Title: Quantum Permutations as Quantum Coordinate Transformations

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Series: Quantum Foundations, Quantum Information

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Quantum Permutations as Quantum Coordinate Transformations

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Motivation: uniting symmetries

Quantum Mechanics	General Relativity
Unitary Symmetry: invariance to **basis change**	Coordinate Symmetry: invariance to **naming of spacetime points**

Quantum Permutations: intuition

Classical Permutations:

Q does j become i?

A yes/no = 0/1

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & & & & \\ \boxed{0} & 0 & & & & \end{pmatrix}$$

permute the N basis elements of $\mathscr{H}_{\text{QM}}$

Quantum Permutations:

Q where does j become i?

A a subspace (of $\tilde{\mathscr{H}}$)

“quantum permute” the N subspaces of $\mathscr{H}_{\text{QM}} \otimes \tilde{\mathscr{H}}$

$$\begin{pmatrix} u_{11} & \dots & u_{1N} \\ \vdots & & \vdots \\ u_{N1} & \dots & \boxed{u_{NN}} \end{pmatrix}$$

Not a group, not closed under composition

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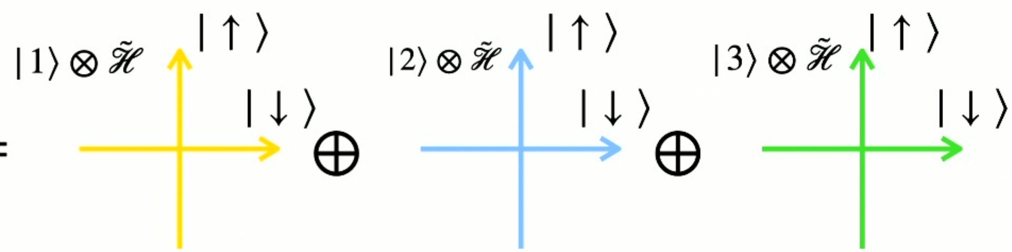
An example for a quantu… …utation

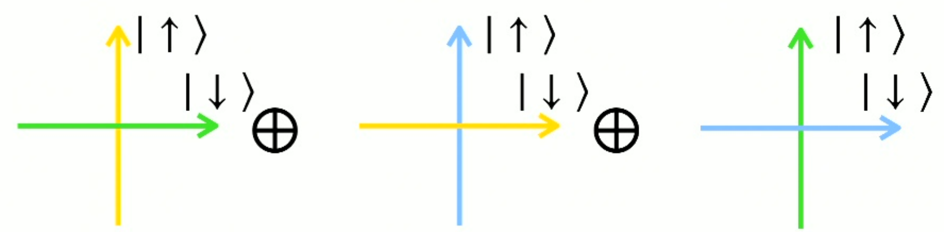
The mathematicians call this 'classical'

$$u = \begin{pmatrix} |\uparrow\rangle\langle\uparrow| & 0 & |\downarrow\rangle\langle\downarrow| \\ |\downarrow\rangle\langle\downarrow| & |\uparrow\rangle\langle\uparrow| & 0 \\ 0 & |\downarrow\rangle\langle\downarrow| & |\uparrow\rangle\langle\uparrow| \end{pmatrix}$$

$$\mathscr{H} = \bigoplus_x |x\rangle \otimes \tilde{\mathscr{H}} =$$



$$u\mathscr{H} = \bigoplus_x |x\rangle \otimes \bigoplus_y u_{xy}\tilde{\mathscr{H}} =$$



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Quantum-controlled permutations

The mathematicians call a quantum permutation u 'classical' if all entries u_{xy} **commute**.

This definition is equivalent to a quantum-controlled permutation:

$$u = \sum_{g \in S_N} \sigma_g \otimes \pi_g$$

c***lassical*** permutations

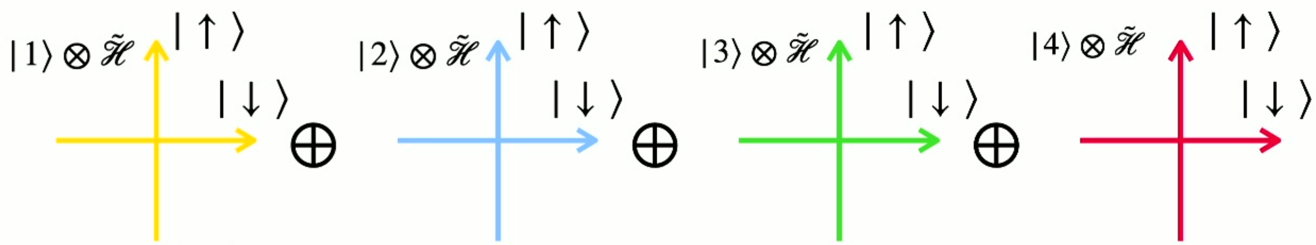
complementary orthogonal projections on $\tilde{\mathscr{H}}$

⟶ **The rest** of the quantum permutations are richer, we will call them **"beyond quantum-controlled" (BQC)**

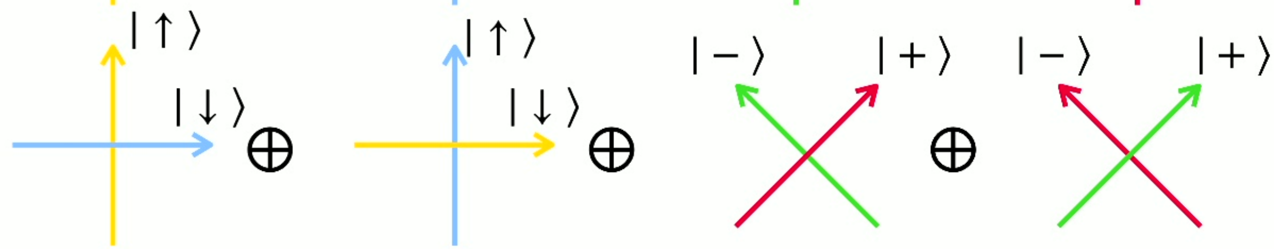
Example for a BQC permutation

$$u = \begin{pmatrix} |\uparrow\rangle\langle\uparrow| & |\downarrow\rangle\langle\downarrow| & 0 & 0 \\ |\downarrow\rangle\langle\downarrow| & |\uparrow\rangle\langle\uparrow| & 0 & 0 \\ 0 & 0 & |-\rangle\langle-| & |+\rangle\langle+| \\ 0 & 0 & |+\rangle\langle+| & |-\rangle\langle-| \end{pmatrix}$$

$$\mathscr{H} = \bigoplus_x |x\rangle \otimes \tilde{\mathscr{H}} =$$



$$u\mathscr{H} = \bigoplus_x |x\rangle \otimes \bigoplus_y u_{xy}\tilde{\mathscr{H}} =$$

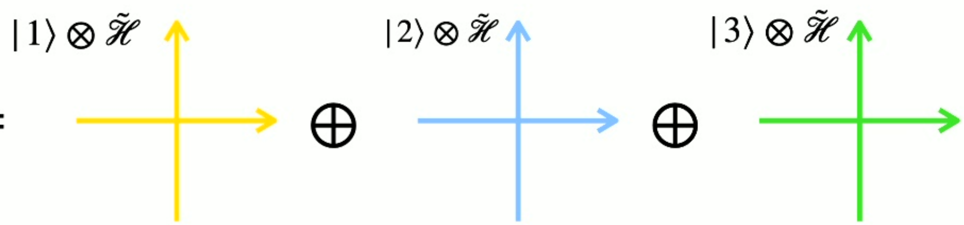


Quantum coordinate systems?

Quantum coordinate systems?

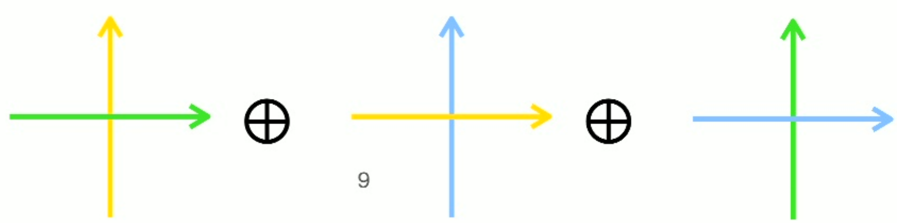
	Usual QM, with $\mathscr{H}_{\mathrm{QM}}$	Here, with $\mathscr{H}_{\mathrm{QM}} \otimes \tilde{\mathscr{H}}$
coordinate system	choice of **1-d eigenspaces** of position	choice of **multi-d eigenspaces** of position
(discrete) coordinate transformations	**classical** permutations	**quantum** permutations = (discrete) **quantum** coordinate transformations

$$\mathscr{H} = \bigoplus_x |x\rangle \otimes \tilde{\mathscr{H}} =$$



a quantum coordinate system

$$u\mathscr{H} = \bigoplus_x |x\rangle \otimes \bigoplus_y u_{xy}\tilde{\mathscr{H}} =$$

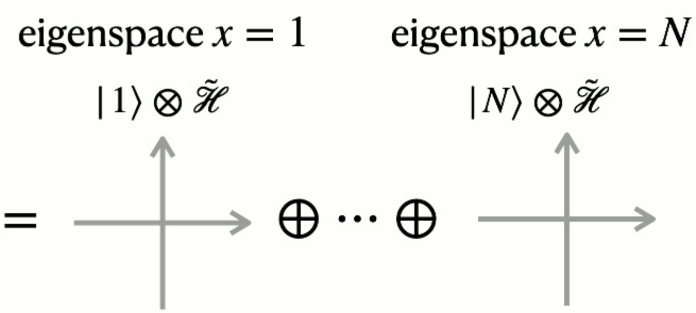


a different quantum coordinate system

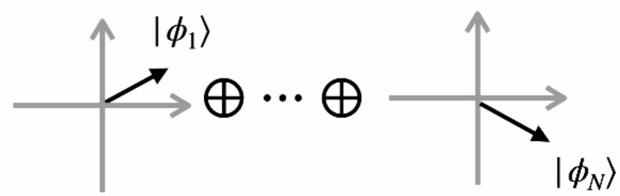
Quantum coordinate systems?

A particle on a line, but with degeneracy in the position operator X.

$$\mathscr{H} = \mathscr{H}_{\mathrm{QM}} \otimes \tilde{\mathscr{H}} =$$



$$|\psi\rangle = \bigoplus_x \psi(x)\,|x\rangle\,|\phi_x\rangle =$$



Quantum coordinate systems?

The action of a magic unitary on a state:

$$u\,|\psi\rangle = \bigoplus_x \psi(x)\,|x\rangle \otimes \sum_y u_{xy}\,|\phi_y\rangle = \begin{pmatrix} u_{11} & \dots & u_{1N} \\ \vdots & & \vdots \\ u_{N1} & \dots & u_{NN} \end{pmatrix} \begin{pmatrix} |1\rangle\,|\phi_1\rangle \\ \vdots \\ |N\rangle\,|\phi_N\rangle \end{pmatrix}$$

Quantum coordinate systems?

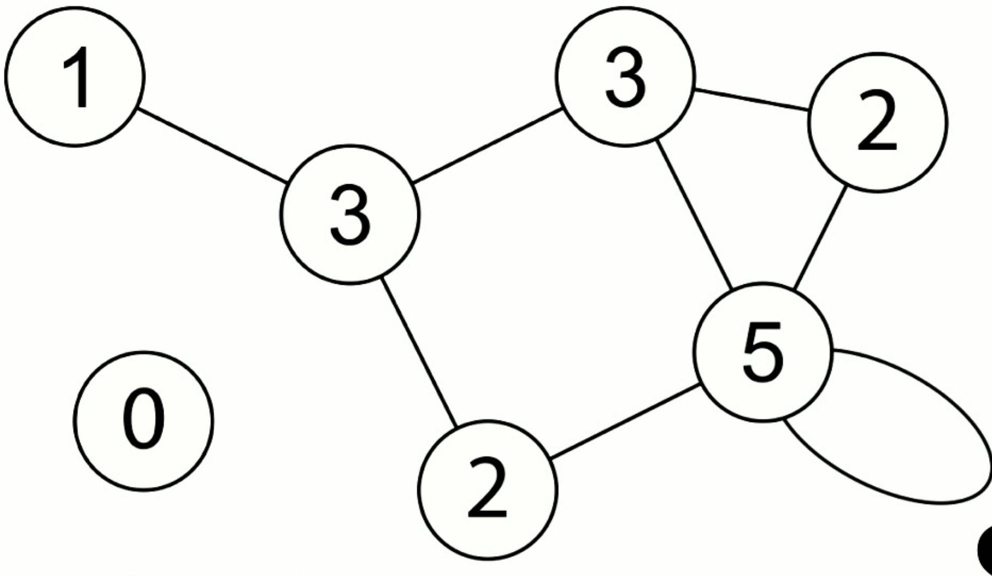
This way, one can "transfer" properties from the state into the descriptive redundancy.

$$|\psi\rangle_{\mathrm{QM}} = \sum_x \psi(x)\,|x\rangle \xrightarrow{\text{Embed in enlarged space}} |\psi\rangle = \sum_x \psi(x)\,|x\rangle\,|e_x\rangle$$

ONB of $\tilde{\mathscr{H}}$

$$\longrightarrow \exists u,\; u\,|\psi\rangle = |x=1\rangle\,|\phi\rangle$$

For some $|\phi\rangle \in \tilde{\mathscr{H}}$



Γ

Symmetries of Graphs

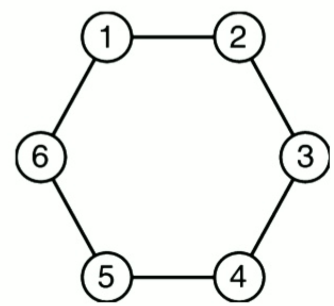
Symmetries of graphs

The classical symmetries of a graph are permutations which preserve the edge structure:

$$\sigma^\dagger A_\Gamma \sigma = A_\Gamma \longleftarrow \quad \text{adjacency matrix}$$

Called automorphisms of the graph, Aut Γ.

e.g. if the graph is a cycle — symmetries include rotations.



Symmetries of graphs

If several permutations σ_i exist, any quantum controlled unitary

$$u = \sum_i \sigma_i \otimes \pi_i$$

is a quantum-controlled permutation which is also symmetry of the graph.

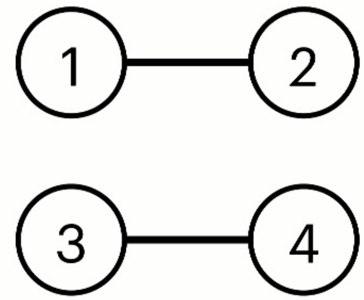
But this is a trivial extension.

For some graphs there exist symmetries u which are **beyond quantum-controlled.**

Symmetries of graphs

E.g. the example we gave earlier is a BQC symmetry of the following graph:

$$u = \begin{pmatrix} |\uparrow\rangle\langle\uparrow| & |\downarrow\rangle\langle\downarrow| & 0 & 0 \\ |\downarrow\rangle\langle\downarrow| & |\uparrow\rangle\langle\uparrow| & 0 & 0 \\ 0 & 0 & |+\rangle\langle+| & |-\rangle\langle-| \\ 0 & 0 & |-\rangle\langle-| & |+\rangle\langle+| \end{pmatrix}$$

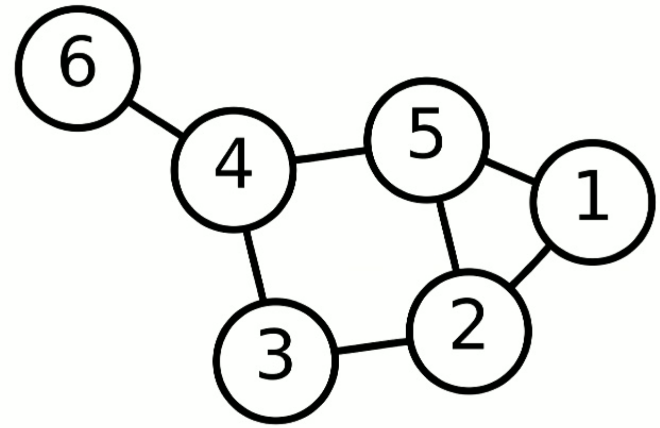


$$u^\dagger A_\Gamma u = A_\Gamma$$

$$u \in \mathrm{QAut}$$

Example: Ising model

Spins on a graph Γ



$$H = \sum_{i \sim_\Gamma j} S_i S_j$$

Symmetries: Aut Γ, classical automorphisms of the graph Γ

$$\sigma H \sigma^\dagger = H$$

Introducing an external space $\tilde{\mathscr{H}}$, the symmetries can be extended to QAut **(*)**

$$u H u^\dagger = H$$

Application: A particle on a graph

<u>Toy example:</u> A d dimensional spin in one of N nodes of a graph Γ.

State: $|\psi\rangle \in \mathscr{H}_N \otimes \mathscr{H}_d,\ |\psi\rangle = \sum_{i\in\Gamma} |i\rangle\,|\psi_i\rangle$

Location Spin

Hamiltonian: amount of "alignment" along edges

$$H = \sum_{i\sim_\Gamma j} |i\rangle\langle j| \otimes 1_d = A_\Gamma \otimes 1_d \quad , \quad \langle\psi|H|\psi\rangle = \frac{1}{2}\sum_{i\sim_\Gamma j} \mathrm{Re}\langle\psi_i|\psi_j\rangle$$

The symmetries of this physical system are both the classical **and** quantum automorphisms of Γ.

Another example in more detail:
A particle on a graph

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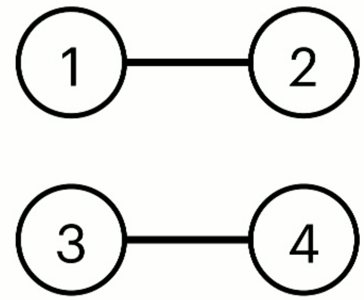
Application: A particle on a graph

$$u = \begin{pmatrix} |\uparrow\rangle\langle\uparrow| & |\downarrow\rangle\langle\downarrow| & 0 & 0 \\ |\downarrow\rangle\langle\downarrow| & |\uparrow\rangle\langle\uparrow| & 0 & 0 \\ 0 & 0 & |+\rangle\langle+| & |-\rangle\langle-| \\ 0 & 0 & |-\rangle\langle-| & |+\rangle\langle+| \end{pmatrix}$$



$$u^\dagger H u = H$$

Classical symmetry is **non-trivially extended into the "beyond quantum-controlled" realm**

Can we use this in QRF transformations?

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Can we use this in QRF transformations?

$$S_{A\to C} = \mathrm{SWAP}_{AC} \sum_{g\in G} |g^{-1}\rangle\langle g|_C \; u_{BC}$$ on a tripartite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

$$\boxed{u_{BC} = \sum_{g\in G} u_g \otimes |g\rangle\langle g|}$$

up to a swap and a reflection: a quantum-controlled magic unitary.

What if we use a BQC u? E.g. is entanglement + coherence still conserved?

Cepollaro et al, arXiv:2406.19448

e.g. incorporating two distinct symmetry groups G_1, G_2 to create the following BQC permutation:

$$u_{BC} = \sum_{g_1\in G_1} u_{g_1} \otimes |g_1\rangle\langle g_1| + \sum_{g_2\in G_2} v_{g_2} \otimes |g_2\rangle\langle g_2|$$

Thanks for listening.

- Magic unitaries/quantum permutations are an interesting generalisation of the concept of permutations into the quantum realm.
- They can be interpreted as ***changes into quantum coordinate systems***
- There are non-trivial, beyond quantum-controlled permutations which ***extend the symmetry groups*** of some Hamiltonians.
- Usual ***QRF transformations are only quantum-controlled***.
- Extending them with BQC permutations might be insightful or useful.
- Might allow to incorporate more than one observable/symmetry group.

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