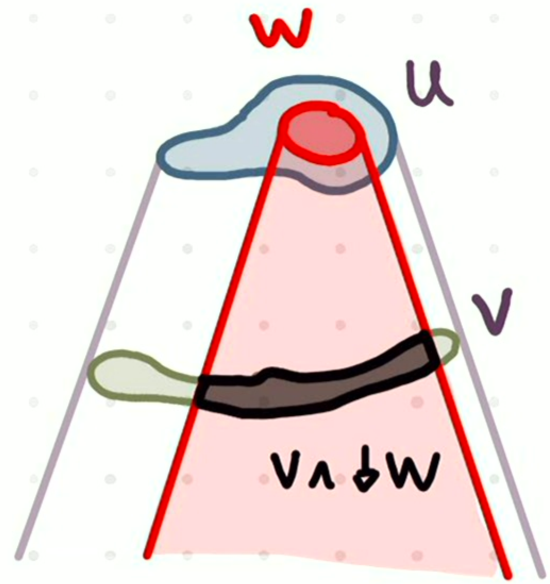
- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V),\ V \in \mathrm{Cov}^-(U)$
  $\implies W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U),\ W \subseteq U$
  $\implies V \wedge \downarrow W \in \mathrm{Cov}^-(W)$



Causal Coverage

LEMMA

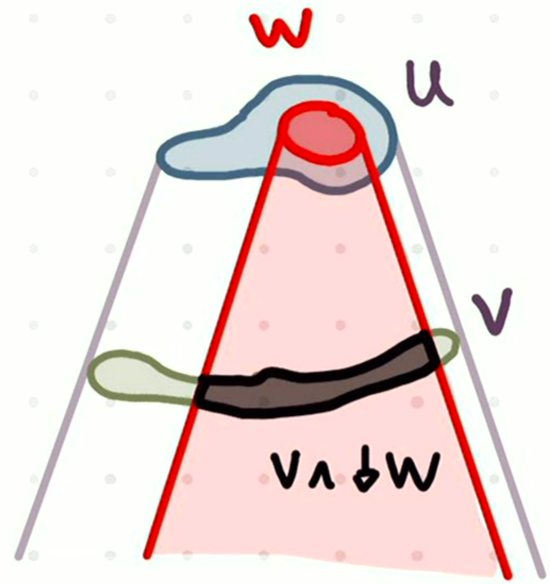
- $\mathcal{U} \in \mathrm{Cov}^-(\mathcal{U})$
- $\downarrow\mathcal{U} \in \mathrm{Cov}^-(\mathcal{U})$
- $\mathcal{W} \in \mathrm{Cov}^-(\mathcal{V}),\ \mathcal{V} \in \mathrm{Cov}^-(\mathcal{U})$
  $\Longrightarrow \mathcal{W} \in \mathrm{Cov}^-(\mathcal{U})$
- $\mathcal{V} \in \mathrm{Cov}^-(\mathcal{U}),\ \mathcal{W} \subseteq \mathcal{U}$
  $\Longrightarrow \mathcal{V} \wedge \downarrow\mathcal{W} \in \mathrm{Cov}^-(\mathcal{W})$



Causal Coverage

LEMMA

- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V),\ V \in \mathrm{Cov}^-(U)$
  $\Longrightarrow W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U),\ W \subseteq U$
  $\Longrightarrow V \wedge \downarrow W \in \mathrm{Cov}^-(W)$

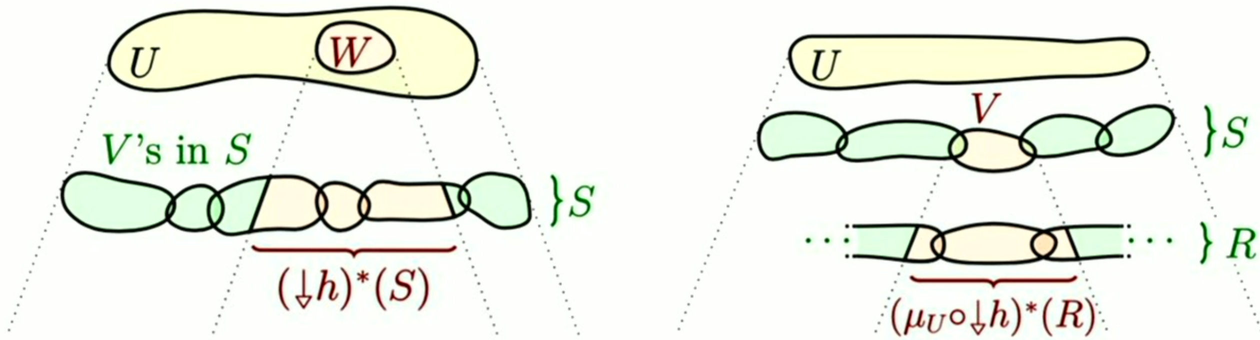


does not hold in [Christensen Crane 05]

Causal Coverage

DEF./THM.

For any ordered locale:
Cou$^-$ naturally determines
a "$\downarrow$-Grothendieck top."



(a) Axiom (ii). (b) Axiom (iii).

Figure 24: Illustrations of the axioms of a $\downarrow$-Grothendieck topology for J^-.

Causal Coverage

DEF./THM.

For any ordered locale:
Cov^- naturally determines
a "$\downarrow$-Grothendieck top."

categorification of $\trianglelefteq$?
tensor topology?

sheaves?

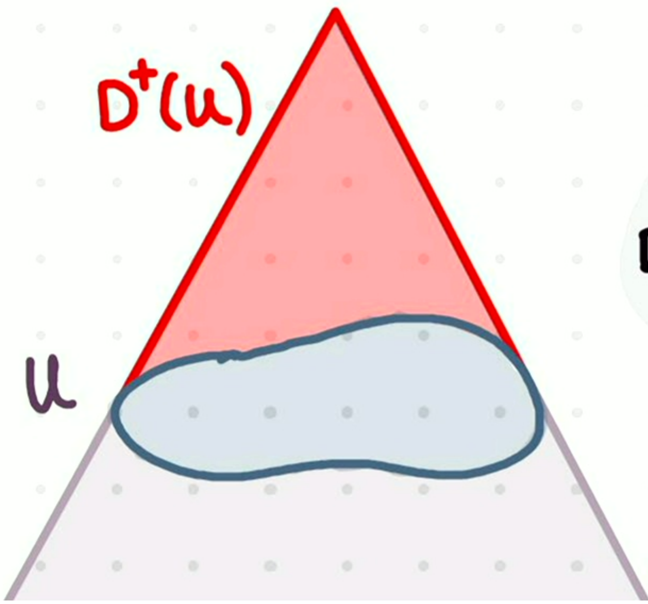
internal temporal logic?
related to modal logic approaches of spacetime? [Goldblatt 80,92]

Causal Coverage

[Geroch 70]

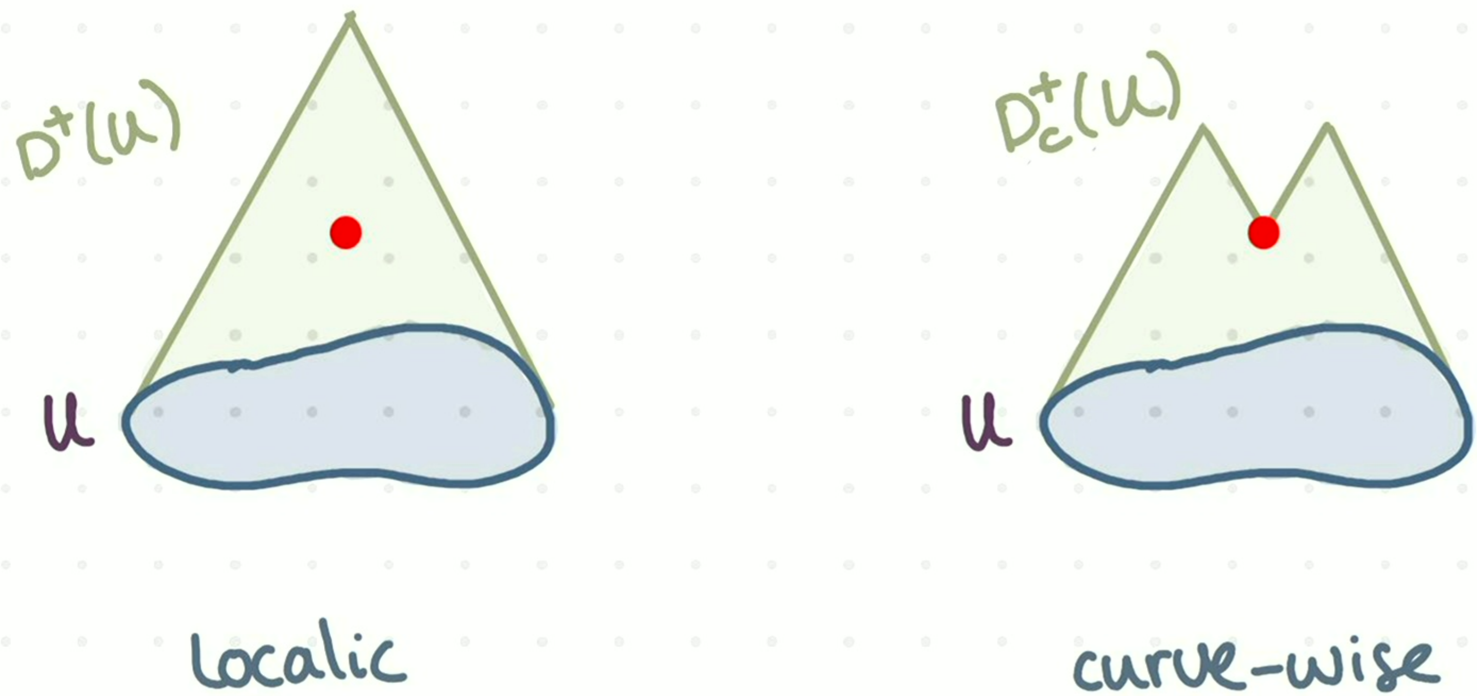
DOMAIN OF DEPENDENCE

$$D^+(U) := \bigvee \{ W \in \mathcal{O}X : U \in \mathrm{Cov}^-(W) \}$$

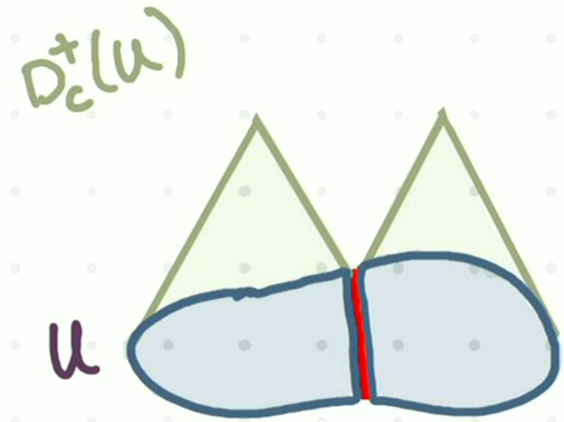


$$D^+_c(A) := \left\{ x \in M : \begin{array}{c} \text{every past inextb.} \\ \text{causal curve through} \\ x \text{ intersects } A \end{array} \right\}$$

Causal Coverage



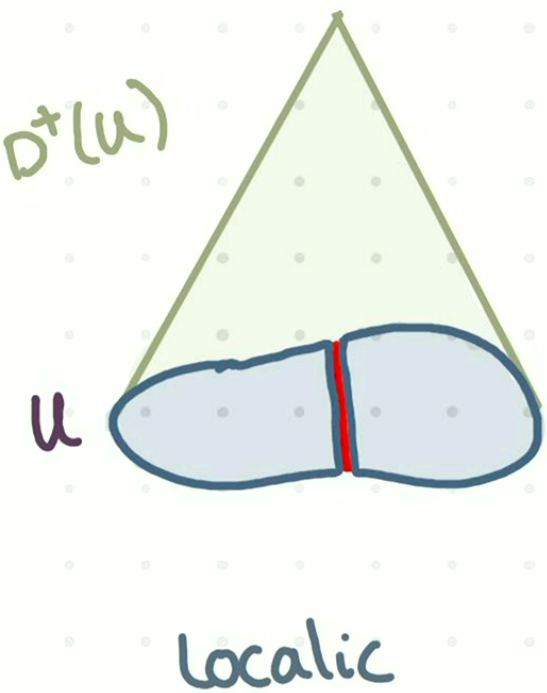
Causal Coverage

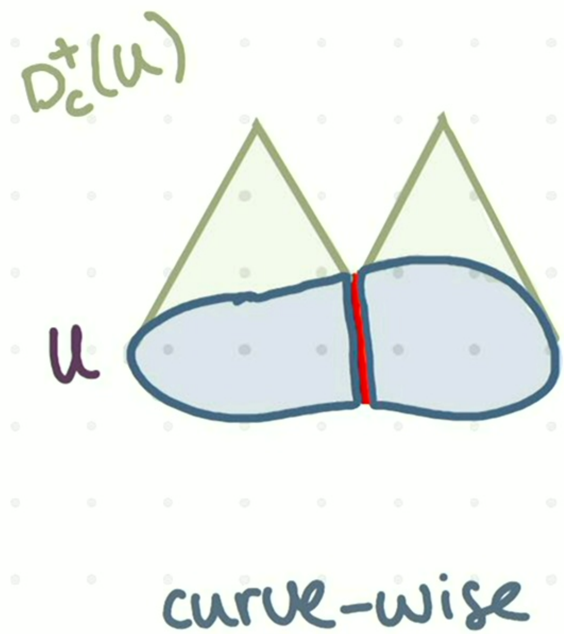


localic

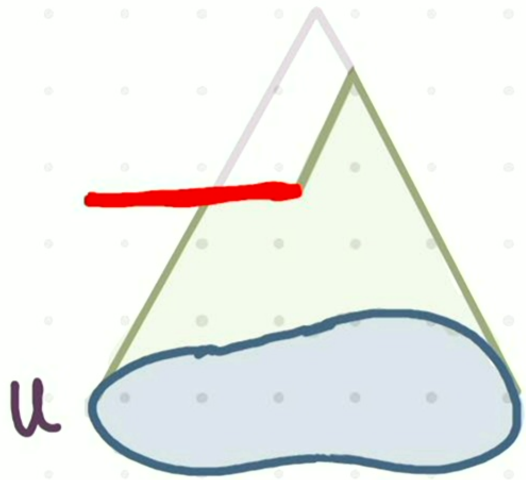
curve-wise

Causal Coverage





Causal Coverage



Q. Problem of holes?

"Even the Minkowski space is holed" [Krasnikov 09]

Sheaves?

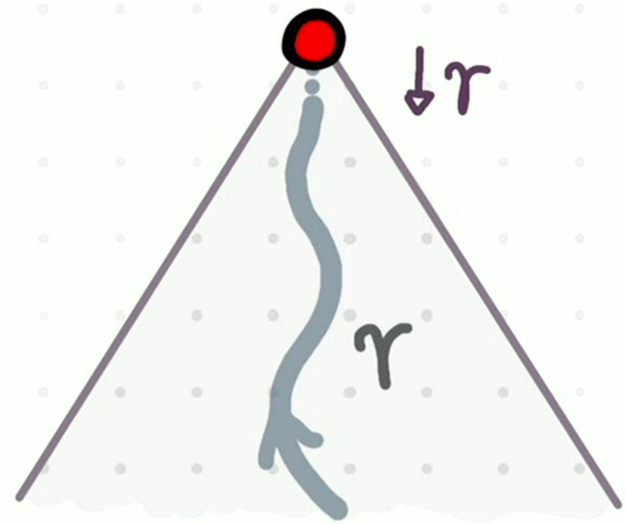
EXAMPLE

Sheaf of solutions to the wave equation on Minkowski space

Causal Boundaries

joint with:
Prakash Panangaden

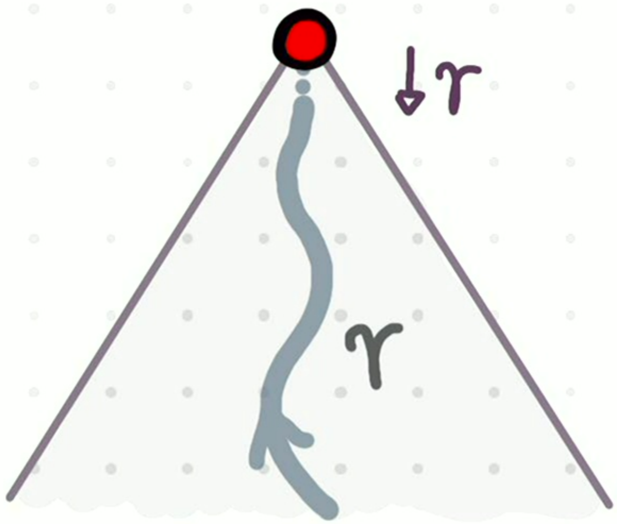
IDEA add ideal points to space(time) via would-be limits of curves γ



Causal Boundaries

joint with:
Prakash Panangaden

DEFN. a **coprime** is $\emptyset \neq P = \downarrow P$ s.t.
$\downarrow A \vee \downarrow B = P \implies P = \downarrow A$ or $\downarrow B$



THEOREM $P \in \mathrm{im}(\downarrow)$ is coprime
iff
$\exists$ causal curve γ s.t. $P = \downarrow\gamma$

[Geroch, Kronheimer, Penrose 72]

Causal Boundaries

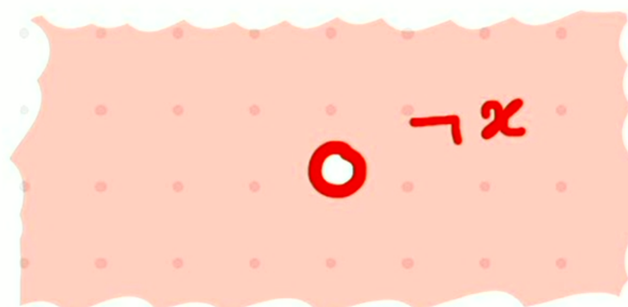
$\mathrm{im}(\downarrow) = \{\downarrow U : U \in \mathcal{O}X\}$
defines a locale X^∇

$$\begin{tikzcd} \mathrm{im}(\downarrow) \arrow[r, shift left=1.5, "\neg"] \arrow[r, phantom, "\perp"] & \mathrm{im}(\uparrow)^{op} \arrow[l, shift left=1.5, "\neg"] \end{tikzcd}$$

THEOREM

$p \in \mathrm{im}(\downarrow)$ is coprime
iff
$\exists$ causal curve γ s.t. $P = \downarrow \gamma$

[Geroch, Kronheimer, Penrose 72]



Causal Boundaries

$\mathrm{im}(\downarrow) = \{\downarrow U : U \in \mathcal{O}X\}$

defines a locale X^{∇}

$$\begin{tikzcd} \mathrm{im}(\downarrow) \arrow[r, shift left, "\neg"] \arrow[r, phantom, "\perp"] & \mathrm{im}(\uparrow)^{op} \arrow[l, shift left, "\neg"] \end{tikzcd}$$

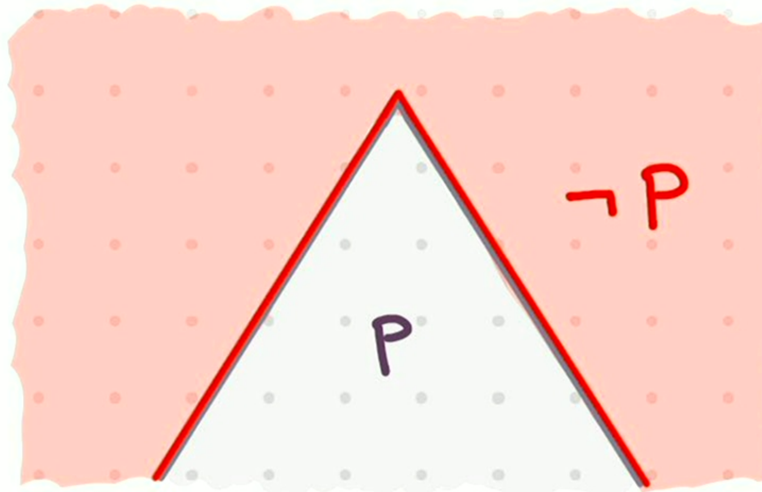
THEOREM

$P \in \mathrm{im}(\downarrow)$ is coprime

iff

$\exists$ causal curve γ s.t. $P = \downarrow\gamma$

[Geroch, Kronheimer, Penrose 72]



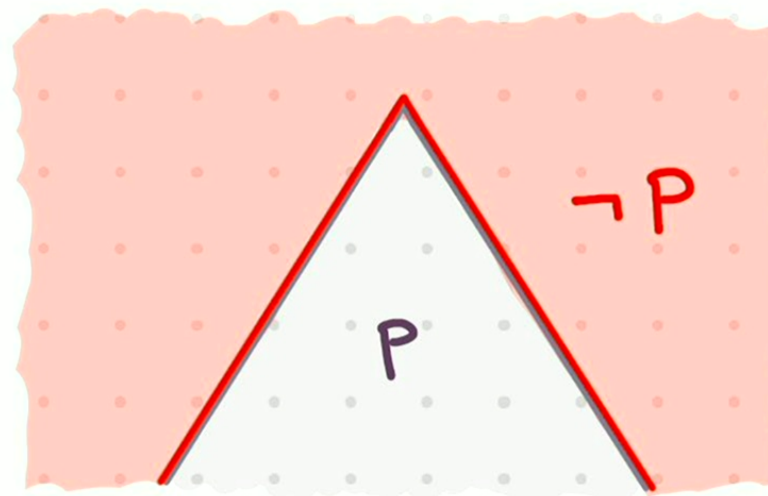
Causal Boundaries

THM: In any spacetime M:

$$\mathrm{IP}(M) \cong \mathrm{pt}(M^{\Delta})$$

$$P \longmapsto \neg P$$

$$\mathrm{im}(\downarrow) \cong \mathrm{im}(\uparrow)^{op}$$



M^{Δ} as localic future causal boundary

Causal Boundaries

LEMMA There is an isomorphism:

$$\left\{\begin{array}{c}\text{"convex" ordered} \\ \text{locales s.t. } \uparrow, \downarrow \\ \text{join-preserving}\end{array}\right\} \cong \left\{\begin{array}{c}\text{"reflective"} \\ \text{biframes}\end{array}\right\}^{op}$$

$$(X, \trianglelefteq) \longmapsto \left(\mathcal{O}X, \mathrm{im}(\uparrow), \mathrm{im}(\downarrow)\right)$$

Q. biframe compactifications for ordered locales?

AQFTs

HAAG-KASTLER

a functor

$$\mathcal{A} : \mathcal{O}X \longrightarrow \mathrm{Alg} \;;$$
$$U \longmapsto \mathcal{A}(U).$$

such that:

if $U, V \in \mathcal{O}X$ space like separated,

then

$$[\mathcal{A}(U), \mathcal{A}(V)] = 0 \quad \text{in } \mathcal{A}(X)$$

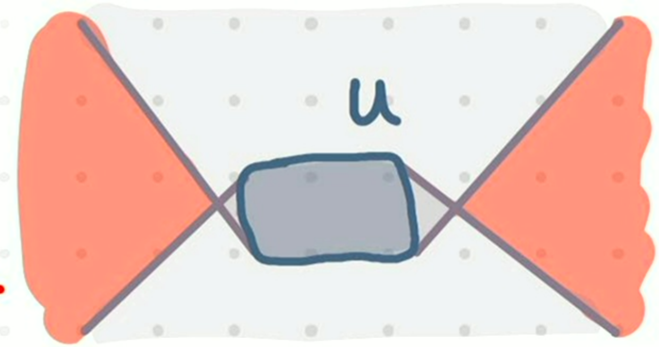
and

$$\mathcal{A}(U) = \mathcal{A}(D^{+}(U)).$$

CAUSAL COMPLMT.

for $U \in \mathcal{O}X$ define:

$$U^{\perp} := \neg(\uparrow U \vee \downarrow U)$$
$$= \neg \uparrow U \wedge \neg \downarrow U.$$



$U^{\perp}$