Title: Causally faithful unitary circuit decompositions

Speakers: Tein van der Lugt

Series: Quantum Foundations, Quantum Information

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Causally faithful circuit decompositions

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Causal decompositions

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Outline

One motivation : quantum causal modelling

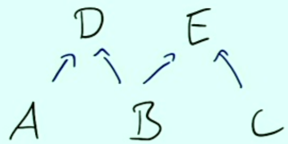
Background

Two recent results

* ⋆ Connection to lattice theory
* ⋆ Parental intersection condition

(upcoming works)

Pearl / Spirtes, Glymour, Scheines/...



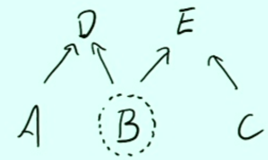
$$P(ABCDE) = P(A)P(B)P(C) \cdot P(D|AB)\, P(E|BC)$$

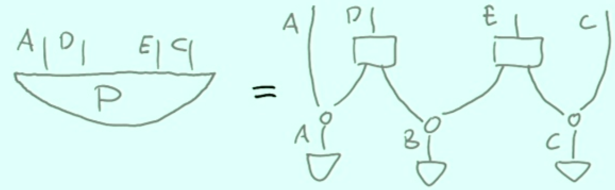
$\Uparrow$

(deterministic evolution + local noise)

$\Longleftrightarrow$

Fong / Jacobs, Kissinger, Zanasi / Lorenz, Tull /...





Allen et al. / Barrett, Lorenz, Oreshkov

" unitarity - first "

(unitary evolution + local noise)

$\Downarrow$

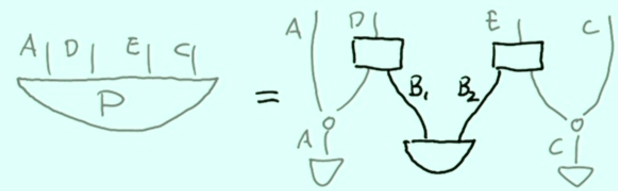
$$\sigma_{ABCDE} = \rho_A \rho_B \rho_C \rho_{D|AB}\, \rho_{E|BC}$$

s.t. $[\rho_{D|AB}, \rho_{E|BC}] = 0$ (etc.)

?

Fritz / Henson, Lal, Pusey /...

" compositionality - first "

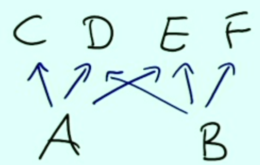


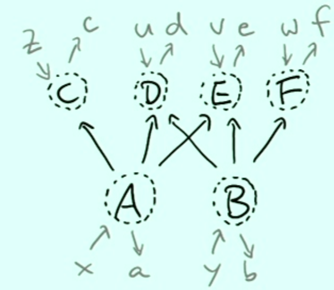
Allen et al./ Barrett, Lorenz, Oreshkov

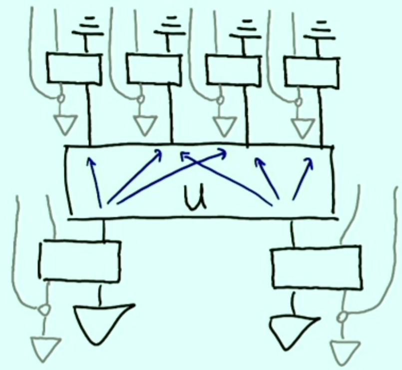
“unitarity-first”

Fritz / Henson, Lal, Pusey/...

“compositionality-first”

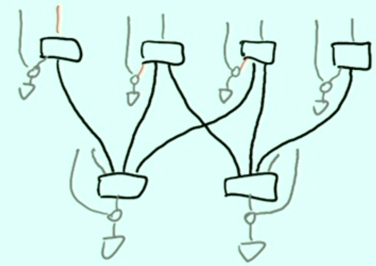






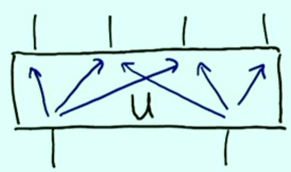
$\Longleftarrow$

$\not\Longrightarrow$

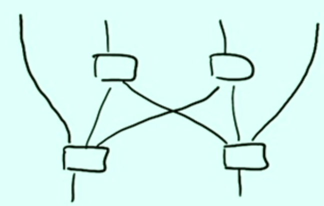


because

$\exists\, U$:



$\neq$

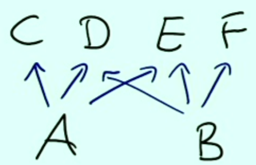


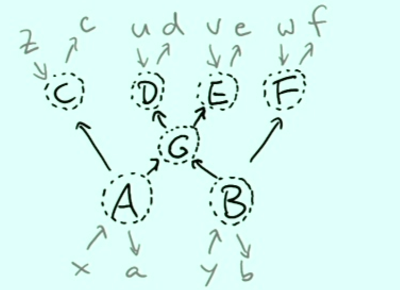
Allen et al./ Barrett, Lorenz, Oreshkov

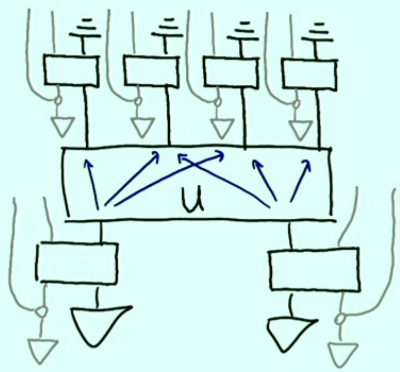
“unitarity-first”

Fritz / Henson, Lal, Pusey/...

“compositionality-first”

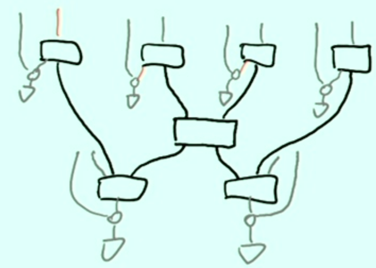






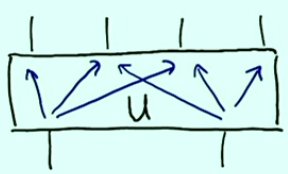
$\Longleftarrow$

$\Longrightarrow$



because

$\forall U:$



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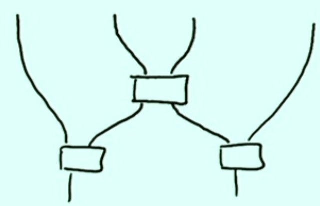
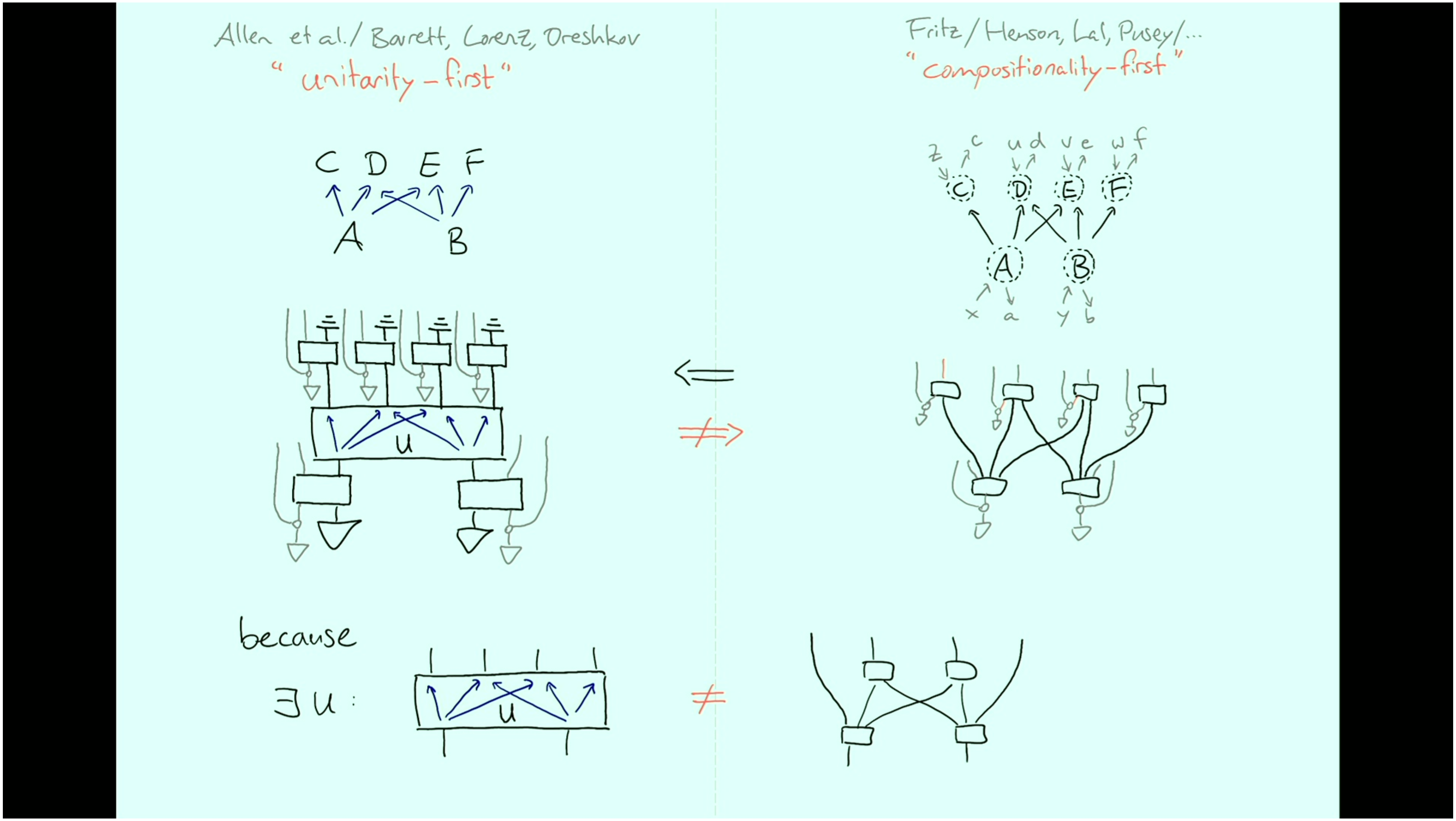
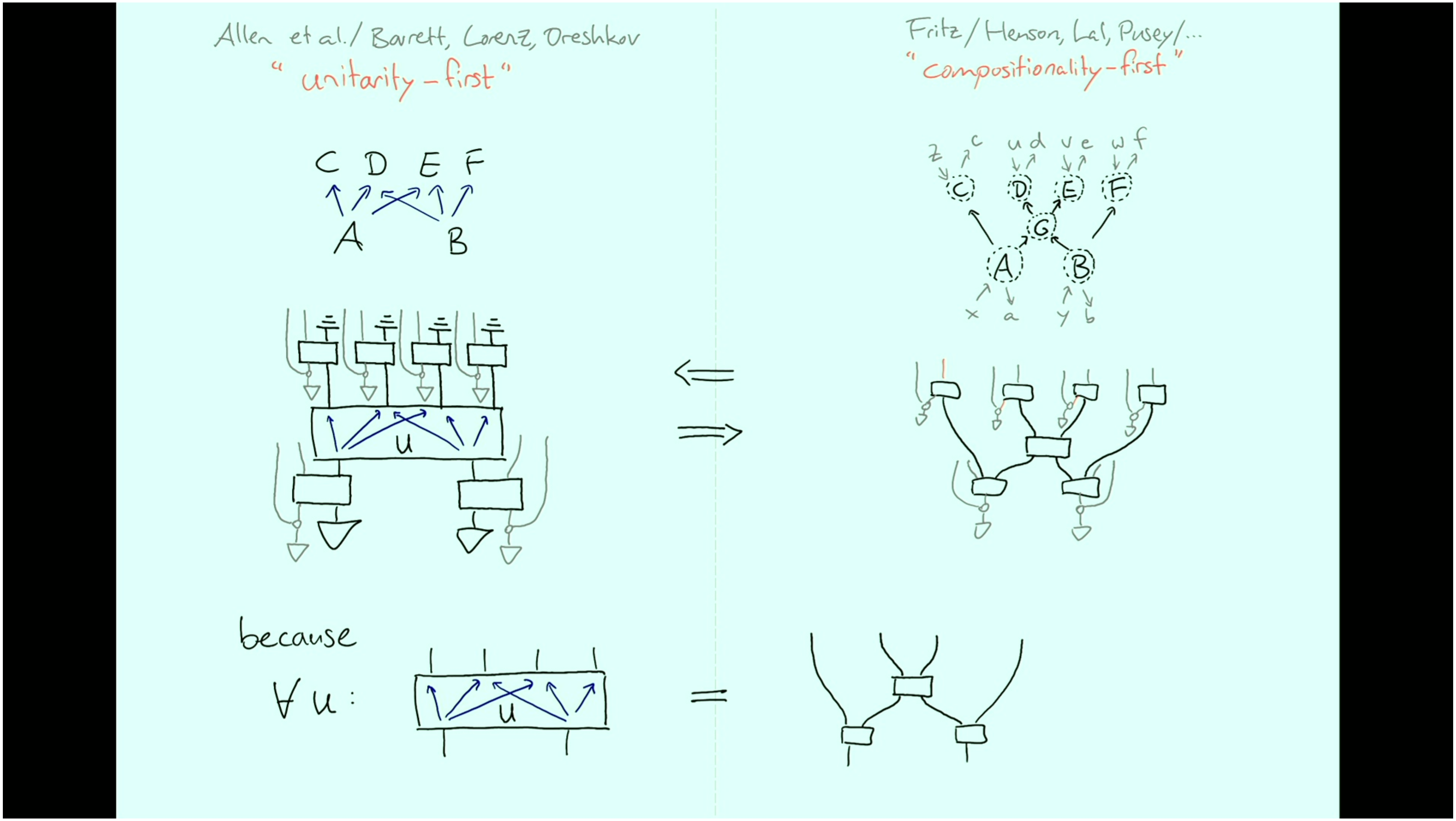



Allen et al./ Barrett, Lorenz, Oreshkov
" unitarity - first "
Fritz / Henson, Lal, Pusey/...
" compositionality - first "
C D E F
A B
z c u d v e w f
C D E F
A B
x a y b
U
because
∃ U :
U


Allen et al./ Barrett, Lorenz, Oreshkov
"unitarity-first"
Fritz/ Henson, Lal, Pusey/...
"compositionality-first"
C D E F
A B
U
z c u d v e w f
C D E F
G
A B
x a y b
⟸
⟹
because
∀ U :
U
=

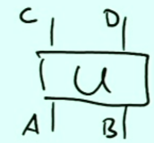
Outline

One motivation : quantum causal modelling

→ **Background**

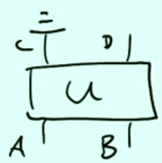
Two new results

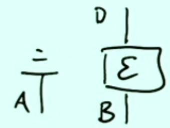
* Connection to lattice theory
* Parental intersection condition

$\boxed{\text{Def}}$ A does <u>not</u> causally influence D through 

,

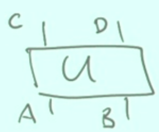
$A \not\to_{U} D$,

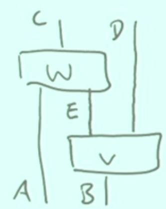
iff 

= 

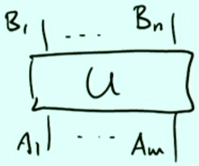
for some $\mathcal{E}$

iff $\mathcal{L}(\mathcal{H}_A) \otimes \mathbb{1}_B \subseteq U^{\dagger}\left(\mathcal{L}(\mathcal{H}_C) \otimes \mathbb{1}_D \right) U$

iff 

= 

for some E, V, W (unitary)

$\boxed{\text{Def}}$ 

$\rightsquigarrow$ causal structure $\to_{U} \subseteq \{A_i\} \times \{B_j\}$

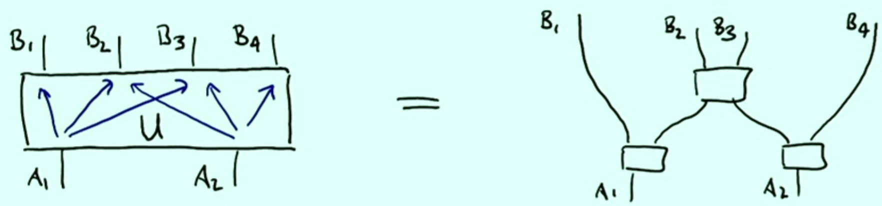
$\boxed{\text{Def}}$ A circuit with overall inputs $A_1, \ldots, A_m$ and outputs $B_1, \ldots, B_n$ is *faithful* to some $\rightarrow \subseteq \{A_i\} \times \{B_j\}$ if

$$\forall i,j: \exists \text{ (directed) path from } A_i \text{ to } B_j \iff A_i \rightarrow B_j.$$

(Lorenz, Barrett, 2021)

$\boxed{\text{Conj}}$ Every unitary U admits a *causal decomposition*, i.e.

a circuit decomposition faithful to its causal structure $\rightarrow_U$.



Applications

- Comparison btw unitarity-first & compositionality-first models
- Give diagrammatic backbone to unitarity-first quantum causal models (quantum causal discovery)

- Partitions of quantum systems
- Quantum cellular automata

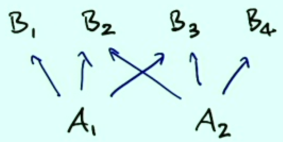
Next talk (Augustin)

- Spacetime realisability of quantum channels (rel. quantum tasks) } talk on Friday

- Indefinite causal order (Barrett Lorenz Oreshkov '21; Vanrietvelde et al. '22)
- ...

faithfulness:

$A_i \to B_j \iff \forall_{ij}: \exists$ (directed) path from A_i to B_j



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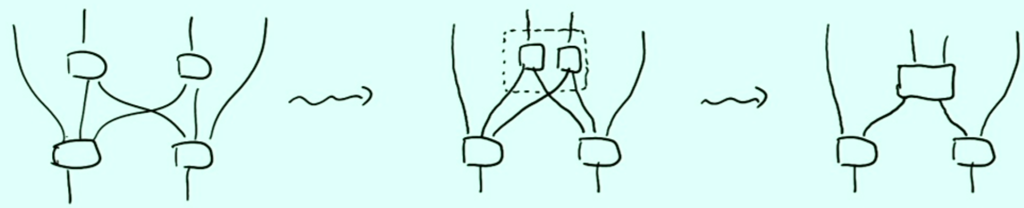
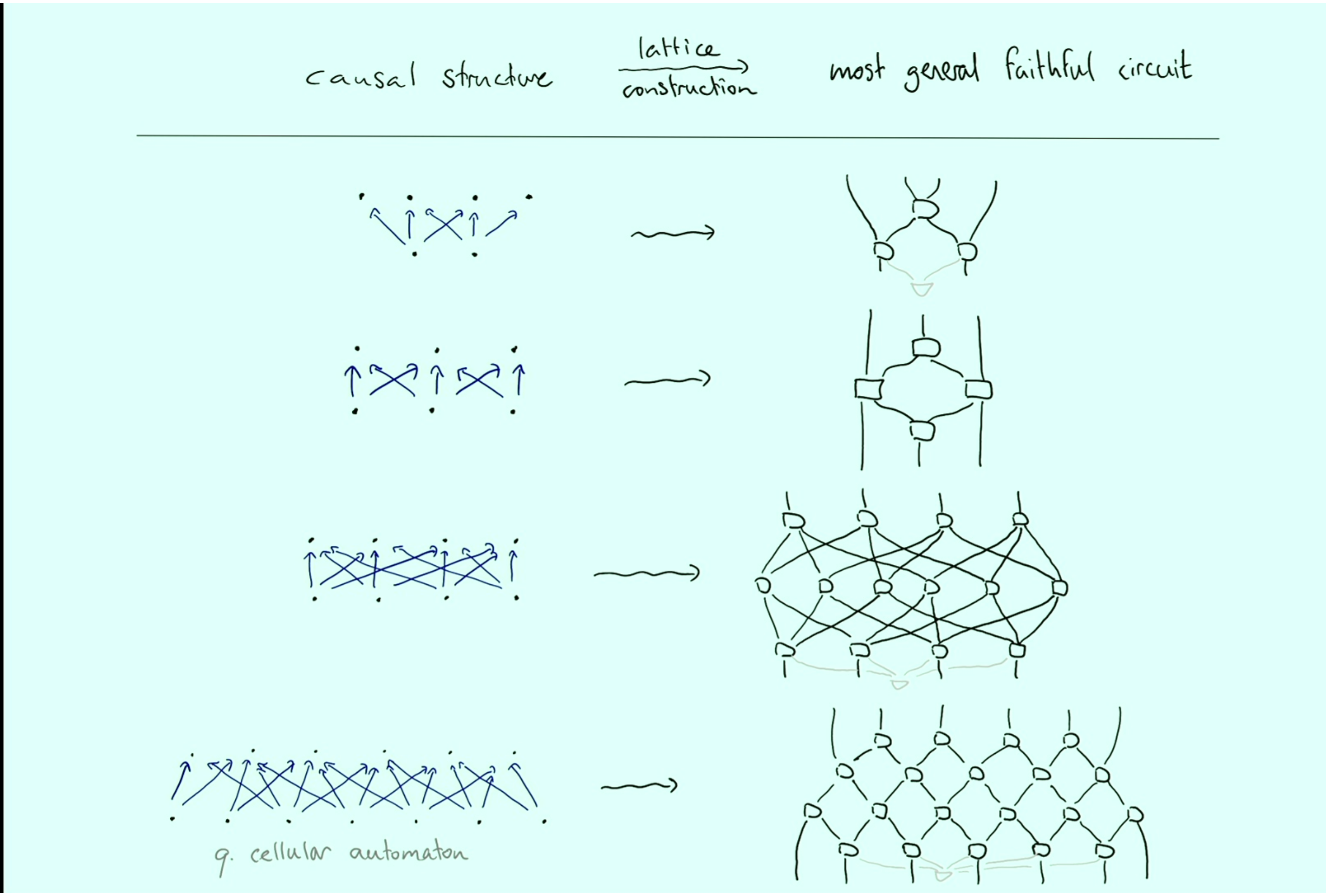
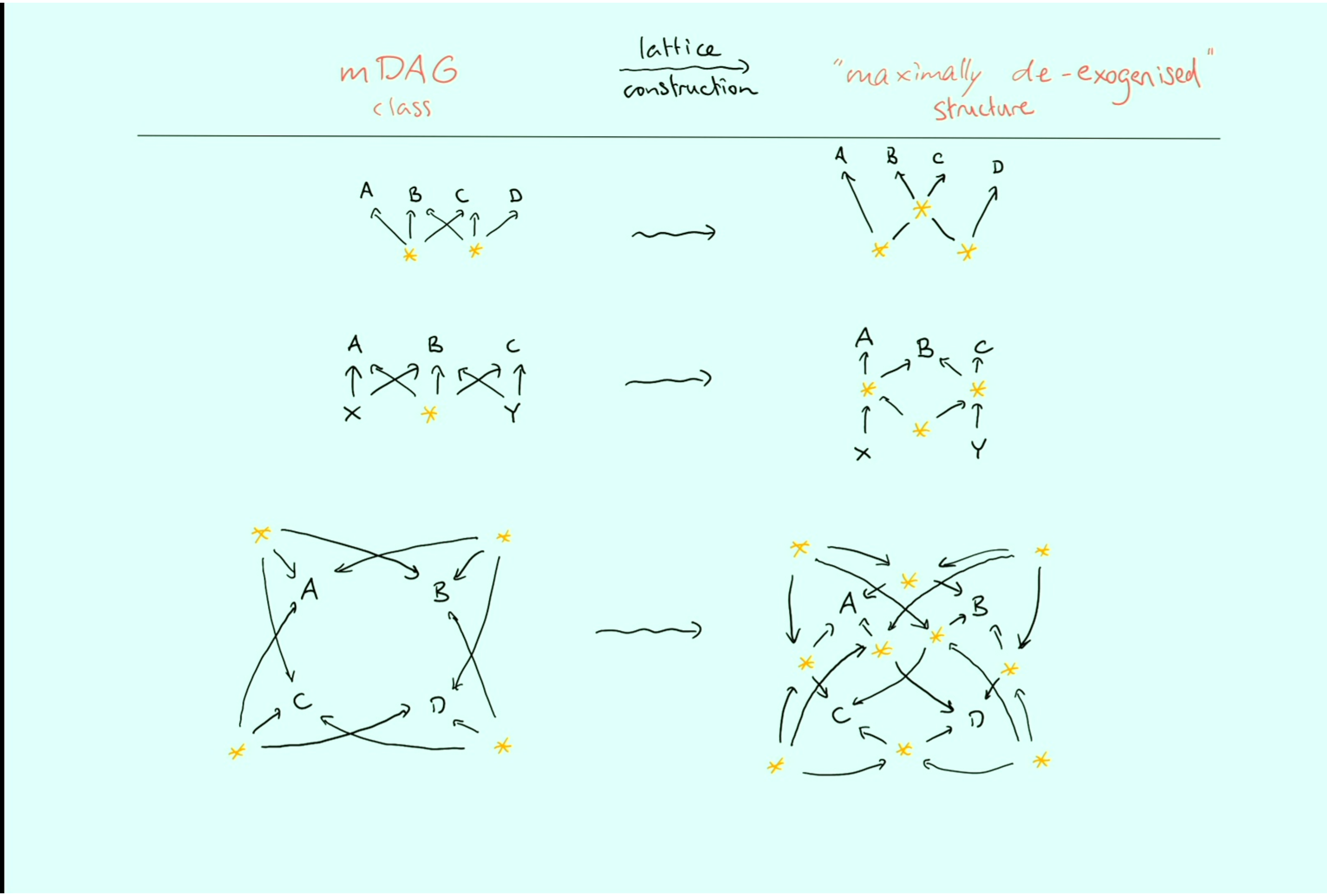


$\left(\{A_i\}, \{B_j\}, \to \subseteq \{A_i\} \times \{B_j\} \right) \rightsquigarrow$ concept lattice $\mathcal{L}_\to$

from formal concept analysis

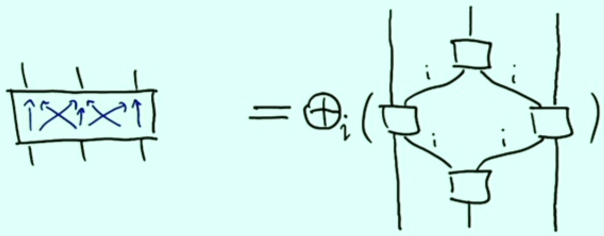
Thm 1 $\mathcal{L}_\to$ is the unique smallest $\to$-faithful circuit shape which every other $\to$-faithful circuit shape can be syntactically rewritten to.


causal structure
lattice
construction
most general faithful circuit
q. cellular automaton


mDAG class
lattice construction
"maximally de-exogenised" structure
A B C D
A B C D
A B C
X Y
A B C
X Y
A B
C D
A B
C D

$\boxed{\cancel{\text{Conj}}}$ $\forall \rightarrow$, every unitary $\mathcal{U}$ with $\rightarrow_{\mathcal{U}} = \rightarrow$ has a causally faithful unitary circuit decomposition.

False – Lorenz, Barrett 2021



$\boxed{\text{Conj}}$ $\forall \rightarrow$, every unitary $\mathcal{U}$ with $\rightarrow_{\mathcal{U}} = \rightarrow$ has a causally faithful routed unitary circuit decomposition.

$\boxed{\text{Thm 2}}$ $\forall \rightarrow$ that satisfy the "parental intersection condition",

every unitary U with $\rightarrow_U = \rightarrow$ has a causally faithful unitary circuit decomposition.

$\boxed{\text{Def}}$ Let $P_{jk} := \{ A_i : \ A_i \rightarrow B_j \text{ and } A_i \rightarrow B_k \}$.

The parental intersection condition (PIC) is that

$$\forall j,k,l: \ P_{jk} \cap P_{kl} = \emptyset \quad \text{or} \quad P_{jk} \subseteq P_{kl} \quad \text{or} \quad P_{jk} \supseteq P_{kl}.$$

$\boxed{\text{Thm 2}}$ $\forall \rightarrow$: $\rightarrow$ satisfies the "parental intersection condition"

$$\Longleftrightarrow$$

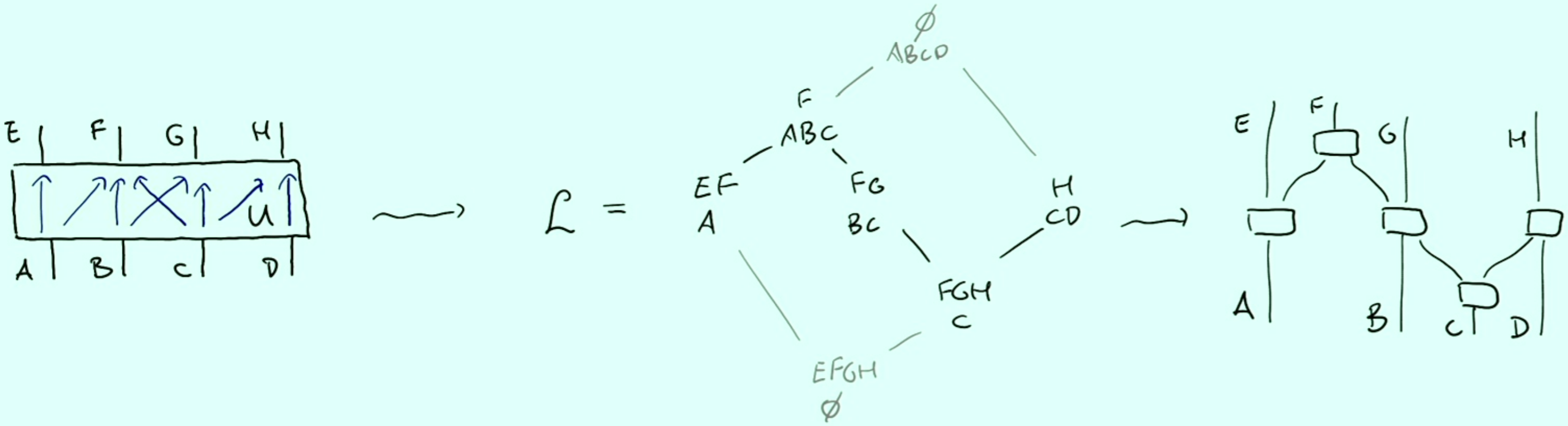
every unitary $\mathcal{U}$ with $\rightarrow_{\mathcal{U}} = \rightarrow$ has a causally faithful unitary circuit decomposition.

(upcoming work w/ Robin Lorenz)

$\boxed{\text{Def}}$ Let $P_{jk} := \{ A_i : A_i \rightarrow B_j \text{ and } A_i \rightarrow B_k \}$.

The parental intersection condition (PIC) is that

$$\forall j,k,\ell: \quad P_{jk} \cap P_{k\ell} = \emptyset \quad \text{or} \quad P_{jk} \subseteq P_{k\ell} \quad \text{or} \quad P_{jk} \supseteq P_{k\ell}.$$



$\boxed{\text{Lemma}}$ Let $\mathcal{H}_C, \mathcal{H}_{X_1}, \dots, \mathcal{H}_{X_n}$ be finite-dim. and $B_j \subseteq \mathcal{L}(\mathcal{H}_{CX_1 \cdots X_n})$ for $j = 1, \dots, n$ be such that

- $[B_j, B_k] = 0 \quad \forall j, k$
- $B_j \subseteq \mathcal{L}(\mathcal{H}_{CX_j}) \otimes \mathbb{1}_{X_{\hat{j}}}$

Then $\exists$ unitary $V : \mathcal{H}_C \xrightarrow{\sim} \bigoplus_k \mathcal{H}_{C_1^k} \otimes \cdots \otimes \mathcal{H}_{C_n^k}$

s.t. $V B_j V^\dagger \subseteq \bigoplus_k \mathcal{L}(\mathcal{H}_{C_j^k X_j})$

Summary

- Causal structure $\neq$ compositional structure
  (of unitaries) (e.g. HLP/F)

  causal decompositions

- "concept lattice" = shape of causal decomposition

- Characterisation of the causal structures that imply an (unrouted) unitary causal decomposition

Next:

... decompositions in terms of routed unitaries / general channels ?

... concept lattices for (quantum/OPT) observational (in)equivalence of compositional structures