Title: Two convergent NPA-like hierarchies for the quantum bilocal scenario

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Characterizing quantum bilocal network scenario with generalized NPA hierarchies

Based on arXiv:2210.09065 [Renou, Xu, Ligthard, 2022]

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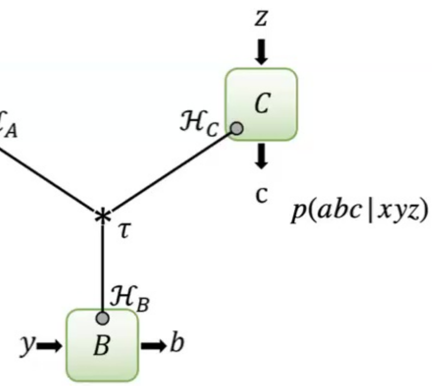
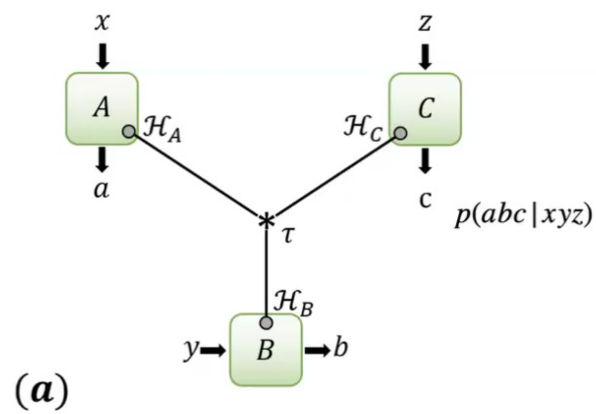
Tripartite Bell scenario: standard QM C_{qa}



(*a*)

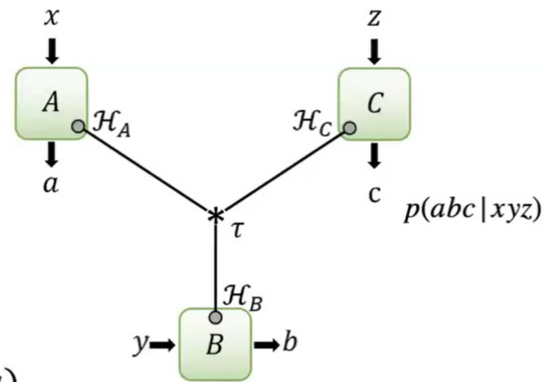
- Hilbert space $H = H_A \otimes H_B \otimes H_C$ with a shared state τ
- PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$
- Born's rule:
  $p(abc\,|\,xyz) = \mathrm{Tr}(\tau(A_{a|x} \otimes B_{b|y} \otimes C_{c|z})) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$

Tripartite Bell scenario: certification problem



- Conversely, given $\overrightarrow{P} = \{p(abc\,|\,xyz)\}$, is it compatible with some tripartite Bell experiment?
- I.e does it exist some H, τ, $\{A_{a|x}\}, \ldots$ such that $p(abc\,|\,xyz) = \ldots$ Is $\overrightarrow{P} \in C_{qa}$?

Tripartite Bell scenario: certification problem

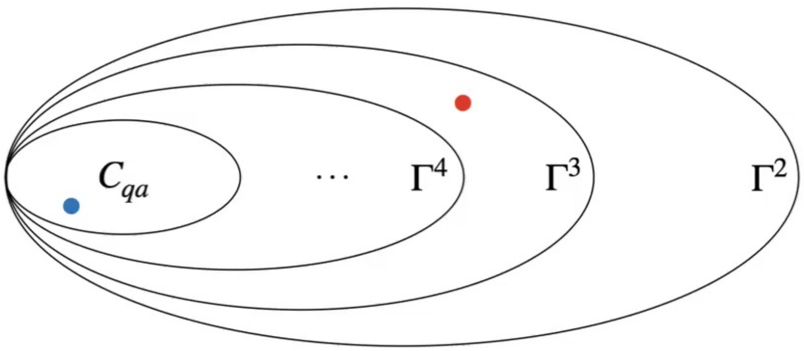


(***a***)

- Inner approximation: calculating all possible $\overrightarrow{P}$ over Hilbert spaces of all dimension, with e.g. gradient-descent.
- Might miss some important distributions!
- Outer approximation: NPA hierarchy [Navascués et al., 2008]

NPA hierarchy: a hierarchy of necessary conditions

- Will sketch, have condition $\Gamma^n, n \geq 2$, such that $\overrightarrow{P} \in C_{qa} \implies \cdots \implies \Gamma^4 \implies \Gamma^3 \implies \Gamma^2$.



- Equivalently, if for some n, Γ^n is not satisfied, then $\overrightarrow{P} \notin C_{qa}$.

- Testing C_{qa} from the outside.

NPA hierarchy: moment matrix Γ^2

- Suppose state&PVMs s.t. $p(abc\,|\,xyz) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$, easy to calculate $\mathrm{Tr}_\tau(A_{a|x}), \mathrm{Tr}_\tau(A^\dagger_{a|x}C_{c|z}), \ldots$
- Put them into a *moment matrix* Γ^2, indexed by $1, A_{a|x}, A_{a|x}A_{a'|x'}, \ldots$ (up to length 2).

$$
\begin{array}{c} \\ \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ \cdots \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathbb{1} & A_{a|x} & B_{b|y} & C_{c|z} & \cdots \end{array} \\
\begin{bmatrix}
1 & \mathrm{Tr}_\tau(A_{a|x}) & \mathrm{Tr}_\tau(B_{b|y}) & \mathrm{Tr}_\tau(C_{c|z}) & \\
 & \mathrm{Tr}_\tau(A_{a|x}) & \mathrm{Tr}_\tau(A_{a|x}B_{b|y}) & \mathrm{Tr}_\tau(A_{a|x}C_{c|z}) & \\
 & & \mathrm{Tr}_\tau(B_{b|y}) & \mathrm{Tr}_\tau(B_{b|y}C_{c|z}) & \\
 & & & \mathrm{Tr}_\tau(C_{c|z}) & \\
 & & & &
\end{bmatrix}
\end{array}
$$

- Rule:
$\Gamma_{B_{b|y},B_{b|y}} = \mathrm{Tr}_\tau(B^\dagger_{b|y}B_{b|y}) = \mathrm{Tr}_\tau(B_{b|y}) = \mathrm{Tr}_\tau(\mathrm{Id}^\dagger \cdot B_{b|y}) = \Gamma_{1,B_{b|y}}$

NPA hierarchy: moment matrix Γ^2

- Zoom out to length 2

$$\Gamma_{A_{a|x}B_{b|y},C_{c|z}} = \mathrm{Tr}_\tau((A_{a|x}B_{b|y})^\dagger C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z}) = p(abc\,|\,xyz)$$

	$\mathbb{1}$	$A_{a\|x}$	$B_{b\|y}$	$C_{c\|z}$	$A_{a\|x}A_{a'\|x'}$	$\cdots$
$\mathbb{1}$	1					
$(A_{a\|x})^\dagger$						
$(B_{b\|y})^\dagger$			$\mathrm{Tr}_\tau(B_{b\|y})$			
$(C_{c\|z})^\dagger$						
$(A_{a\|x}A_{a'\|x'})^\dagger$				$\mathrm{Tr}_\tau(A_{a'\|x'}A_{a\|x}C_{c\|z})$		
$(A_{a\|x}B_{b\|y})^\dagger$				$p(abc\|xyz)$		
$\cdots$						

- Γ^2 is semidefinite positive, symmetric, satisfies many linear constraints…

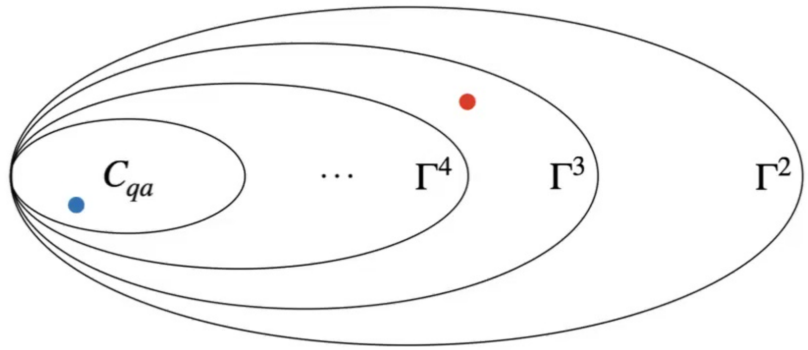
NPA hierarchy: moment matrix Γ^3 and more

- Longer indices, such as $A_{a|x}B_{b|y}C_{c|z}$, of length 3, to get a bigger matrix Γ^3.

	$\mathbb{1}$	$A_{a\|x}$	$B_{b\|y}$	$C_{c\|z}$	$\cdots$
$\mathbb{1}$	1				
$(A_{a\|x})^\dagger$					
$(B_{b\|y})^\dagger$			$\mathrm{Tr}_\tau\left(B_{b\|y}\right)$		
$(C_{c\|z})^\dagger$					
$(A_{a\|x}A_{a'\|x'})^\dagger$				$\mathrm{Tr}_\tau\left(A_{a'\|x'}A_{a\|x}C_{c\|z}\right)$ $p(abc\|xyz)$	
$(A_{a\|x}B_{b\|y})^\dagger$					
$\cdots$					
$(A_{a\|x}B_{b\|y}C_{c\|z})^\dagger$		$\mathrm{Tr}_\tau\left(A_{a\|x}B_{b\|y}C_{c\|z}\right)$			
$\cdots$					

- Containing Γ^2 as a submatrix: $\Gamma^3 \implies \Gamma^2$.

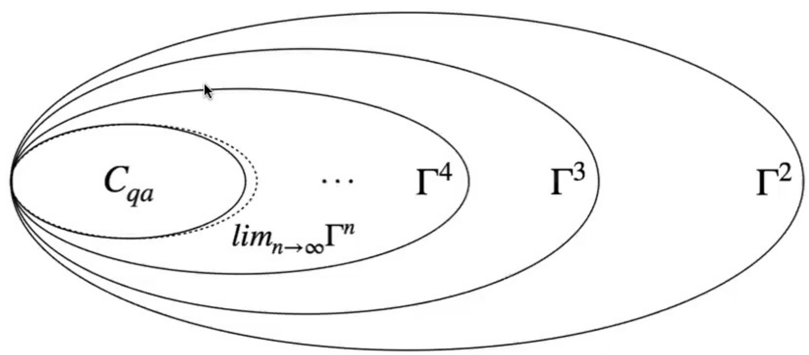
NPA hierarchy is necessary



- If $\overrightarrow{P} \in C_{qa}$, then for every n there exists compatible moment matrix Γ^n.
- If $\overrightarrow{P}$ does not admit Γ^n for some n, then $\overrightarrow{P} \notin C_{qa}$.
- Semidefinite program (SDP): checking if Γ^n exists can be done with computers!

Is NPA hierarchy sufficient for C_{qa}?

- What if $\overrightarrow{P}$ admits all Γ^n? I.e. what is the limit $lim_{n\to\infty}\Gamma^n$?
- Can we say $lim_{n\to\infty}\Gamma^n = C_{qa}$?



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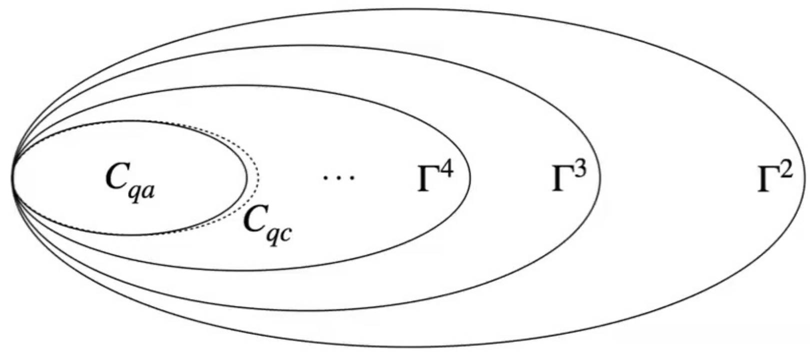
NPA hierarchy is sufficient for C_{qc}

- Theorem: If $\overrightarrow{P}$ admits Γ^n for all $n \to \infty$, then $\overrightarrow{P} \in C_{qc}$ the *commutator quantum distribution:*
    1. A global Hilbert space H with pure density operator τ
    2. PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$ mutually commute
    3. $p(abc \mid xyz) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$

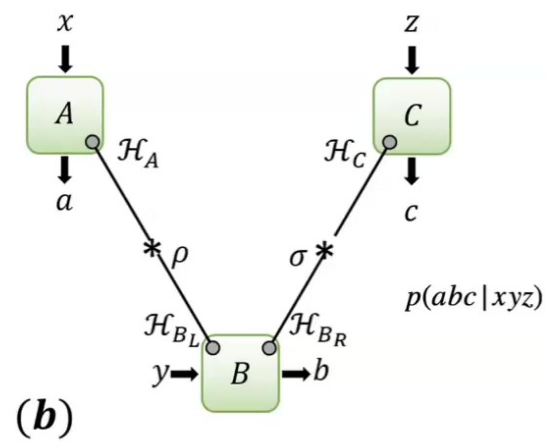
- Tensor C_{qa} vs commutator C_{qc}? Known as Tsirelson's problem.
- $A_{a|x} \otimes id_{BC}$ commutes with $id_{AB} \otimes C_{c|z}$, so $C_{qa} \subset C_{qc}$.
- We know $C_{qa} \subsetneq C_{qc}$ [Ji et al., 2021], but they do agree in finite dimension [Fritz, 2012].

NPA hierarchy: summary



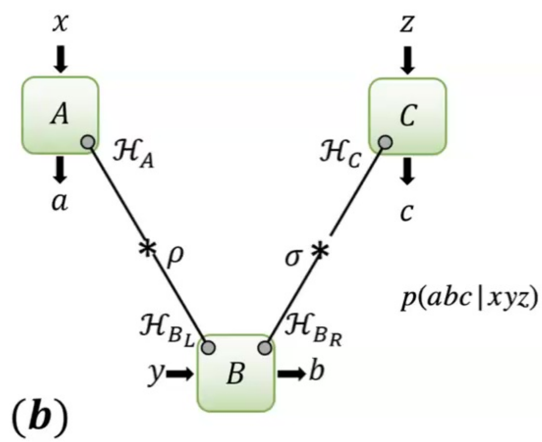
- A hierarchy Γ^n converges to commutator quantum model C_{qc} from the outside.
- In finite dimension, it converges to the usual quantum model with tensor product C_{qa}.
- Each step can be solved by computers via SDP.

Quantum bilocal scenario



- The simplest network scenario beyond the Bell scenario.
- Entanglement swapping [Branciard et al., 2012], real quantum theory can be falsified experimentally [Renou et al., 2021], etc.

Bilocal scenario: standard QM $Q_{bilocal}$



- $H = H_A \otimes H_{B_L} \otimes H_{B_R} \otimes H_C$, $\tau = \rho_{AB_L} \otimes \sigma_{B_R C}$
- PVMs $\{A_{a|x}\}$, $\{B_{b|y}\}$, $\{C_{c|z}\}$, s.t. $[A_{a|x}, \sigma] = [\rho, C_{c|z}] = 0$
- Alice and Charlie are independent: $\mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z})$, similarly for any products of $A_{a|x}$, $C_{c|z}$.

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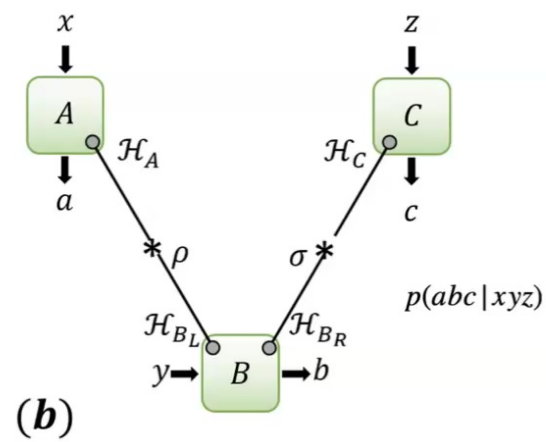
Bilocal scenario $\mathcal{Q}_{bilocal}$ vs Bell scenario



- We have $\mathcal{Q}_{bilocal} \subsetneq C_{qa}$
- Bilocal scenario is always Bell (let $\tau = \rho \otimes \sigma$).
- Converse is not true, e.g. GHZ state cannot be separate. In fact, $\mathcal{Q}_{bilocal}$ is not convex.

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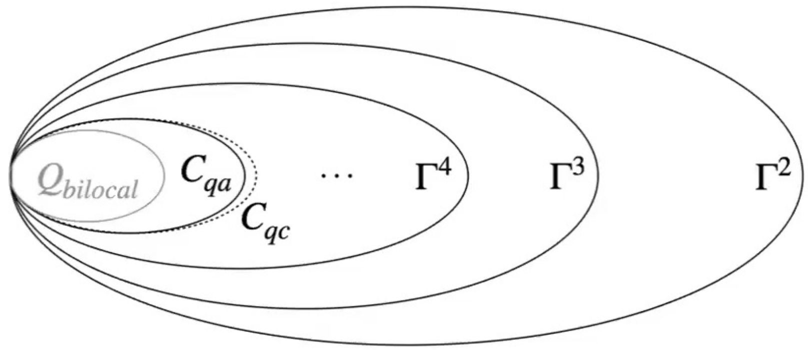
Bilocal scenario $\mathcal{Q}_{bilocal}$ vs NPA hierarchy



- Already an outer approximation to C_{qa}, standard NPA hierarchy is too unrestricted for $\mathcal{Q}_{bilocal}$.
- More constraint/stronger tests are needed. Adding more constraints?
- $\mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z})$ and any product of A, C!

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Factorisation bilocal hierarchy $\tilde{\Gamma}^n$

$$
\begin{array}{c}
\mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathbb{1} & A_{a|x} & B_{b|y} & C_{c|z} & \cdots \end{array} \\
\begin{bmatrix}
1 & & & \mathrm{Tr}_\tau(C_{c|z}) & \\
\mathrm{Tr}_\tau(A_{a|x}) & & & & \\
 & & & & \\
 & \mathrm{Tr}_\tau(A_{a|x}C_{c|z}) & & & \\
 & & & & \\
 & & & & \\
 & & & &
\end{bmatrix}
\end{array}
$$

- Given bilocal $\overrightarrow{P} \in Q_{bilocal}$, we get a moment matrix $\tilde{\Gamma}^n$ for any n the usual way. Almost the same as standard Γ^n.
- But for bilocal, also have factorisation constraints: e.g. $\tilde{\Gamma}^n_{A_{a|x},C_{c|z}} = \mathrm{Tr}_\tau(A_{a|x}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Tr}_\tau(C_{c|z}) = \tilde{\Gamma}^n_{A_{a|x},1} \cdot \tilde{\Gamma}^n_{1,C_{c|z}}$ and arbitrary products.

Factorisation bilocal hierarchy $\tilde{\Gamma}^n$

	$\mathbb{1}$	$A_{a\|x}$	$B_{b\|y}$	$C_{c\|z}$	$\cdots$
$\mathbb{1}$	1			$\mathrm{Tr}_\tau(C_{c\|z})$	
$(A_{a\|x})^\dagger$	$\mathrm{Tr}_\tau(A_{a\|x})$				
$(B_{b\|y})^\dagger$					
$(C_{c\|z})^\dagger$		$\mathrm{Tr}_\tau(A_{a\|x}C_{c\|z})$			
$(A_{a\|x}A_{a'\|x'})^\dagger$					
$(A_{a\|x}B_{b\|y})^\dagger$					
$\cdots$					

- Given bilocal $\overrightarrow{P} \in Q_{bilocal}$, we get a moment matrix $\tilde{\Gamma}^n$ for any n the usual way. Almost the same as standard Γ^n.
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- Define $\tilde{\Gamma}^n = \Gamma^n$ + factorisation constraints

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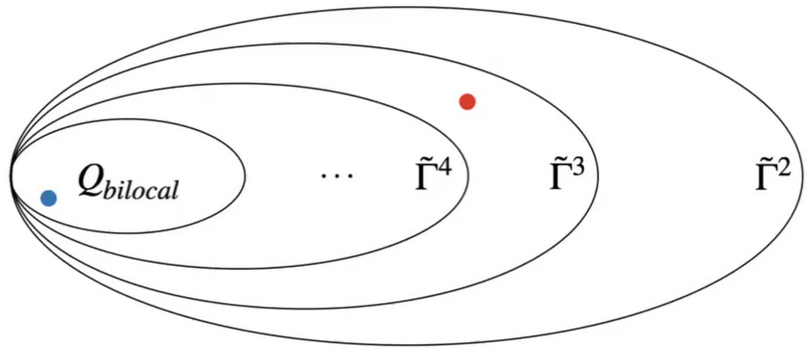
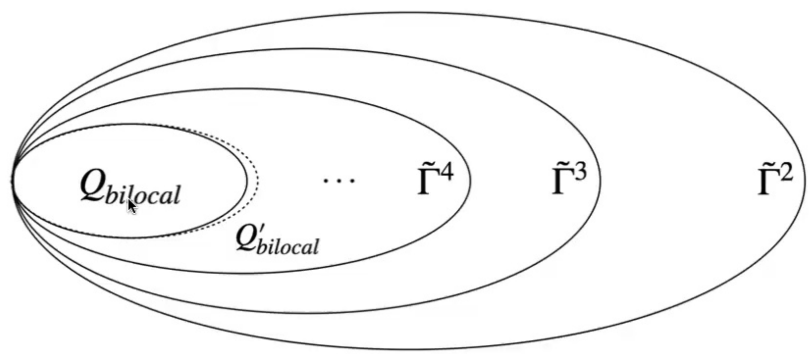
Factorisation hierarchy is necessary



- New hierarchy $\tilde{\Gamma}^n = \Gamma^n +$ factorisation constraints.
- If $\overrightarrow{P} \in Q_{bilocal}$, then for all n there exists a compatible $\tilde{\Gamma}^n$.
- If $\overrightarrow{P}$ does not admit $\tilde{\Gamma}^n$ for some n, then $\overrightarrow{P} \notin Q_{bilocal}$.
- But nonlinear, it is *not* SDP!

Is factorisation hierarchy sufficient?

- What is $Q'_{bilocal} = lim_{n\to\infty}\tilde{\Gamma}^n$?
- Can we say $Q'_{bilocal} = Q_{bilocal}$? Analogous to C_{qa} vs C_{qc}?



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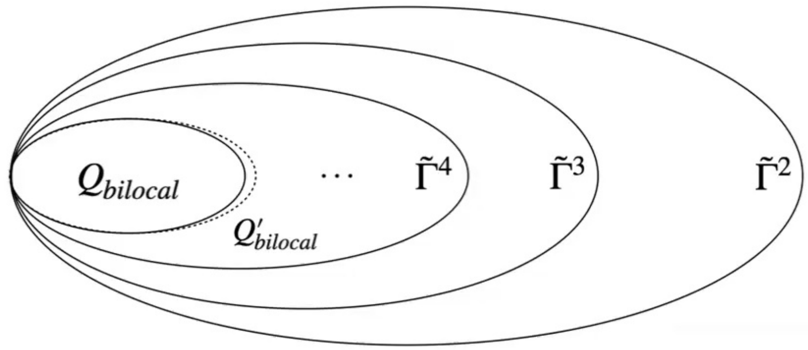
Factorisation hierarchy: sufficiency

- Main Theorem: If $\overrightarrow{P}$ admits $\tilde{\Gamma}^n$ for all $n \to \infty$, then $\overrightarrow{P} \in Q'_{bilocal}$ the *projector bilocal quantum distribution:*

    1. A global Hilbert space H with pure density operator τ;
    2. PVMs $\{A_{a|x}\}, \{B_{b|y}\}, \{C_{c|z}\}$ mutually commute;
    3. $p(abc \mid xyz) = \mathrm{Tr}_\tau(A_{a|x}B_{b|y}C_{c|z})$;
    4. Projectors ρ, σ on H such that $\tau = \rho \cdot \sigma = \sigma \cdot \rho$;
    5. $[A_{a|x}, \sigma] = [\rho, C_{c|z}] = 0$.

- Bilocal Tsirelson: [Ligthart and Gross, 2023], shows that $Q'_{bilocal}$ agrees $Q_{bilocal}$ in finite dimension

Factorisation bilocal hierarchy: summary



- A hierarchy $\tilde{\Gamma}^n$ converges to projector bilocal quantum $Q'_{bilocal}$ from the outside.
- $Q'_{bilocal}$ is equivalent to standard bilocal quantum model $Q_{bilocal}$ in finite dimension.
- Not SDP, cannot be solved by computers.

Scalar extension: linearise the hierarchy

Problem: factorisation constraints are not linear.

[Pozas-Kerstjens et al., 2019] introduces the original scalar extension:

- New commutative variables to Alice: $\kappa_{A_{a|x}} = \mathrm{Tr}_\tau(A_{a|x})\mathrm{Id}$

$$
\begin{array}{c} \mathbb{1} \\ (A_{a|x})^\dagger \\ (B_{b|y})^\dagger \\ (C_{c|z})^\dagger \\ (A_{a|x}A_{a'|x'})^\dagger \\ (A_{a|x}B_{b|y})^\dagger \\ \cdots \end{array}
\overset{\begin{array}{ccccc} \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} \;\cdots \end{array}}{
\begin{bmatrix} 1 & & & & \\ & & & & \\ & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & \\ & & & & \end{bmatrix}}
$$

Scalar extension: first idea

$$
\begin{array}{c|ccccc}
 & \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} \quad \cdots \\
\hline
\mathbb{1} & 1 & & & & \\
(A_{a|x})^\dagger & & & & & \\
(B_{b|y})^\dagger & & & & & \\
(C_{c|z})^\dagger & & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & \\
(A_{a|x}A_{a'|x'})^\dagger & & & & & \\
(A_{a|x}B_{b|y})^\dagger & & & & & \\
\cdots & & & & &
\end{array}
$$

- New commutative variables to Alice: $\kappa_{A_{a|x}} \neq \mathrm{Tr}_\tau(A_{a|x})\mathrm{Id}$
- Then

  But we have a fix!

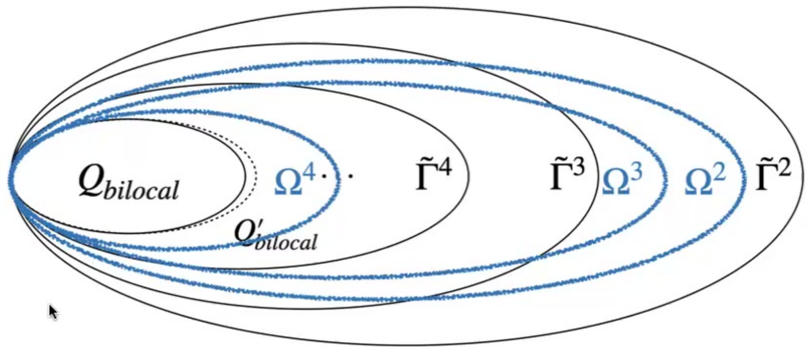
  $\mathrm{Tr}_\tau(\kappa_{A_{a|x}}C_{c|z}) = \mathrm{Tr}_\tau(\mathrm{Tr}_\tau(A_{a|x})\mathrm{Id} \cdot C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x}) \cdot \mathrm{Tr}_\tau(C_{c|z})$
- Hence, factorisation constraints should be imposed via letting $\mathrm{Tr}_\tau(\kappa_{A_{a|x}}C_{c|z}) = \mathrm{Tr}_\tau(A_{a|x}C_{c|z})$?

Scalar extension: on the carpet

$$
\begin{array}{c|cccccc}
 & \mathbb{1} & A_{a|x} & \cdots & \kappa_{A_{a|x}} & \kappa_{A_{a|x}A_{a'|x'}} & \cdots \\
\hline
\mathbb{1} & 1 & & & & & \\
(A_{a|x})^\dagger & & & & & & \\
(B_{b|y})^\dagger & & & & & & \\
(C_{c|z})^\dagger & & \mathrm{Tr}_\tau\left(A_{a|x}C_{c|z}\right) & \cdots & \mathrm{Tr}_\tau\left(\kappa_{A_{a|x}}\hat{C}_{c|z}\right) & & \\
(A_{a|x}A_{a'|x'})^\dagger & & & & & & \\
(A_{a|x}B_{b|y})^\dagger & & & & & & \\
\cdots & & & & & &
\end{array}
$$

- Need new commutative variables: $\kappa_{A_{a|x}}$, $\kappa_{A_{a|x}B_{b|y}}$, $\kappa_{A_{a|x}A_{a'|x'}C_{c|z}}\cdots$
- Need more complicated constraints
- A convergent scalar extension hierarchy Ω^n to $Q'_{bilocal}$

Bilocal network: summary on two hierarchies



- Two hierarchies, $\tilde{\Gamma}^n$ and Ω^n, both converge to projector commutator model $Q'_{bilocal}$ from the outside.
- In finite dimension, they both characterise the standard QM bilocal network correlations.
- Scalar extension hierarchy Ω^n is SDP.

Bilocal network: the wider context

- Our first draft motivate the following two, which in turn help us fix a mistake:
- [Ligthart and Gross, 2023] show bilocal Tsirelson's problem and provide another convergent hierarchy based on polarisation technique.
- [Klep et al., 2023] from theory of noncommutative polynomials. They prove scalar extension in more generality as *state polynomials*.
- Together, bilocal network scenario (actually, stars) can be completely characterized in the C^*-algebraic/Heisenberg picture.
- But, more general networks remain open. Quantum inflation [Wolfe et al., 2021]?