Title: Knot invariants and indefinite causal order

Speakers: Samuel Fedida

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Knot invariants and indefinite causal order

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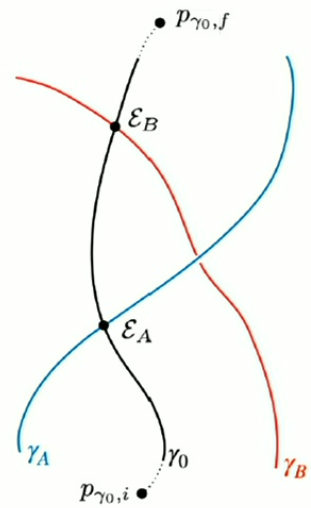
September 2024

Events in spacetime

Events

The crossing of two worldlines within a spacetime $(\mathcal{M}, g)$ at a spacetime point $p_{\mathcal{E}} \in \mathcal{M}$ defines an *event* $\mathcal{E}$. The event defined by the crossing of a worldline γ_0 with a worldline γ_a is denoted $\mathcal{E}_a$, $a = 1, ..., N$ for N events.



See this[1] paper for a treatment of events in spacetimes in superposition.

[1] Anne-Catherine de la Hamette et al. *Quantum diffeomorphisms cannot make indefinite causal order definite*. 2022. arXiv: 2211.15685 [quant-ph].

Events in spacetime

Events

The crossing of two worldlines within a spacetime $(\mathcal{M}, g)$ at a spacetime point $p_{\mathcal{E}} \in \mathcal{M}$ defines an *event* $\mathcal{E}$. The event defined by the crossing of a worldline γ_0 with a worldline γ_a is denoted $\mathcal{E}_a$, $a = 1, ..., N$ for N events.

We parameterise γ_0 by the proper time of the test particle and define the proper time τ_a associated to the event $\mathcal{E}_a$ by

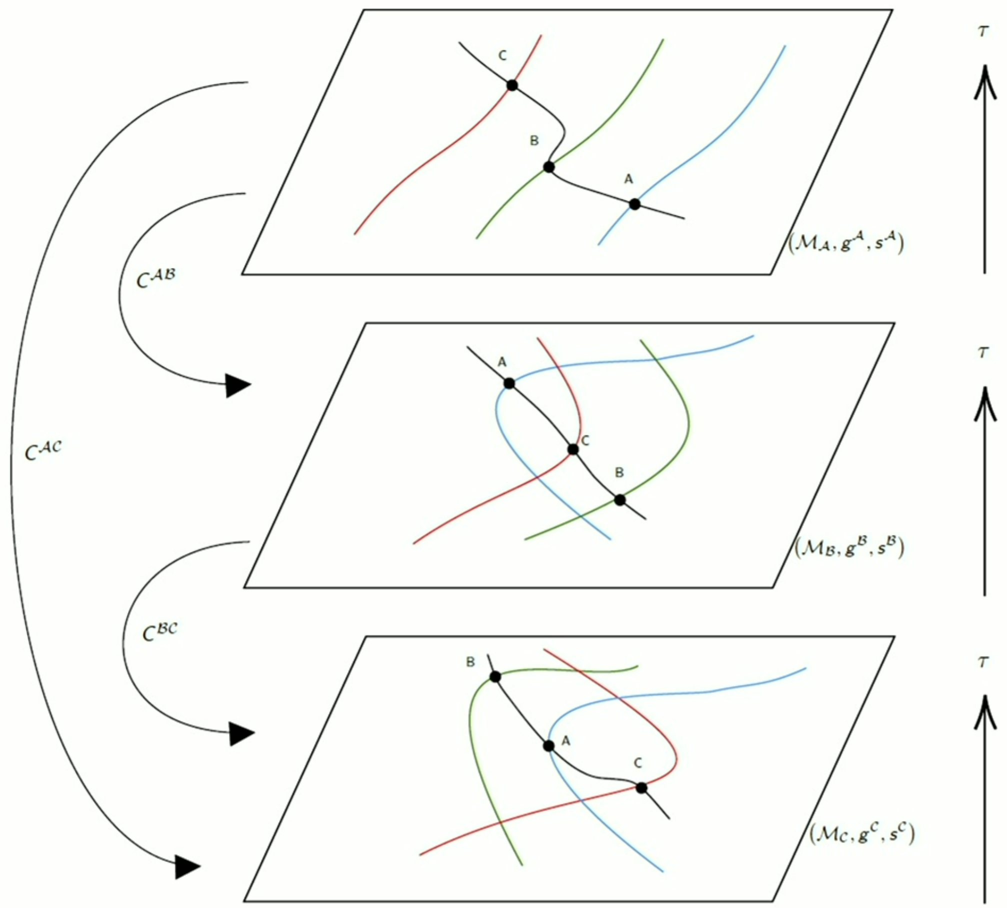
$$\tau_a = \int_{p_{\rm init}}^{p_{\mathcal{E}_a}} \frac{ds}{c} = \int_{p_{\rm init}}^{p_{\mathcal{E}_a}} \frac{\sqrt{-g_{\mu\nu} dx^\mu dx^\nu}}{c},$$

where $p_{\rm init}$ denotes an arbitrary initial point of the test particle's worldline. For two distinct points $p_{\mathcal{E}_a}$ and $p_{\mathcal{E}_b}$ we have that $\tau_a \neq \tau_b$. Let

$$\Delta\tau = \tau_b - \tau_a = s_{ab}^{\mathcal{M}} |\tau_b - \tau_a|$$

so that $s_{ab}^{\mathcal{M}} = \text{sign}(\tau_b - \tau_a)$ is the *causal order* of the events, with $s = 1$ if $\mathcal{E}_2$ lies in the future of $\mathcal{E}_1$ and $s = -1$ if it lies in the past.

Spacetime representation



Identification of points across superpositions of spacetimes

Identifying points across spacetimes is problematic in quantum gravity:

- If spacetime is non-classical, the inner product in the superposition

$$\psi(x) := \langle x|\psi\rangle = \alpha_1\psi_1(x) + \alpha_2\psi_2(x) \equiv \alpha_1 \langle x|\psi_1\rangle + \alpha_2 \langle x|\psi_2\rangle$$

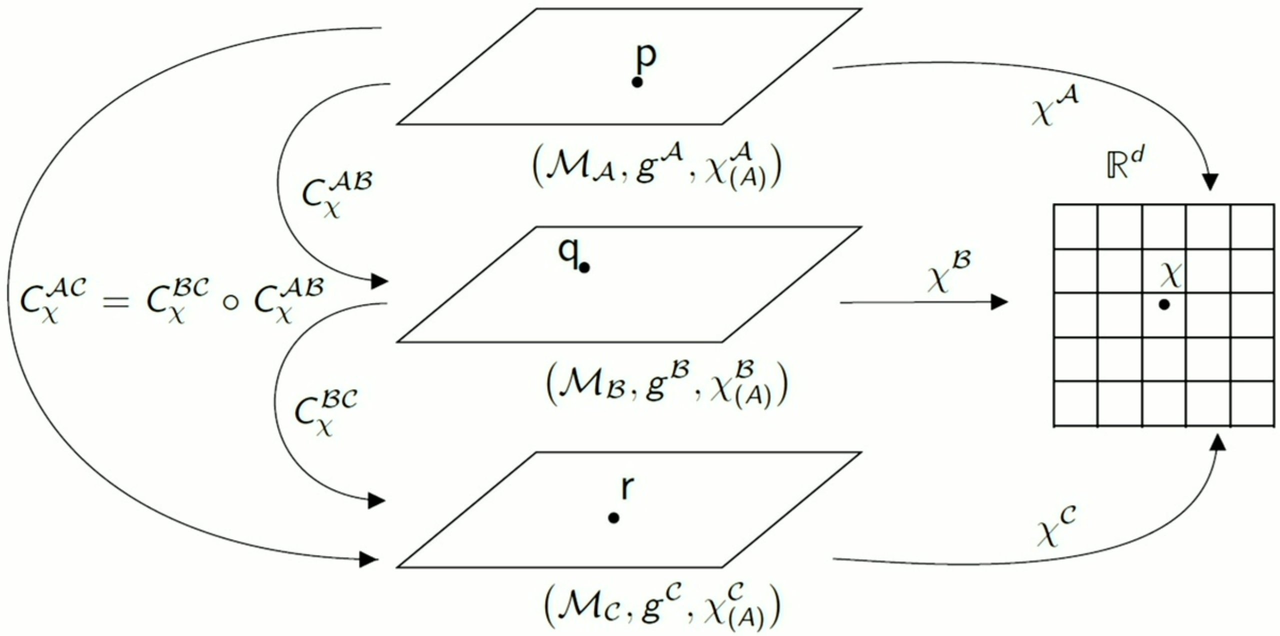
  is ill-defined (quantum diffeomorphisms / different topologies).
- There are infinitely many ways to associate points across spacetimes through arbitrarily many different projections[2].

How does one do simple interference experiments in a lab?

[2] Flaminia Giacomini and Časlav Brukner. "Einstein's Equivalence principle for superpositions of gravitational fields and quantum reference frames". In: (2023). arXiv: 2012.13754 [quant-ph].

Identification of points across superpositions of spacetimes

See[3] and Anne-Catherine de la Hamette's talk on Friday.



[3]Viktoria Kabel et al. *Identification is Pointless: Quantum Reference Frames, Localisation of Events, and the Quantum Hole Argument*. 2024. arXiv: 2402.10267 [quant-ph].

Identification of events across superpositions of spacetimes

We can now identify the N events $\mathcal{E}_a^{\mathcal{X}}$ across a superposition of M spacetimes $\mathcal{X} = \mathcal{A}, \mathcal{B}, ...$ for $a = 1, ..., N$. For given a, all instances $\mathcal{E}_a^{\mathcal{A}}, \mathcal{E}_a^{\mathcal{B}}, ...$ refer to the same physical event: the crossing of the worldlines γ_0 with γ_a in different branches of the superposition.

This gives us a collection of causal orders $\{s_{ab}^{\mathcal{X}}\}_{a,b=1,...,N}^{\mathcal{X}=\mathcal{A},\mathcal{B},...}$, from which we can write an *ordered collection of events*

$$S_{\mathcal{X}} := \{\mathcal{E}_1^{\mathcal{X}} \prec \mathcal{E}_2^{\mathcal{X}} \prec ....\}$$

where the total order is given by $\mathcal{E}_a^{\mathcal{X}} \prec \mathcal{E}_b^{\mathcal{X}}$ if $s_{ab}^{\mathcal{X}} = 1$.

Causal order quantifiers

Causal order quantifiers

- Pairwise causal order: are two events in indefinite causal order (ICO)?

$$\mathfrak{s}_{ab}^{\mathcal{A}\mathcal{B}} = s_{ab}^{\mathcal{A}} s_{ab}^{\mathcal{B}}$$

with $\mathfrak{s}_{ab}^{\mathcal{A}\mathcal{B}} = -1$ for ICO, $\mathfrak{s}_{ab}^{\mathcal{A}\mathcal{B}} = +1$ otherwise[4].

- Causal indefiniteness: how many pairs of events are in pairwise ICO?

$$\delta(\mathcal{A},\mathcal{B}) := \sum_{1 \leq a < b}^{N} \left| s_{ab}^{\mathcal{A}} - s_{ab}^{\mathcal{B}} \right|$$

- Total causal indefiniteness: how many events are in pairwise ICO across the entire superposition of spacetimes?

$$\Delta := \sum_{1 \leq \mathcal{X} < \mathcal{Y}}^{M} \delta(\mathcal{X},\mathcal{Y})$$

[4] Anne-Catherine de la Hamette et al. *Quantum diffeomorphisms cannot make indefinite causal order definite*. 2022. arXiv: 2211.15685 [quant-ph].

Properties of causal indefiniteness

- $\delta(\mathcal{A},\mathcal{B}) = 0$ for definite pairwise causal order across all events

$$S_{\mathcal{A}} = \{A \prec B \prec C \prec ... \prec N\}$$
$$S_{\mathcal{B}} = \{A \prec B \prec C \prec ... \prec N\}$$

- $\delta(\mathcal{A},\mathcal{B}) = \binom{N}{2}$ for indefinite pairwise causal order across all events

$$S_{\mathcal{A}} = \{A \prec B \prec C \prec ... \prec N\}$$
$$S_{\mathcal{B}} = \{N \prec ... \prec C \prec B \prec A\}$$

$\rightarrow$ *maximally indefinite causal order*

- Additivity under the concatenation of subsequences of events: if

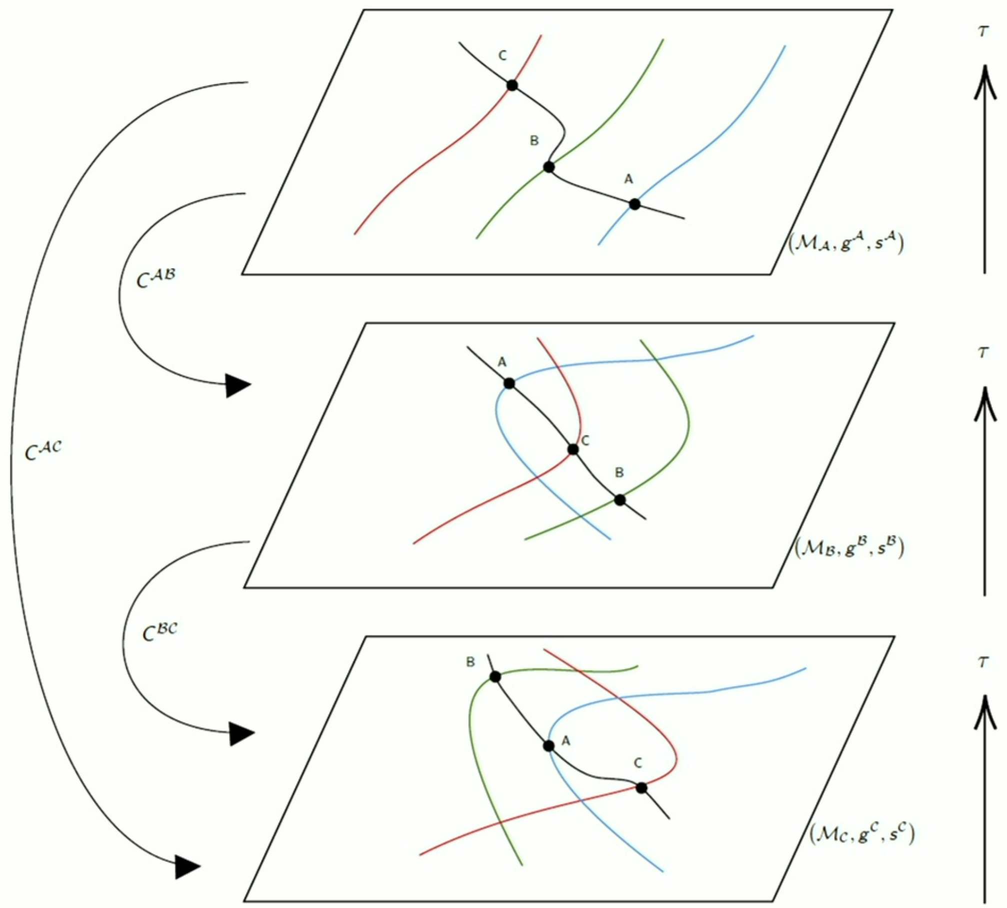
$$S_{\mathcal{A}} = S_{\mathcal{A}}^{(1)} \sqcup S_{\mathcal{A}}^{(2)} \sqcup ...$$
$$S_{\mathcal{B}} = S_{\mathcal{B}}^{(1)} \sqcup S_{\mathcal{B}}^{(2)} \sqcup ...$$

then

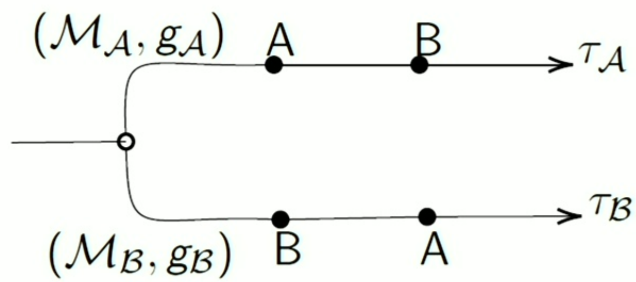
$$\delta(\mathcal{A},\mathcal{B}) = \delta^{(1)}(\mathcal{A},\mathcal{B}) + \delta^{(2)}(\mathcal{A},\mathcal{B}) + ...$$

Diagrammatic representation

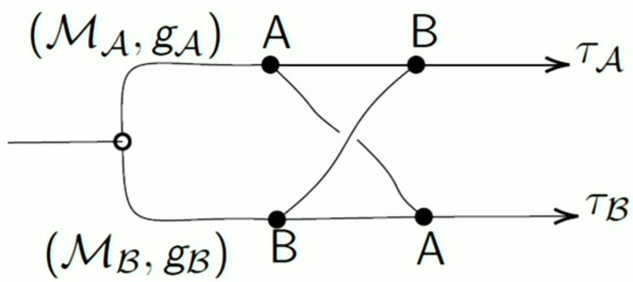
Spacetime representation



Diagrammatic representation



Diagrammatic representation



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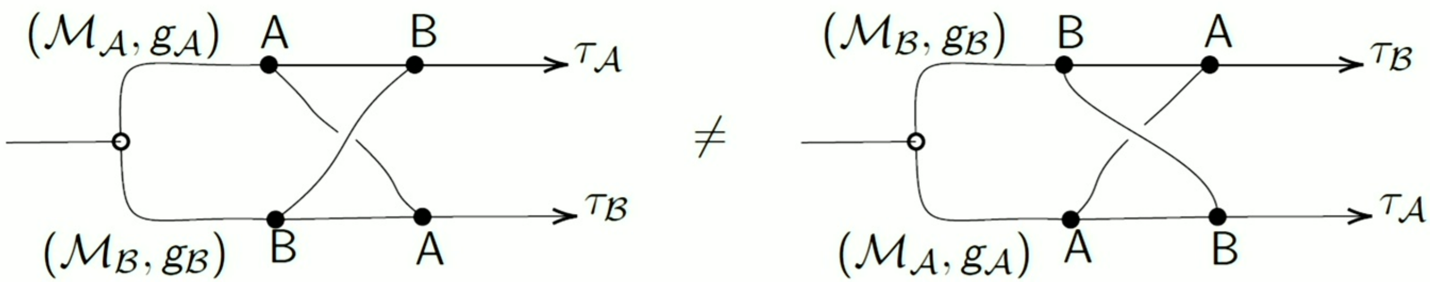

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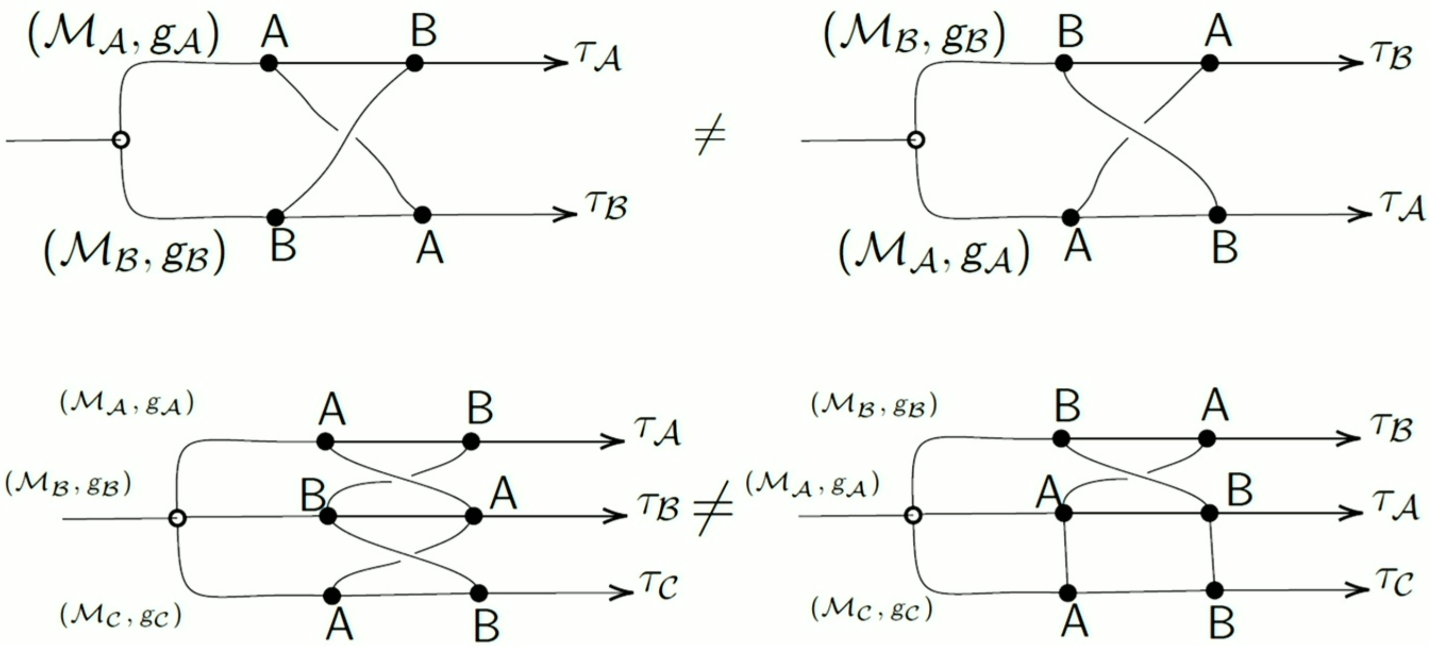

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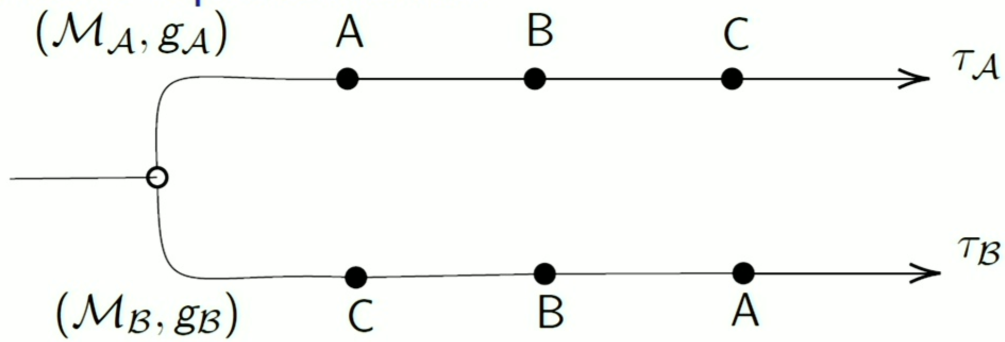

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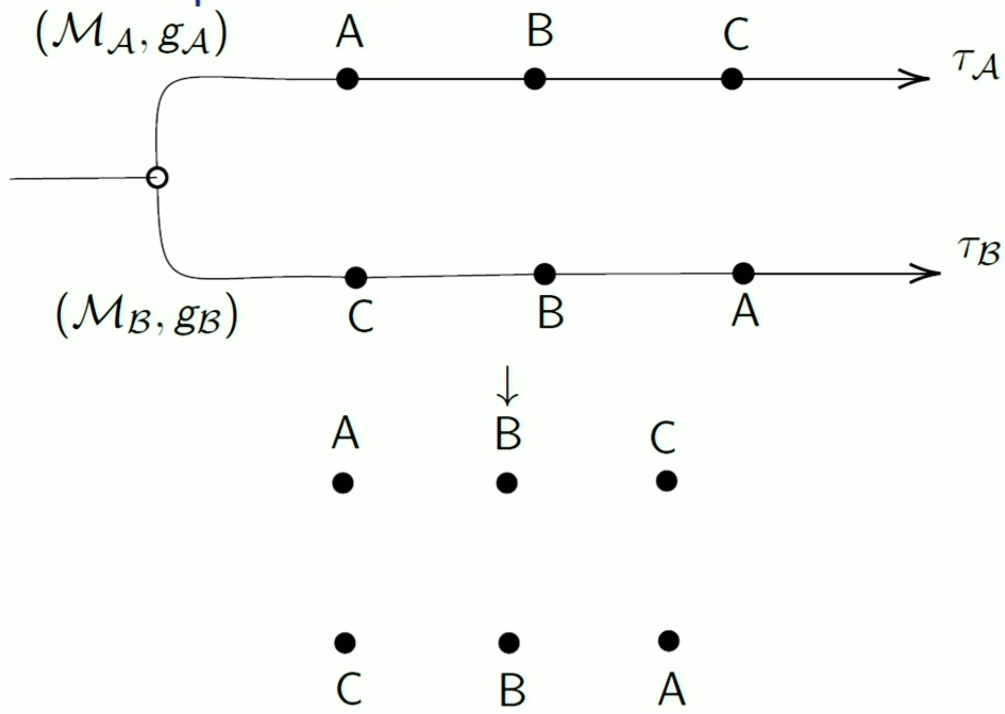

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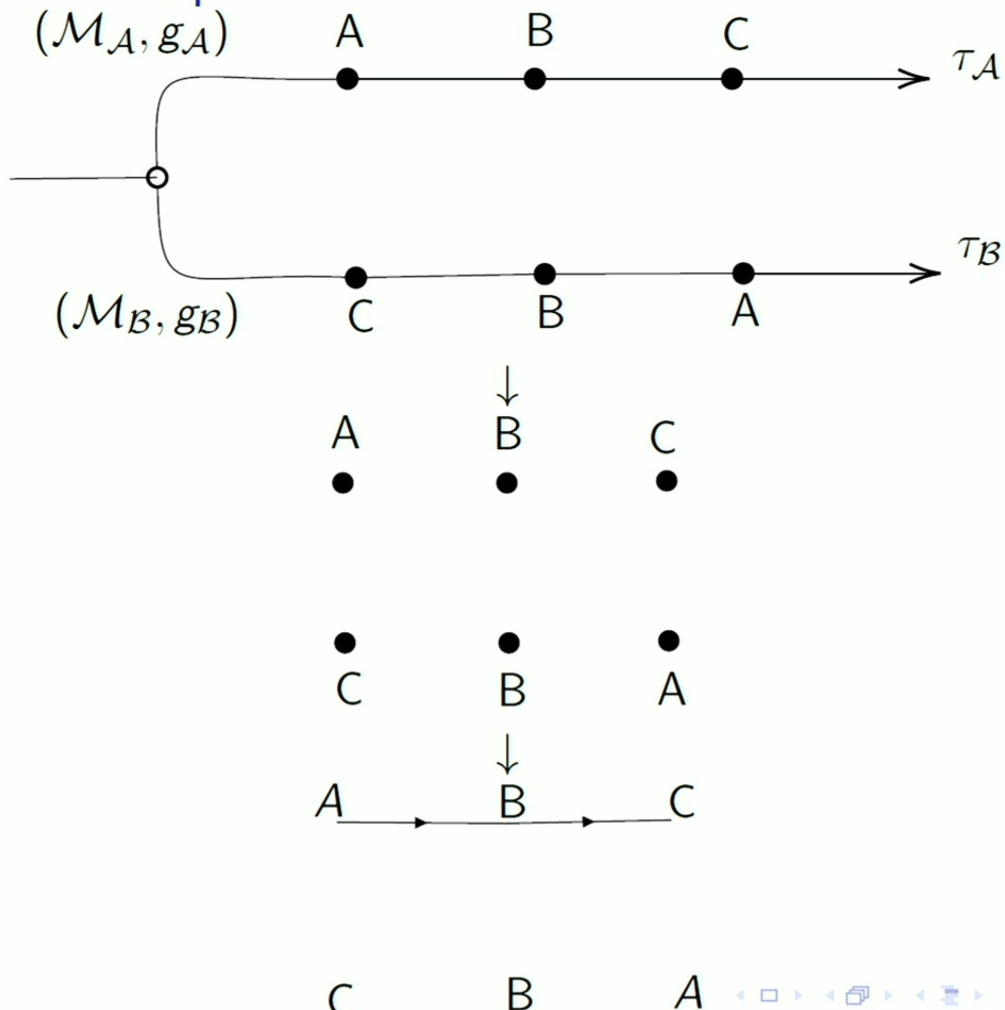

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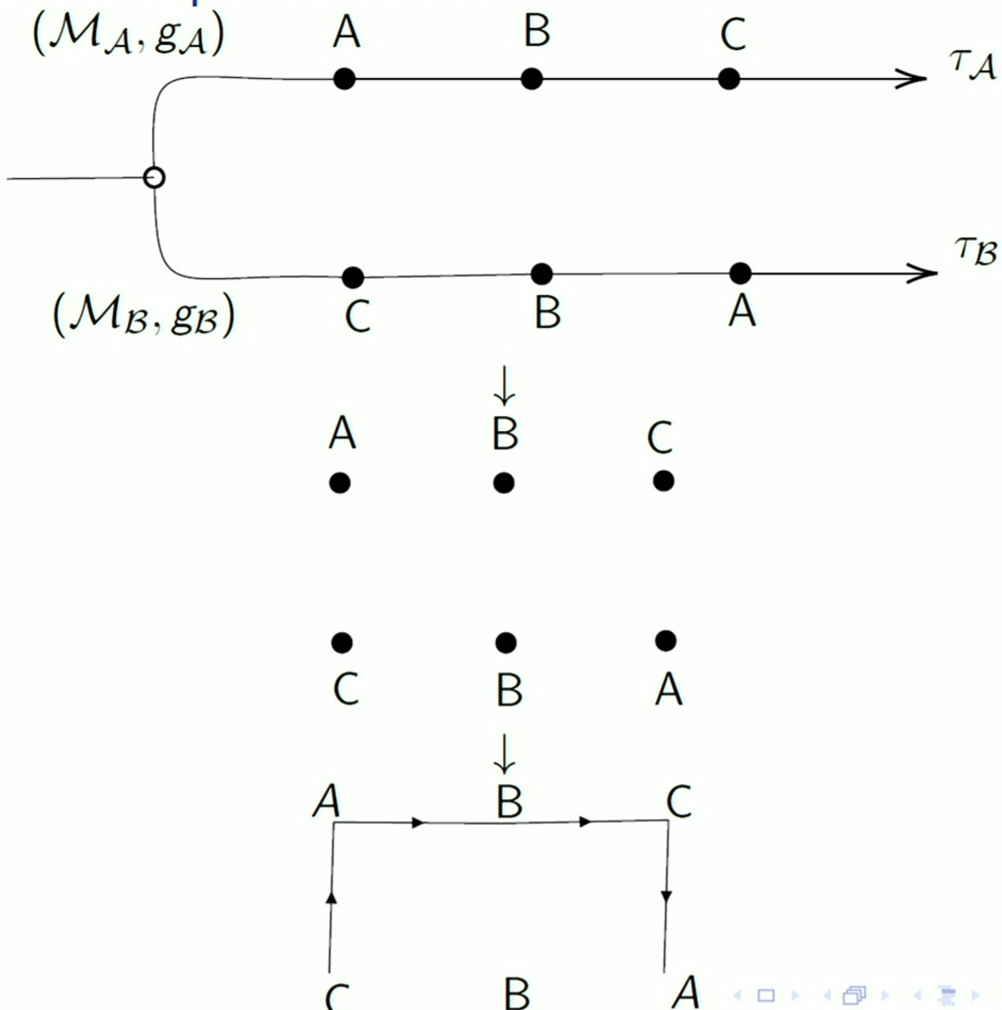

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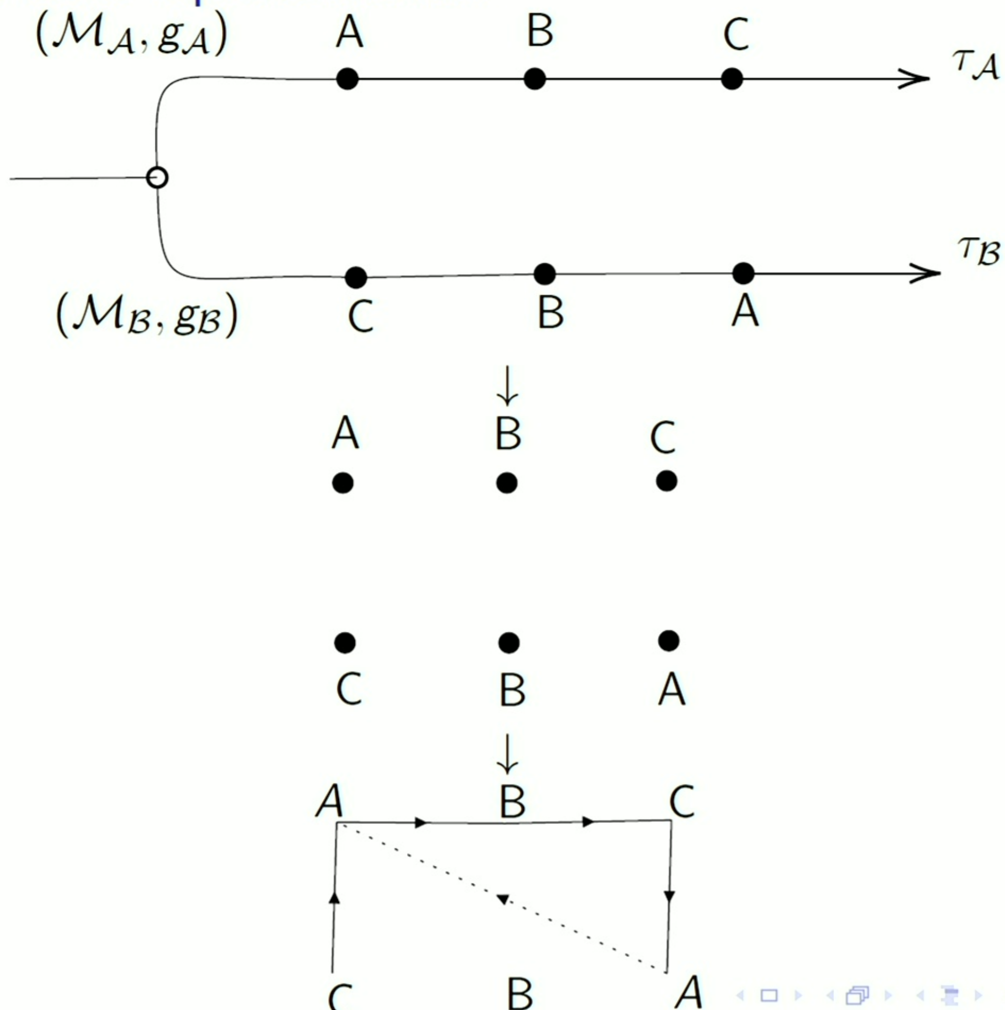

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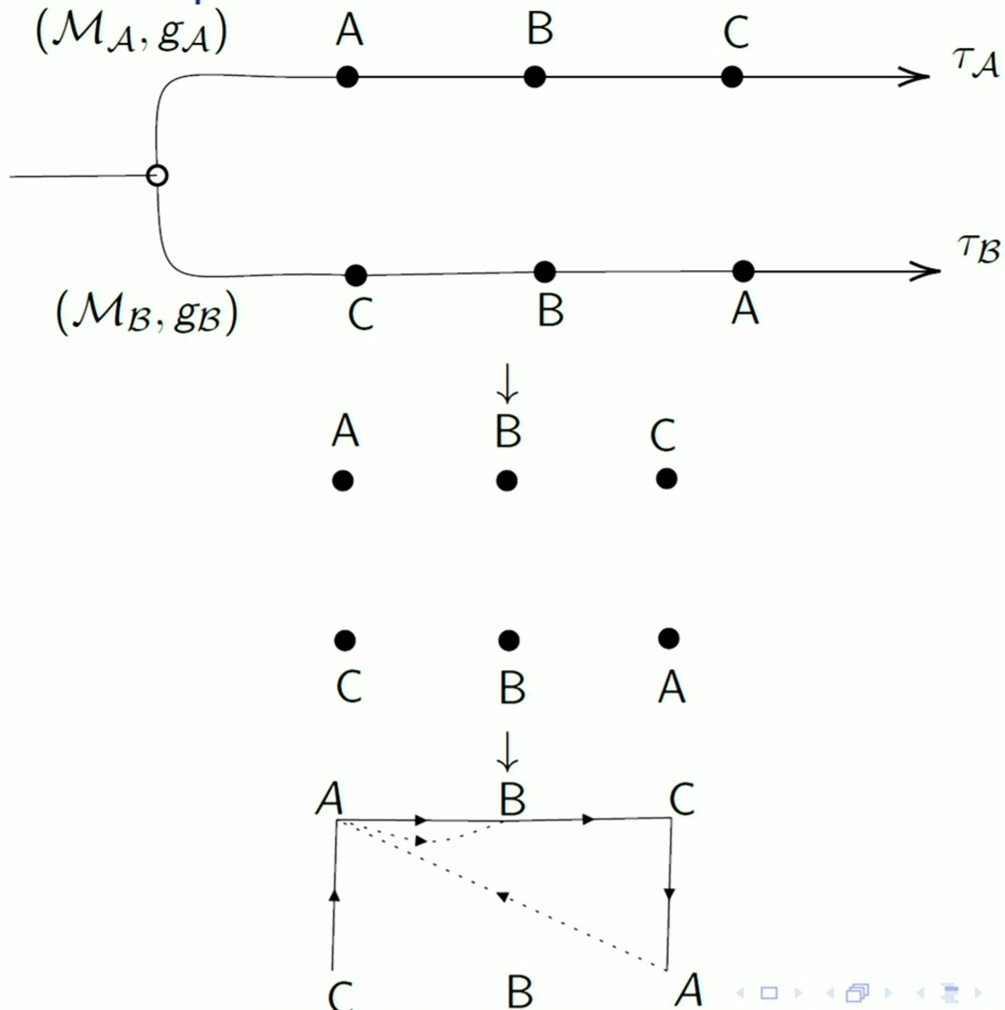

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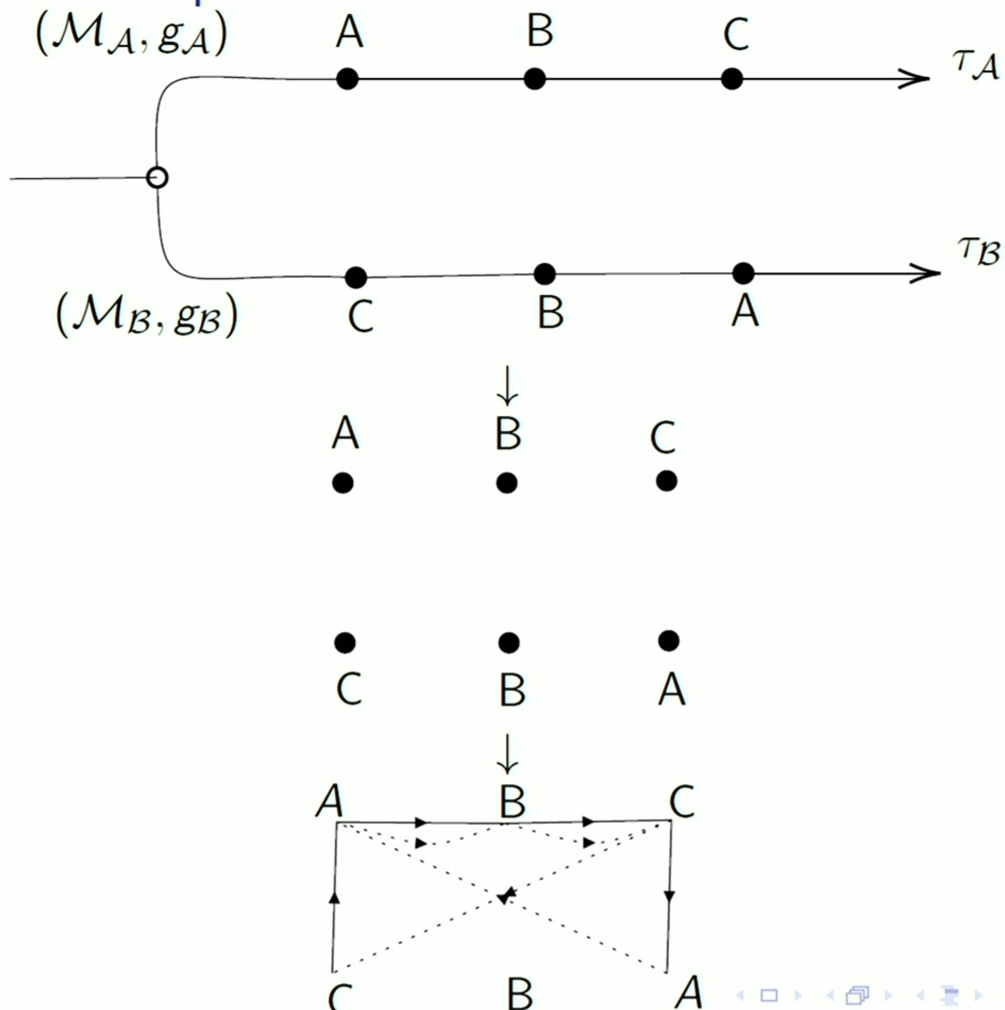
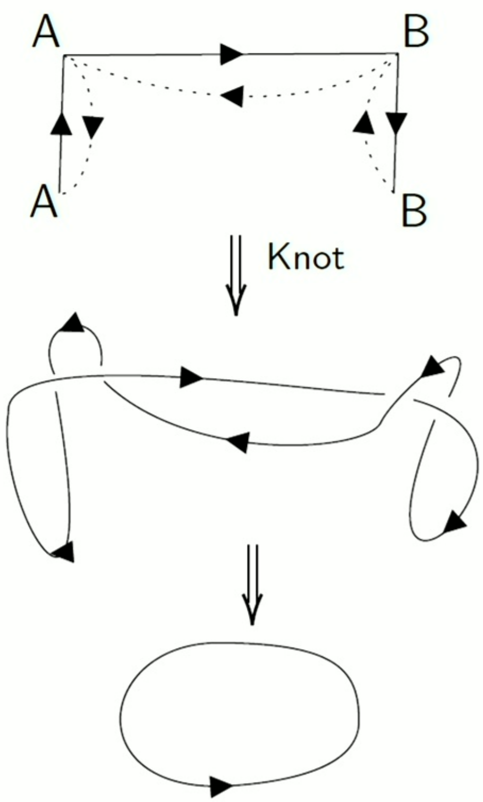
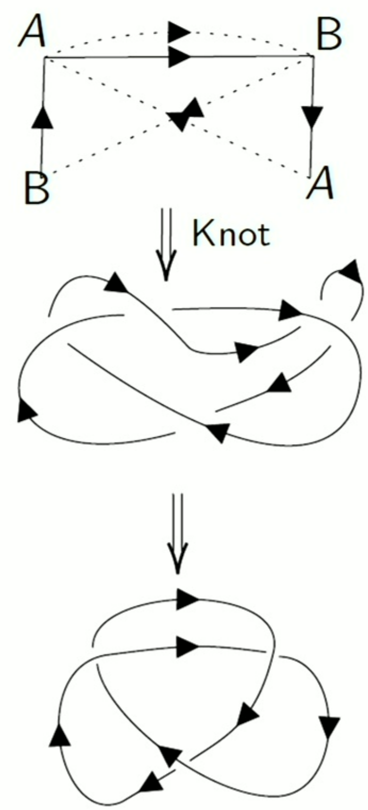


Knot representation

Definite causal order



Indefinite causal order



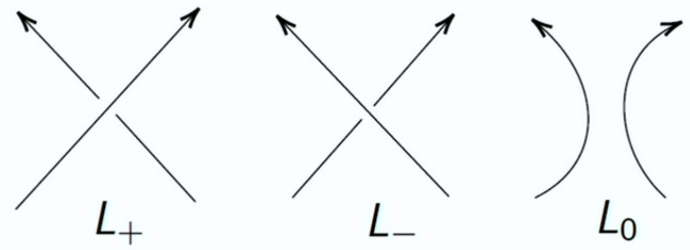
Knot invariants and indefinite causal order

Alexander-Conway polynomial – skein relations

Knot invariant built recursively:

- $\nabla(O) = 1$
- $\nabla(L_+) = \nabla(L_-) + z\nabla(L_0)$

  E.g. for a trefoil knot
  $$\nabla(z) = 1 + z^2$$



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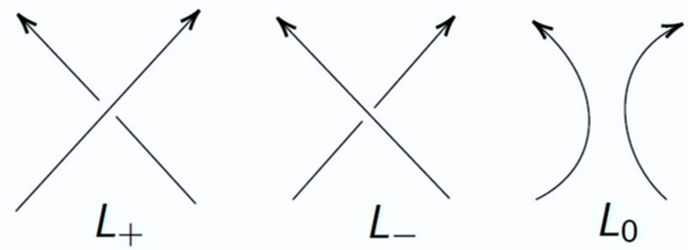


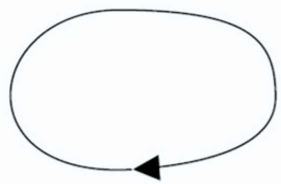
Every coefficient in the Alexander-Conway polynomial has a topological meaning[5]:

- $\nabla(z)_0 = 1$ if it is a knot, 0 otherwise (e.g. a link)
- $\nabla(z)_1 = \mathrm{lk}(K_1, K_2)$ for a link of two knots K_1 and K_2 ($= 0$ for a knot)
- $\nabla(z)_2$ is a measure of "self-linking" of the knot

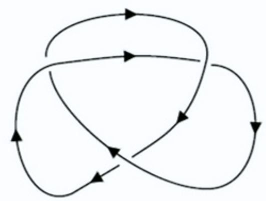
[5] L. H. Kauffman. "On Knots". In: vol. 115. Princeton University Press, 1987, pp. 25–28.

An example: $M = 2, N = 3$ indefinite causal order

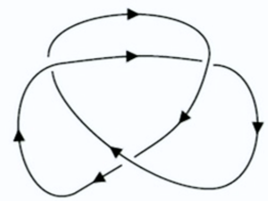
Classification of $M = 2, N = 3$ indefinite causal order



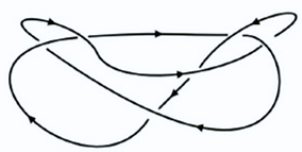
(a) ABC-ABC, $\delta = 0 = \nabla(z)_2$



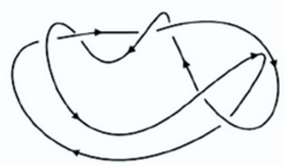
(b) ABC-ACB, $\delta = 1 = \nabla(z)_2$



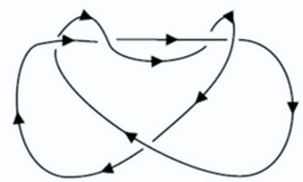
(c) ABC-BAC, $\delta = 1 = \nabla(z)_2$



(d) ABC-BCA, $\delta = 2 = \nabla(z)_2$



(e) ABC-CAB, $\delta = 2 = \nabla(z)_2$



(f) ABC-CBA, $\delta = 3 = \nabla(z)_2$

Knot invariants and indefinite causal order

Theorem

The causal indefiniteness of a superposition of two spacetimes $(\mathcal{M}_{\mathcal{A}}, g_{\mathcal{A}})$ and $(\mathcal{M}_{\mathcal{B}}, g_{\mathcal{B}})$ with N events whose subsequences are either in definite causal order or in maximally indefinite causal order is related to its knot representation as

$$\nabla(z)_2 = \delta(\mathcal{A}, \mathcal{B})$$

Proof (sketch)

- Definite causal order $\Rightarrow$ unknot
- Lemma: Maximally indefinite causal order $\Rightarrow$ $(2, 2N-1)$-torus knots.
- Alexander-Conway polynomial of torus knots: Fibonacci sequence

$$\Rightarrow \nabla_{(2,2N-1)}(z)_2 = \frac{N(N-1)}{2} \equiv \delta \text{ for maximally ICO}$$

- Concatenation of subsequences $\Leftrightarrow$ knot sum $\Rightarrow$ Alexander-Conway polynomial additive in quadratic term. But δ is also additive.

Knot invariants and indefinite causal order

Conjecture

The causal indefiniteness of a superposition of two spacetimes $(\mathcal{M}_{\mathcal{A}}, g_{\mathcal{A}})$ and $(\mathcal{M}_{\mathcal{B}}, g_{\mathcal{B}})$ with N events for a pair of irreducible causal sequences is related to its knot representation by

$$\nabla(z)_2 = \delta(\mathcal{A}, \mathcal{B})$$

Theorem

Provided this conjecture is true, the causal indefiniteness of a superposition of two spacetimes $(\mathcal{M}_{\mathcal{A}}, g_{\mathcal{A}})$ and $(\mathcal{M}_{\mathcal{B}}, g_{\mathcal{B}})$ with N events is given by

$$\nabla(z)_2 = \delta(\mathcal{A}, \mathcal{B})$$

where $\nabla(z)_2$ is the quadratic term of the Alexander-Conway polynomial of the knot representation of the causal sequence.

Knot invariants and indefinite causal order

Conjecture

The causal indefiniteness of a superposition of two spacetimes $(\mathcal{M}_{\mathcal{A}}, g_{\mathcal{A}})$ and $(\mathcal{M}_{\mathcal{B}}, g_{\mathcal{B}})$ with N events for a pair of irreducible causal sequences is related to its knot representation by

$$\nabla(z)_2 = \delta(\mathcal{A}, \mathcal{B})$$

Proof ideas

- (Lemma) The knot associated to irreducible causal sequences is prime.
- For an arbitrary genus m knot, the quadratic coefficient of the Alexander-Conway polynomial is given by[6]

$$\nabla(z)_2 = \sum_{i \leq i \leq j}^{m} |A_{ij}| \text{ (note similarity with } \sum_{1 \leq i < j}^{N} \left| s_{ij}^{\mathcal{A}} - s_{ij}^{\mathcal{B}} \right| \text{)}$$

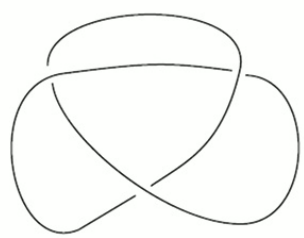
where $[A_{ij}]$ is a *Seifert matrix* associated to the knot.

[6] Chichen M. Tsau. "On the topology of the coefficients of the Alexander-Conway polynomials of knots". In: *Journal of Knot Theory and its Ramifications* 25 (2 2016). ISSN: 02182165. DOI: 10.1142/S0218216516500085

Conclusions

- We associated events across superpositions of spacetimes
- We quantified the degree to which causal sequences are in ICO
- We provided diagrammatic and knot representations of such sequences of events, providing a way to classify them pictorially
- We gave a topological meaning to causal indefiniteness, and related it to a knot invariant – the Alexander-Conway polynomial

See also operational encoding of the causal order, quantum mechanical quantifiers of indefinite causality and more on our paper (out very soon!)



Thank you!