Title: Simulating the quantum switch using causally ordered circuits requires at least an exponential overhead in query complexity

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Simulating the quantum switch using standard quantum circuits requires at least an exponential overhead in quantum query complexity

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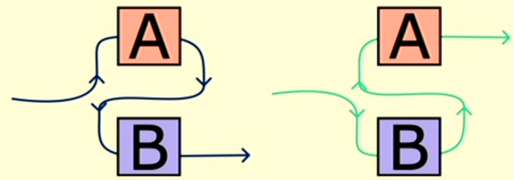
Indefinite causal order

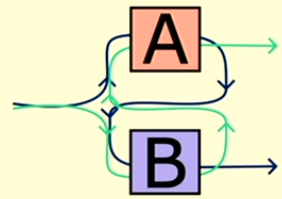
Everyday world: events happen in a well-defined causal order

$A \prec B$ $\quad\quad$ $B \prec A$



Quantum switch[1]: coherent superposition of causal orders

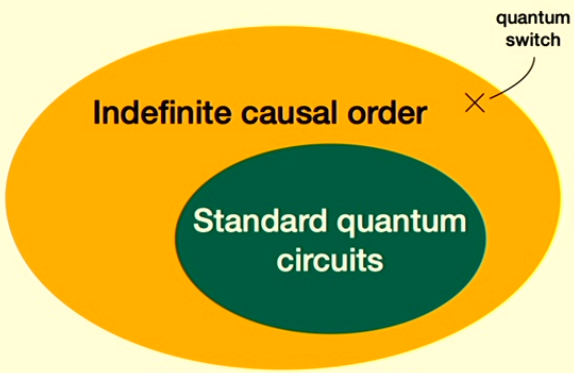
$$A \prec B \oplus B \prec A$$



[1] Chiribella et al. 2009; Chiribella et al. 2013.

Motivations

New model of computation beyond standard quantum circuits



- Challenges fundamental view of causality, time and gravity.
- Advantages in quantum information processing, e.g. computation, communication, multipartite games, metrology, thermodynamics, etc.

Advantages in quantum computation

In this talk, focus on advantages in computational query complexity

- **Channel discrimination**: distinguish whether two unknown unitary channels either commute or anti-commute.
  - standard quantum circuits require two calls to at least one of the unitaries
  - the quantum switch only requires one call to each unitary[2].
- **Fourier promise problems**: given N unitaries satisfying one of $N!$ properties, find which property is satisfied.
  - standard quantum circuits: $O(N^2)$ calls
  - quantum N-switch (superposition of all possible causal orders of N gates): $O(N)$ calls[3] $\rightarrow$ quadratic advantage!

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[2] Chiribella 2012.
[3] Araújo, Costa, and Brukner 2014.

This talk

So far: not clear if indefinite causal order processes can always be efficiently simulated by standard quantum circuits, by using extra calls to the channels.

Research question:

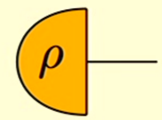
Is there an *exponential* separation in query complexity between indefinite causal order processes and standard quantum circuits?

Outline of the talk

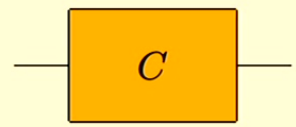
- Framework: quantum states, channels and supermaps
- Definite and indefinite causal order
- Query complexity of quantum supermaps
- The quantum switch and its simulations
- Problem setting and main result: exponential separation between the quantum switch and causally ordered circuits

Quantum states, channels and supermaps

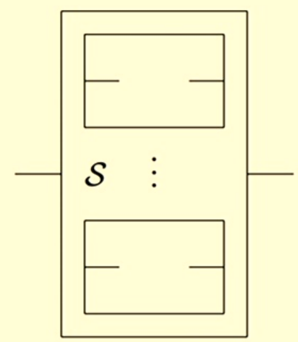
Quantum state: linear operator on Hilbert space
$\rho \in \mathbb{L}(\mathcal{H})$, $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$



Quantum channel: linear map from states to states
$\mathcal{C} : \mathbb{L}(\mathcal{H}^I) \to \mathbb{L}(\mathcal{H}^O)$, **CP**, **TP**



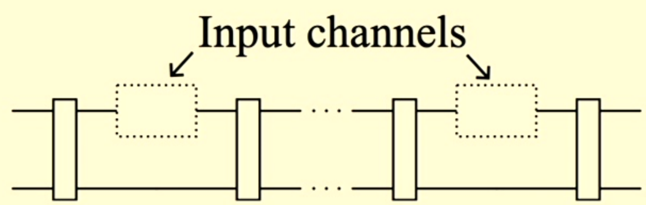
Quantum supermap: multilinear map from M channels to channels[a]
$\mathcal{S} : \bigotimes_{i=1}^{M}[\mathbb{L}(\mathcal{H}^{I_i}) \to \mathbb{L}(\mathcal{H}^{O_i})] \to [\mathbb{L}(\mathcal{H}^P) \to \mathbb{L}(\mathcal{H}^F)]$, **CCPP**, **TPP**



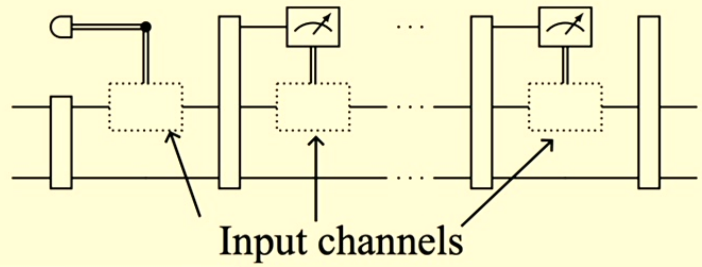
[a] Chiribella, D'Ariano, and Perinotti 2008.

Supermaps with definite causal order

Standard quantum circuits
$\doteq$ Quantum circuits with fixed order (QC-FOs / quantum 'combs')[4]



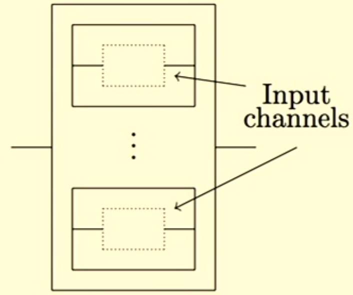
Quantum circuits with classical control of causal order (QC-CCs)[5]

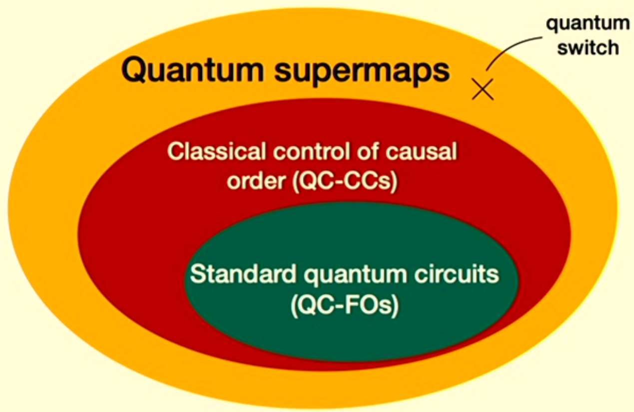


[4] Chiribella, D'Ariano, and Perinotti 2008.
[5] Wechs et al. 2021.

Supermaps with indefinite causal order

Most general supermaps:
indefinite causal order





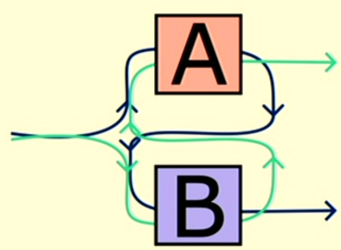
The quantum switch

Quantum switch: 2-slot quantum supermap defined by

$$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{U},\mathcal{V})(\cdot) = S \cdot S^\dagger, \tag{1}$$

$$S = VU \otimes |0\rangle\langle 0| + UV \otimes |1\rangle\langle 1|\,, \tag{2}$$

for any unitary channels $\mathcal{U} := U(\cdot)U^\dagger$ and $\mathcal{V} := V(\cdot)V^\dagger$.



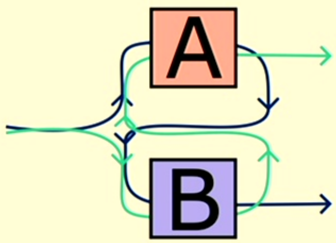
The quantum switch

Quantum switch on general channels:

$$\mathcal{S}_{\texttt{SWITCH}}(\mathcal{A},\mathcal{B})(\cdot) = \sum_{ij} S_{ij} \cdot S_{ij}^{\dagger}, \qquad (1)$$

$$S_{ij} = B_j A_i \otimes |0\rangle\langle 0| + A_i B_j \otimes |1\rangle\langle 1| \qquad (2)$$

for any quantum channels $\mathcal{A}(\cdot) = \sum_i A_i \cdot A_i^{\dagger}$, $\mathcal{B}(\cdot) = \sum_j B_j \cdot B_j^{\dagger}$



Simulating the quantum switch for unitaries

Clearly, the one-sided quantum query complexity of the action of the quantum switch w.r.t. all supermaps is 1.

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Query complexity of higher-order computation

Function f maps pair of channels $\mathcal{A}, \mathcal{B}$ to channel $f(\mathcal{A}, \mathcal{B})$

$f : [\mathbb{L}(\mathcal{H}^I) \to \mathbb{L}(\mathcal{H}^O)] \otimes [\mathbb{L}(\mathcal{H}^{I'}) \to \mathbb{L}((\mathcal{H}^{O'})] \to [\mathbb{L}(\mathcal{H}^P) \to \mathbb{L}(\mathcal{H}^F)]$

Definition: A quantum supermap $\mathcal{S}$ *simulates* the function f deterministically and exactly if, given M black-box queries to the quantum channel $\mathcal{A}$ and N black-box queries to the quantum channel $\mathcal{B}$, $\mathcal{S}(\mathcal{A}^{\otimes M}, \mathcal{B}^{\otimes N}) = f(\mathcal{A}, \mathcal{B})$.

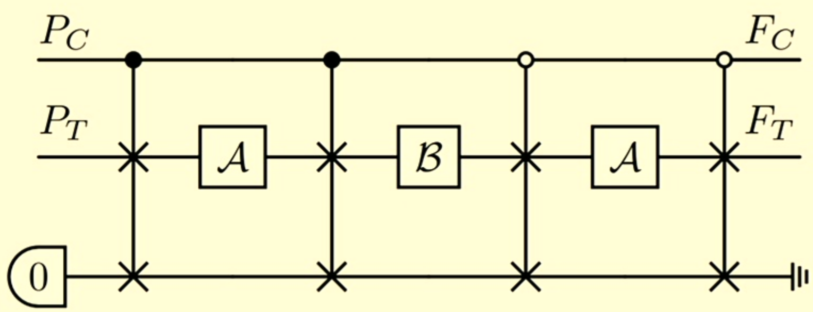
Definition: The *one-sided quantum query complexity* of a function f, with respect to a class of supermaps $\mathbb{S}$, is the minimum number of queries M while $N = 1$, over all supermaps $\mathcal{S} \in \mathbb{S}$ such that $\mathcal{S}$ simulates f.

Simulating the quantum switch for unitaries

Clearly, the one-sided quantum query complexity of the action of the quantum switch w.r.t. all supermaps is 1.

For unitary channels, the quantum switch can be simulated by a QC-FO with just one extra call: one-sided q. query complexity of 2.

For all $\mathcal{A}, \mathcal{B}$, the circuit



simulates the quantum switch if $\mathcal{A}$ and $\mathcal{B}$ are unitary channels.

Simulating the quantum switch for unitaries

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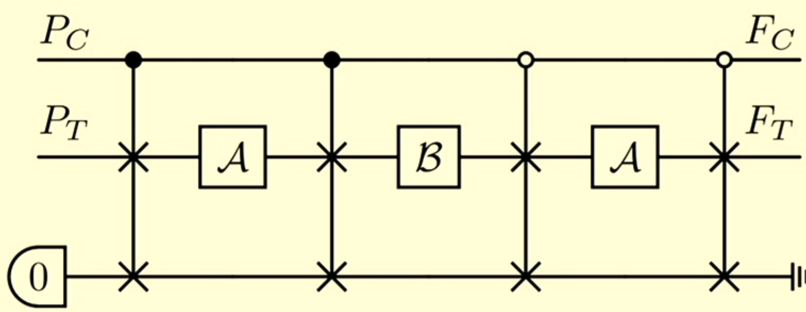
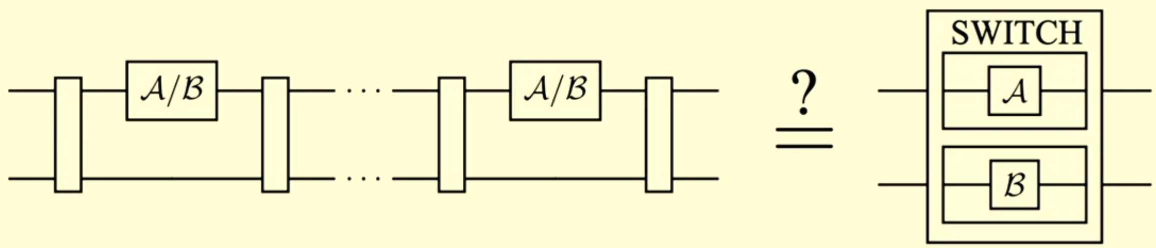


simulates the quantum switch if $\mathcal{A}$ and $\mathcal{B}$ are unitary channels.

$\rightarrow$ **Doesn't work for general channels!**

Problem setting

Question:
How many copies of the input quantum channels are needed to simulate the quantum switch using fixed causal order ?



Proof sketch

3 steps:

1. Linearity argument
2. Uniqueness
3. Contradiction with QC-CC conditions

Main result

Theorem
There is no $(M+1)$-slot supermap with classical control of causal order $\mathcal{C}$ satisfying

$$\mathcal{C}(\underbrace{\mathcal{A},\ldots,\mathcal{A}}_{M},\mathcal{B}) = \mathcal{S}_{\texttt{SWITCH}}(\mathcal{A},\mathcal{B}) \tag{3}$$

for all n-qubit channels $\mathcal{A}$ and $\mathcal{B}$, if $M \leq \max(2, 2^n - 1)$.

Remark

- No-go on deterministic and exact simulation
- Lower bound on the one-sided quantum query complexity of the action of the quantum switch, with respect to all causally ordered supermaps

Proof sketch (0. Choi representation)

Definition (Choi representation)
Choi matrix of a linear map $\mathcal{Q} : \mathbb{L}(\mathcal{H}^A) \to \mathbb{L}(\mathcal{H}^B)$:

$$Q := \sum_{ij} |i\rangle\langle j|^A \otimes \mathcal{Q}(|i\rangle\langle j|) \in \mathbb{L}(\mathcal{H}^A \otimes \mathcal{H}^B), \qquad (4)$$

where $\{|i\rangle\}$ is the computational basis of $\mathcal{H}^A$.

Choi matrix of unitary operation $\mathcal{U}(\cdot) = U \cdot U^\dagger$ is represented as a rank-1 operator

$$|U\rangle\rangle\langle\langle U| \qquad (5)$$

where $|U\rangle\rangle$ is a Choi vector defined by $|U\rangle\rangle := \sum_i |i\rangle^A \otimes U |i\rangle$.

Proof sketch

3 steps:

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Proof sketch

3 steps:

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Logical flow:

- Assume that $\mathcal{C}$ simulates the quantum switch
  $\Rightarrow$ Restrict the form of C (steps 1, 2)
- The restricted form does not satisfy QC-CC conditions (step 3)

Proof sketch (1. Linearity argument)

For simplicity, we consider $M = 2$ case

1. Linearity argument

Assume that

$$\mathcal{C}(\mathcal{A}, \mathcal{A}, \mathcal{B}) = \mathcal{S}_{\texttt{SWITCH}}(\mathcal{A}, \mathcal{B}) \quad (6)$$

for $\mathcal{A} = \mathcal{U}_1, \mathcal{U}_2, \frac{\mathcal{U}_1+\mathcal{U}_2}{2}$ and $\mathcal{B} = \mathcal{V}$ for unitary operations $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}$.

We show that this implies that

$$\begin{aligned} &C * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|) \\ \leq &|S_{\texttt{SWITCH}}\rangle\rangle\langle\langle S_{\texttt{SWITCH}}| * [(|U_1\rangle\rangle\langle\langle U_1| + |U_2\rangle\rangle\langle\langle U_2|) \otimes |V\rangle\rangle\langle\langle V|] \quad (7) \end{aligned}$$

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Proof sketch (1. Linearity argument)

$$C * (|U_1\rangle\rangle\langle\langle U_1| \otimes |U_2\rangle\rangle\langle\langle U_2| \otimes |V\rangle\rangle\langle\langle V|)$$
$$\leq |S_{\texttt{SWITCH}}\rangle\rangle\langle\langle S_{\texttt{SWITCH}}| * [(|U_1\rangle\rangle\langle\langle U_1| + |U_2\rangle\rangle\langle\langle U_2|) \otimes |V\rangle\rangle\langle\langle V|]$$

Since $C \geq 0$, C can be written as $C = \sum_i |C_i\rangle\rangle\langle\langle C_i|$.

Thus,

$$|C_i\rangle\rangle * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) = \sum_{l=1}^{2} \xi_i^{(l)}(U_1, U_2, V)|S_{\texttt{SWITCH}}\rangle\rangle * (|U_l\rangle\rangle \otimes |V\rangle\rangle) \quad (8)$$

for some $\xi_i^{(l)}(U_1, U_2, V) \in \mathbb{C}$.

Proof sketch (2. Uniqueness)

2. Uniqueness

We show that

$$|C_i\rangle\rangle * (|U_1\rangle\rangle \otimes |U_2\rangle\rangle \otimes |V\rangle\rangle) = \sum_{l=1}^{2} \xi_i^{(l)}(U_1, U_2, V)|S_{\texttt{SWITCH}}\rangle\rangle * (|U_l\rangle\rangle \otimes |V\rangle\rangle)$$

implies that

$$|C_i\rangle\rangle = \sum_{l=1}^{2} |S_{\texttt{SWITCH}}\rangle\rangle^{l3} \otimes |\xi_i^{(l)}\rangle\rangle^{\bar{l}} \tag{9}$$

for $\xi_i^{(l)}(U_1, U_2, V) = |\xi_i^{(l)}\rangle\rangle * |U_{\bar{l}}\rangle\rangle$ by differentiating with respect to a parametrisation of U_1, U_2[6].

[6] Odake, Yoshida, and Murao 2024.

Proof sketch (3. Contradiction with QC-FO conditions)

3. Contradiction with QC-CC conditions

If $\mathcal{C}$ is a QC-CC, then C should satisfy affine conditions.

As shown in steps 1 and 2, if $\mathcal{C}$ simulates the quantum switch, then $C = \sum_i |C_i\rangle\rangle\langle\langle C_i|$ for $|C_i\rangle\rangle = \sum_{l=1}^{2} |S_{\texttt{SWITCH}}\rangle\rangle^{l3} \otimes |\xi_i^{(l)}\rangle\rangle^{\bar{l}}$.

Proof sketch (3. Contradiction with QC-FO conditions)

3. Contradiction with QC-CC conditions

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As shown in steps 1 and 2, if $\mathcal{C}$ simulates the quantum switch, then $C = \sum_i |C_i\rangle\rangle\langle\langle C_i|$ for $|C_i\rangle\rangle = \sum_{l=1}^{2} |S_{\mathtt{SWITCH}}\rangle\rangle^{l3} \otimes |\xi_i^{(l)}\rangle\rangle^{\bar{l}}$.

$\rightarrow$ Contradiction!

Future works

- Approximate or probabilistic settings?
- More relaxed settings (e.g. only simulating reduced quantum switch)?
- Multiple copies of both input channels $\mathcal{A}$ and $\mathcal{B}$?
- Is it possible to exactly simulate the action of the quantum switch by using exponentially many copies of the input channels?

$\rightarrow$ We investigate the first three with SDP in a companion paper[7].

[7] Bavaresco et al., In preparation

Conclusion

Take-home

Simulation of the quantum switch is (at least) exponentially hard

Therefore, there exists an exponential separation (as a function of system size) in a type of quantum query complexity between indefinite causal order processes and causally ordered circuits

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Proof technique

Linear algebra + differentiation technique

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Possible topics within quantum information and foundations:

- **Causality** in quantum theory
- **Higher-order quantum computation**/ quantum functional programming
- Foundational approaches to **ML with quantum data** (possible co-supervision at Mila)
- Applications to **simulation** of physical systems

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