Title: Can the quantum switch be deterministically simulated?

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Series: Quantum Foundations, Quantum Information

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Abstract: Higher-order transformations that act on a certain number of input quantum channels with an indefinite causal order, such as the quantum switch, cannot be described by standard quantum circuits that use the same number of calls of the input quantum channels. But could they be simulated, i.e., could their action on their input channels be deterministically reproduced, for all arbitrary inputs, by a quantum circuit that uses on a larger number of calls of the input channels? In this work, we prove that, when only one extra call of each input channel is available, the quantum switch cannot be simulated. We demonstrate the robustness of this result by showing that even when probabilistic and approximate simulations are considered, higher-order transformations that are close to the quantum switch can be at best simulated with a probability strictly less than one. This result stands in stark contrast with the known fact that, when the quantum switch acts exclusively on unitary channels, its action can be simulated. We also show other particular cases where a restricted simulation of the quantum switch is possible. Finally, we discuss the implications of our findings to the analysis of experiments based on the quantum switch.

# Can the quantum switch be deterministically simulated?

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*17 September 2024, Causalworlds, Waterloo*

UNIVERSITÉ DE GENÈVE

Fonds national suisse

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(in preparation)

# Formalism: Higher-order operations

**Quantum computing**

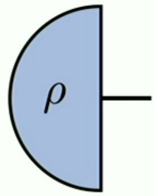
how can we efficiently manipulate and process quantum data?

**Higher-order quantum computing**

how can we efficiently manipulate and process quantum **functions**?

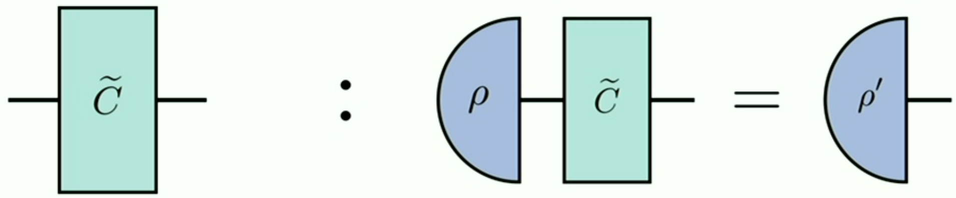
# Formalism: Higher-order operations

**quantum data:**
quantum states



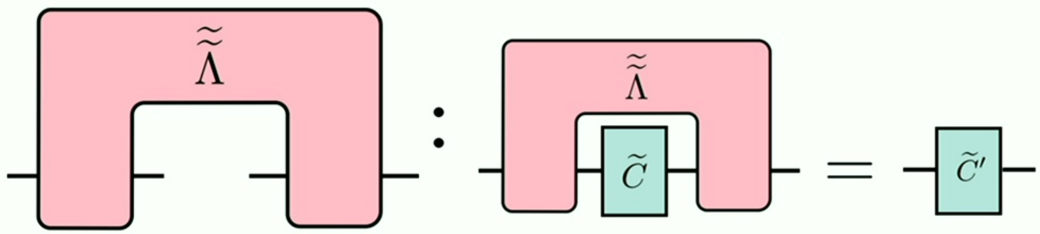

$\rho \in \mathcal{L}(\mathcal{H}_{\text{in}})$

**quantum functions:**
quantum operations
(quantum channels)




$\widetilde{C} : \mathcal{L}(\mathcal{H}_{\text{in}}) \to \mathcal{L}(\mathcal{H}_{\text{out}})$

**higher-order quantum operations:**
quantum processes



$\widetilde{\widetilde{\Lambda}} : [\mathcal{L}(\mathcal{H}_{\text{in}}) \to \mathcal{L}(\mathcal{H}_{\text{out}})]$
$\to [\mathcal{L}(\mathcal{H}_{\text{in}'}) \to \mathcal{L}(\mathcal{H}_{\text{out}'})]$

**"functions of functions"**

# Formalism: Higher-order operations

- *Toward a general theory of quantum games*
  Gutoski, Watrous
  STOC 2007, pages 565-574

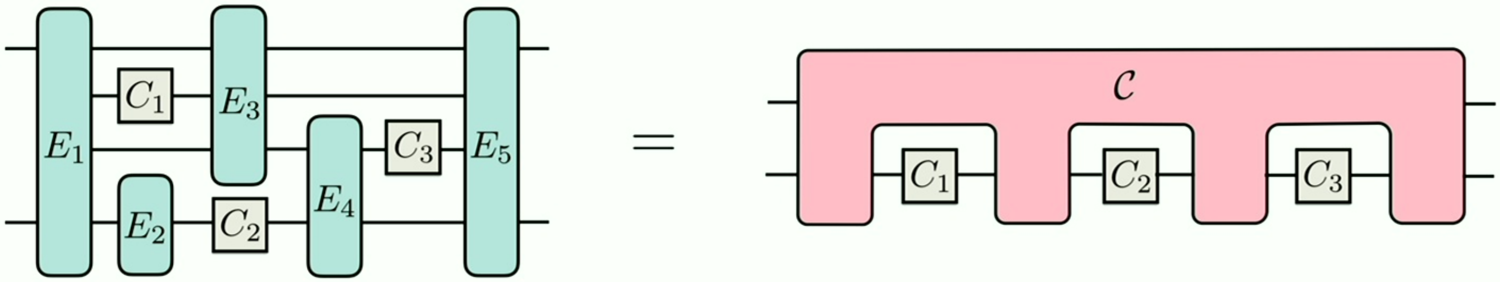
- *Quantum circuit architecture*
  Chiribella, D'Ariano, Perinotti
  PRL 101, 060401 (2008)

- *Transforming quantum operations: quantum supermaps*
  Chiribella, D'Ariano, Perinotti
  Europhys. Lett. 83, 30004 (2008)

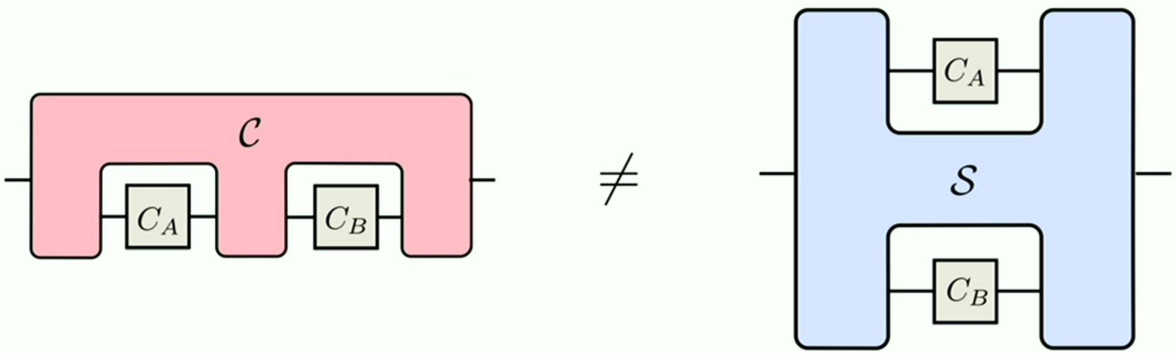
- *Quantum correlations with no causal order*
  Oreshkov, Costa, Brukner
  Nat. Commun. 3, 1092 (2012)

# Key features: Higher-order operations

**1)** Higher-order quantum operations efficiently describe quantum circuits



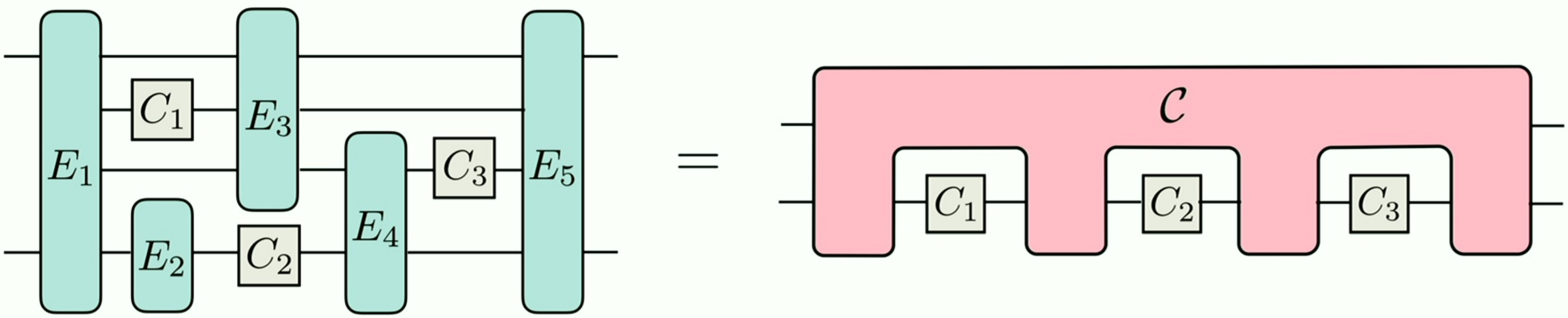

**2)** Higher-order quantum operations go beyond the quantum circuit model



# Key features: Higher-order operations

**1)** Higher-order quantum operations efficiently describe quantum circuits

**Component-wise optimization** | **Semidefinite Programming (SDP)**

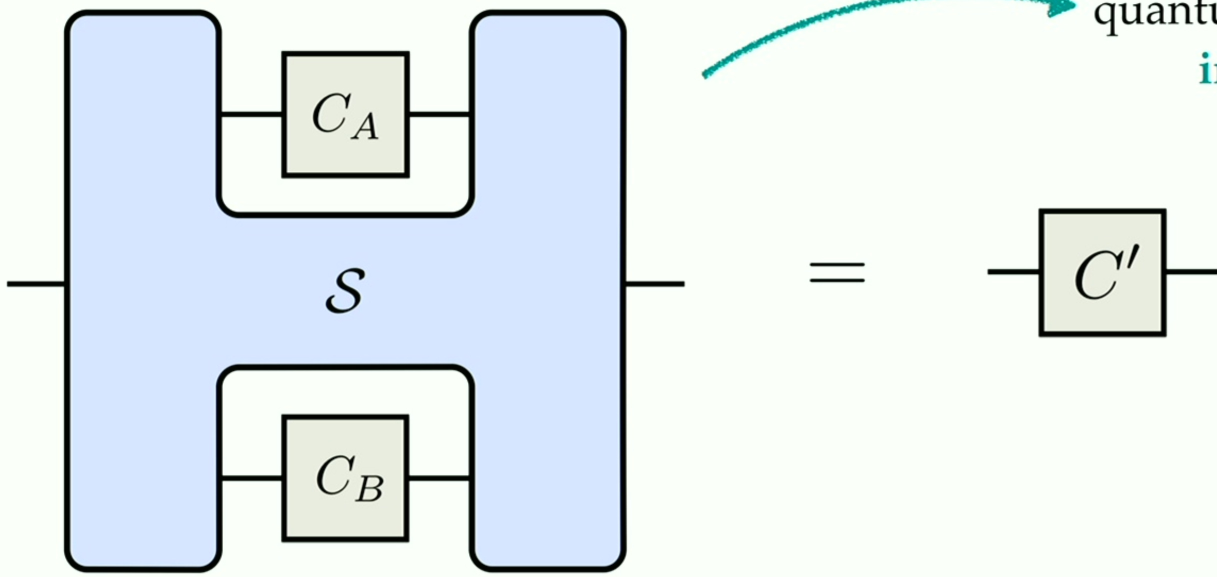


# Key features: Higher-order operations

**2)** Higher-order quantum operations go beyond the quantum circuit model

$$(C_A, C_B) \mapsto C'$$

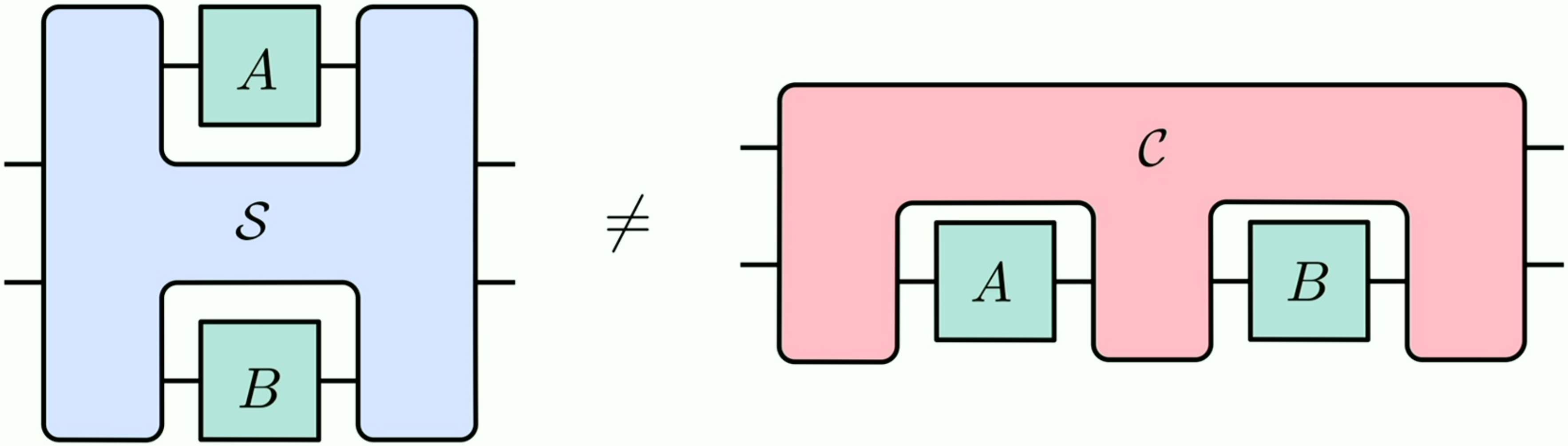
Admissible transformations of quantum operations may have an **indefinite causal order**



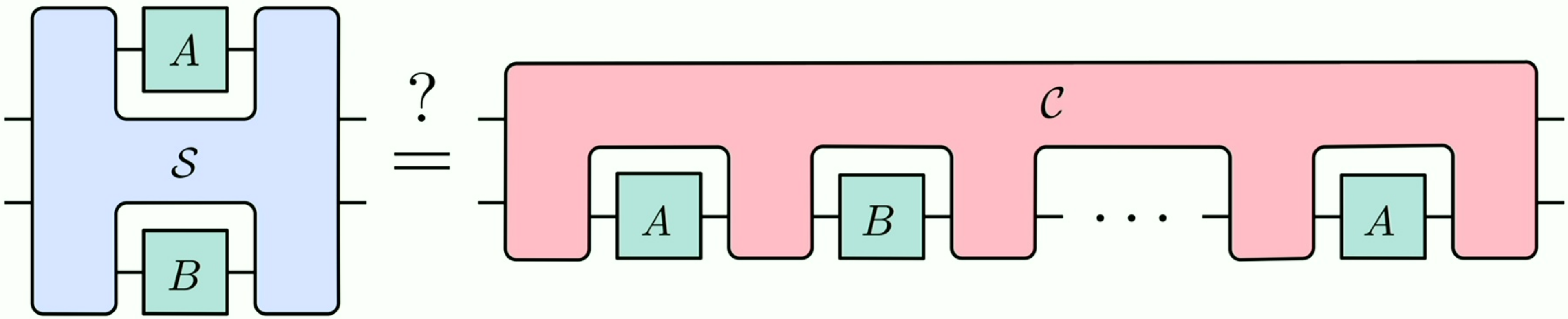
# Advantages of indefinite causal order

- **Channel discrimination**
  Chiribella, PRA 86, 040301 (2012)
  Araújo, NJP 17, 102001 (2015)
  Bavaresco et al, PRL 127, 200504 (2021), J. Math. Phys. 63, 042203 (2022)
- **Computational advantage**
  Araújo et al, PRL 113, 250402 (2014)
  Renner et al, PRL 128, 230503 (2022)
  Abbott et al, PRResearch 6, L032020 (2024)
- **Universal unitary transformations**
  Quintino et al, PRL 123, 210502 (2019), PRA 100, 062339 (2019)
  Yoshida et al, Quantum 7, 957 (2023)
- **Metrology**
  Zhao et al, PRL 124, 190503 (2020)
  Liu et al, PRL 130, 070803 (2023)

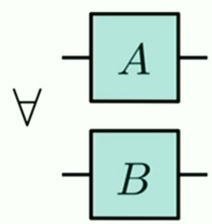
# Advantages of indefinite causal order



# Simulation of indefinite causal order



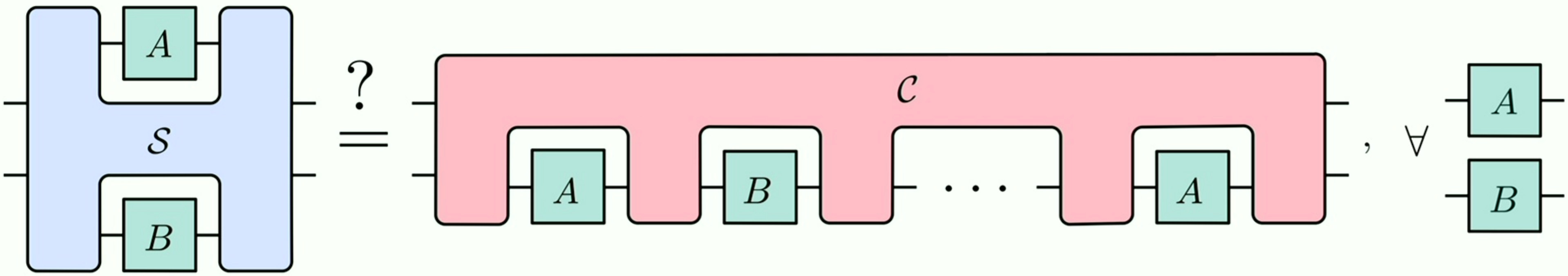

$$\mathcal{C}(A^{\otimes k_A}, B^{\otimes k_B}) = \mathcal{S}(A, B) \quad \forall\ A, B$$



# Simulation of indefinite causal order

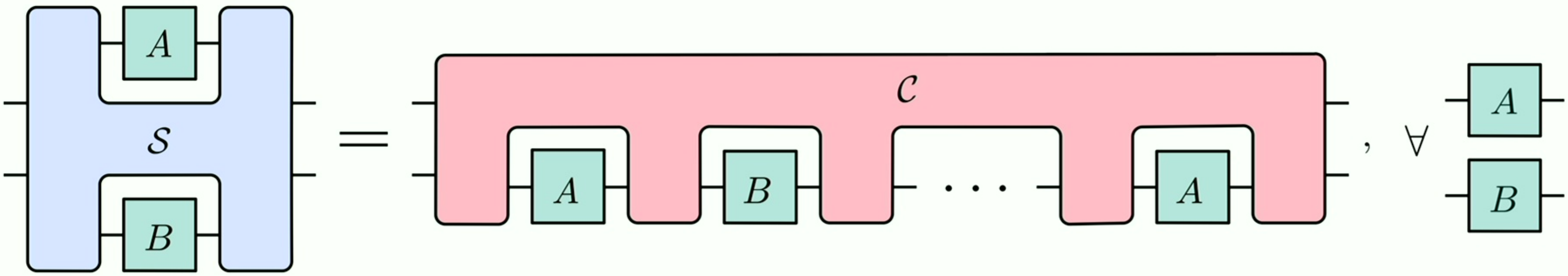
**Cost: quantum query complexity**

$$(k_A, k_B)$$



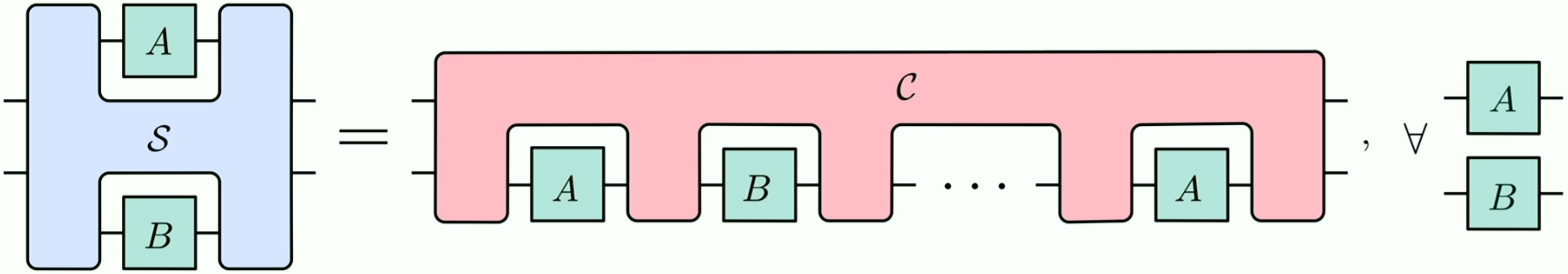

$$\mathcal{C}(A^{\otimes k_A}, B^{\otimes k_B}) = \mathcal{S}(A, B) \quad \forall\ A, B$$

# Simulation of indefinite causal order




**Possible strategies for simulation:**

# Simulation of indefinite causal order



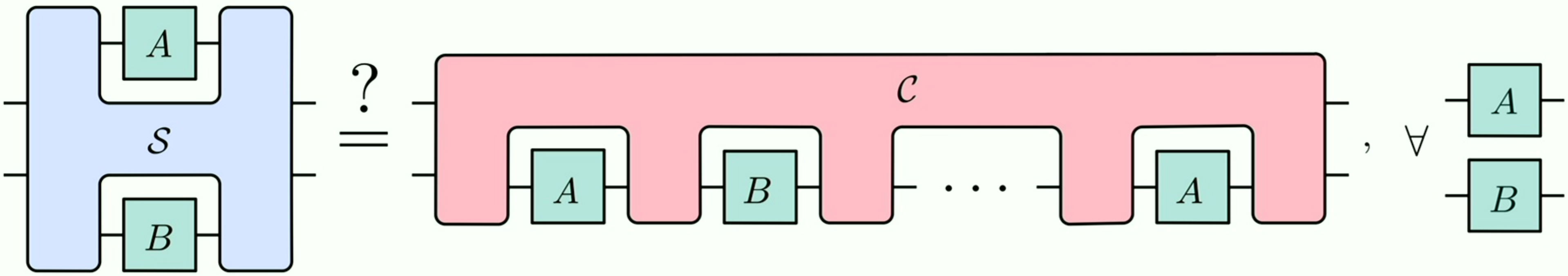

**Possible strategies for simulation:**

- Infinite number of queries = process tomography
- Approximate / probabilistic simulation [1,2]
- Particular inputs [1]


[1] Chiribella, D'Ariano, Perinotti, Valiron, PRA 88, 022318 (2013)

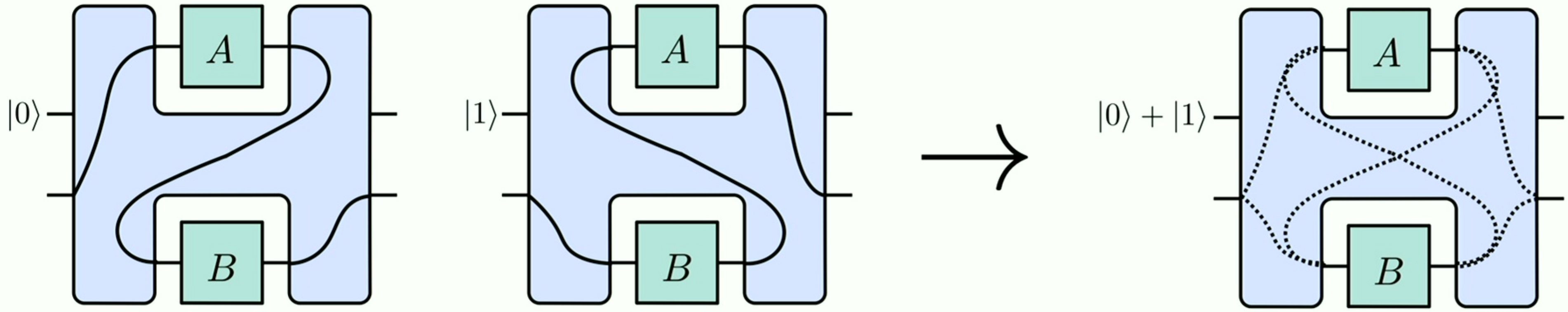
[2] Milz, Pollock, Le, Chiribella, Modi, NJP 20, 033033 (2018), Quintino, Dong, Shimbo, Soeda, Murao, PRA 100, 062339 (2019)

# Rules of the simulation



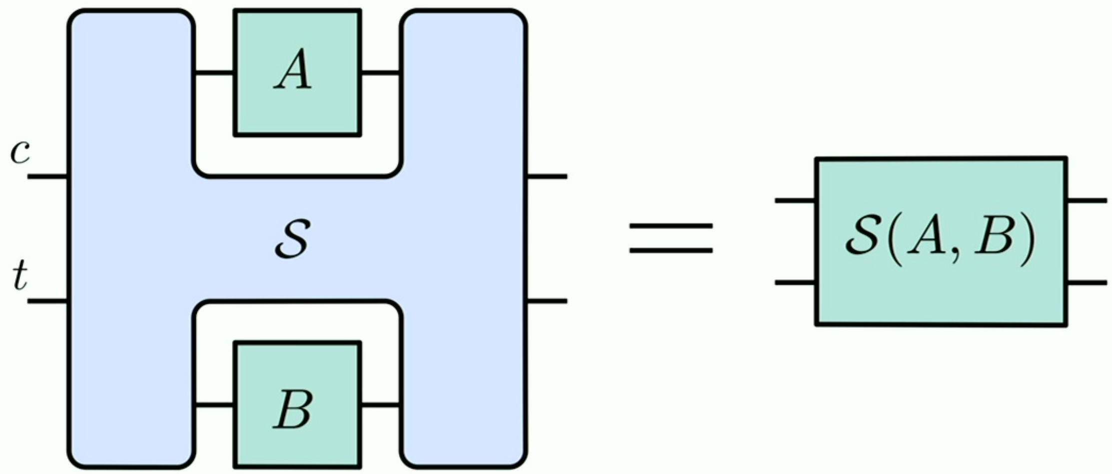

- Universal (blackbox inputs)
- Deterministic
- Finite number of calls

# The quantum switch transformation




Chiribella, D'Ariano, Perinotti, Valiron, PRA 88, 022318 (2013)

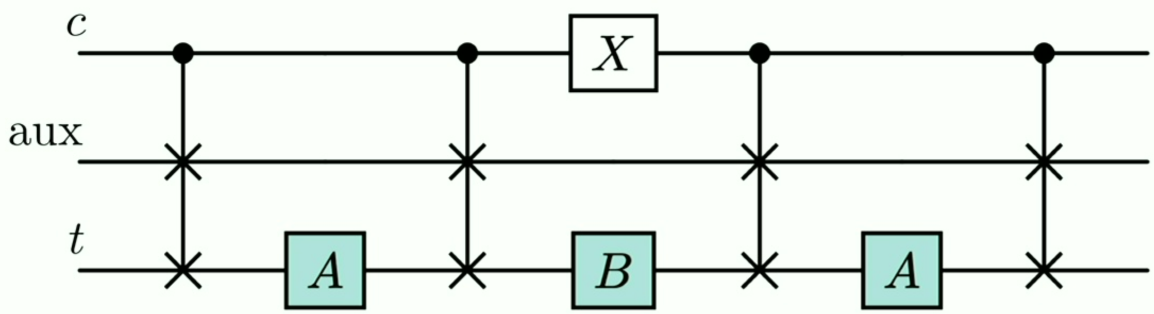
# The quantum switch transformation




$$\mathcal{S}(A,B)[\,\sigma_c \otimes \rho_t\,] := \sum_{i,j} S_{ij}(\sigma_c \otimes \rho_t)S_{ij}^{\dagger}$$

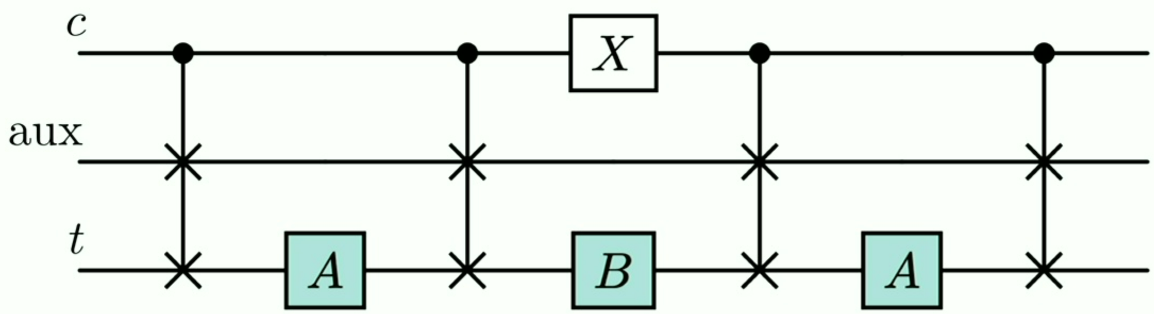
Chiribella, D'Ariano, Perinotti, Valiron, PRA 88, 022318 (2013)

# Special-case simulation: unitary channels




Chiribella, D'Ariano, Perinotti, Valiron, PRA 88, 022318 (2013)

# Special-case simulation: unitary channels



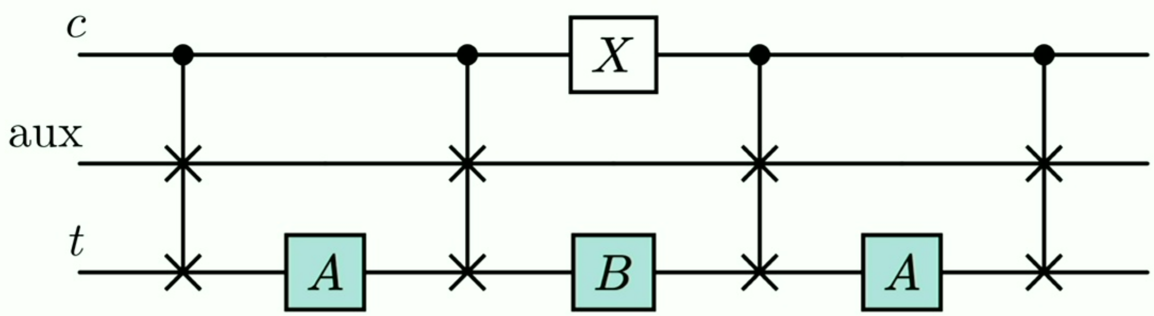

$$C_{000} = |0\rangle\langle 0| \otimes A_0 \otimes B_0 A_0 + |1\rangle\langle 1| \otimes A_0 \otimes A_0 B_0 \qquad \text{(quantum circuit)}$$

**For unitary channels:**

$$S_{00} := |0\rangle\langle 0| \otimes B_0 A_0 + |1\rangle\langle 1| \otimes A_0 B_0 \qquad \text{(quantum switch)}$$

Chiribella, D'Ariano, Perinotti, Valiron, PRA 88, 022318 (2013)

# Special-case simulation: unitary channels



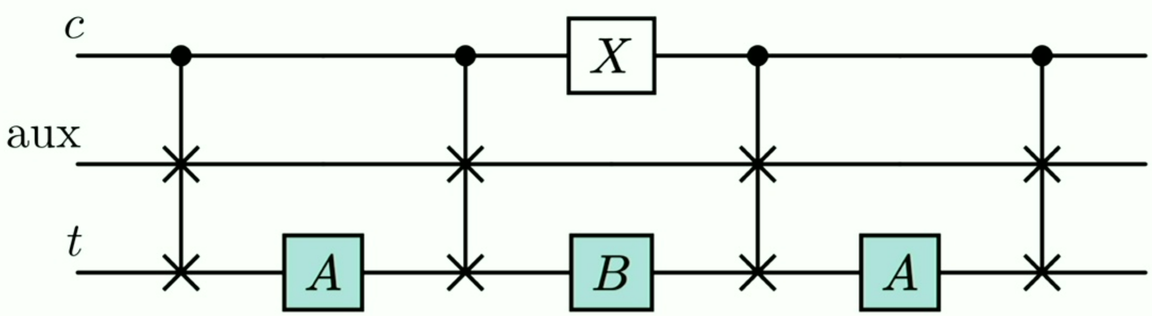

$$C_{iji'} = |0\rangle\langle 0| \otimes A_{i'} \otimes B_j A_i + |1\rangle\langle 1| \otimes A_i \otimes A_{i'} B_j$$ (quantum circuit)

$$\neq$$

**For general channels:** $$S_{ij} := |0\rangle\langle 0| \otimes B_j A_i + |1\rangle\langle 1| \otimes A_i B_j$$ (quantum switch)

Chiribella, D'Ariano, Perinotti, Valiron, PRA 88, 022318 (2013)

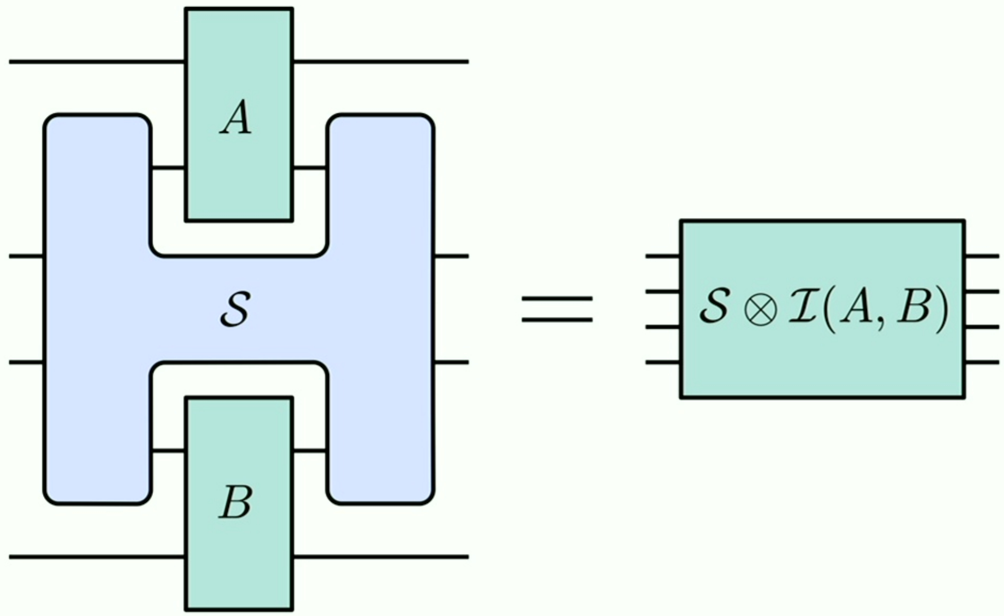
# Special-case simulation: unitary channels




Not a simulation of the quantum switch in general.

But it can be generalized to a certain extent.

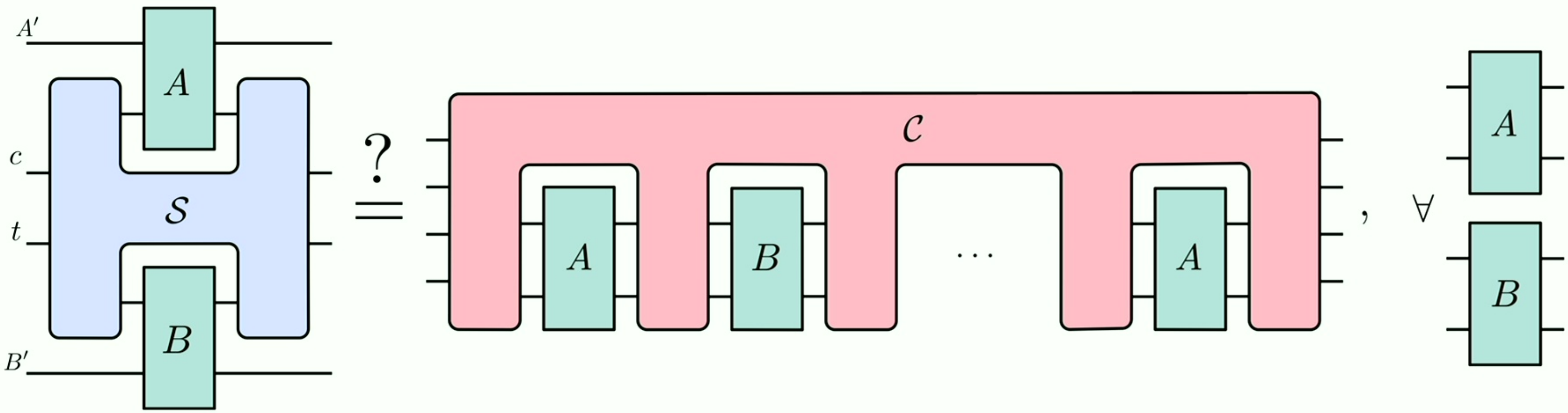
# The general quantum switch transformation




$$\mathcal{S} \otimes \mathcal{I}(A, B)[\,\sigma_c \otimes \rho_t \otimes \omega_{A'B'}\,] = \sum_{i,j} S_{ij}(\sigma_c \otimes \rho_t \otimes \omega_{A'B'})S_{ij}^\dagger$$

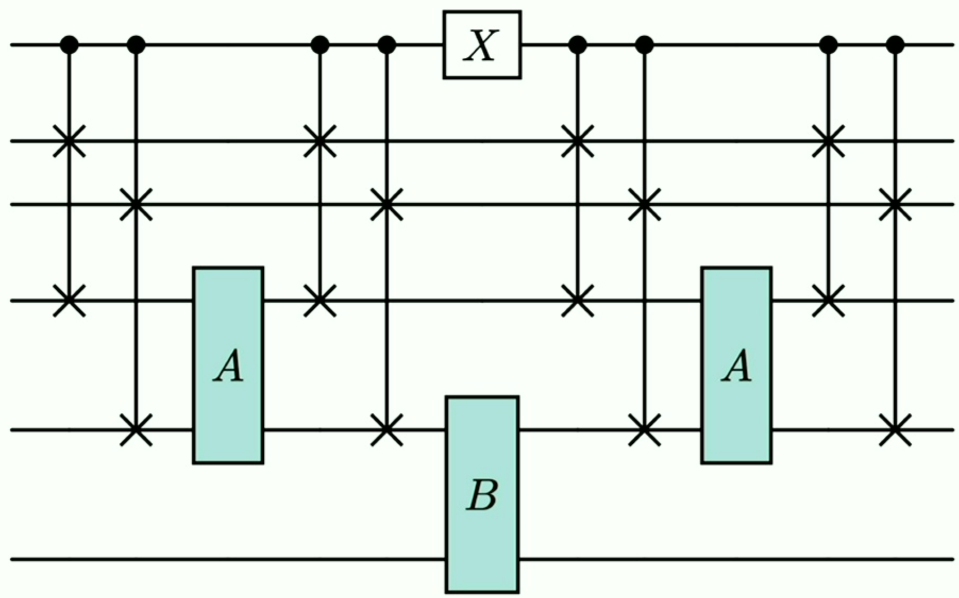
definition implied by linearity

# The general quantum switch transformation




$$\mathcal{C}(A^{\otimes k_A}, B^{\otimes k_B}) = \mathcal{S} \otimes \mathcal{I}(A, B) \quad \forall\ A, B$$

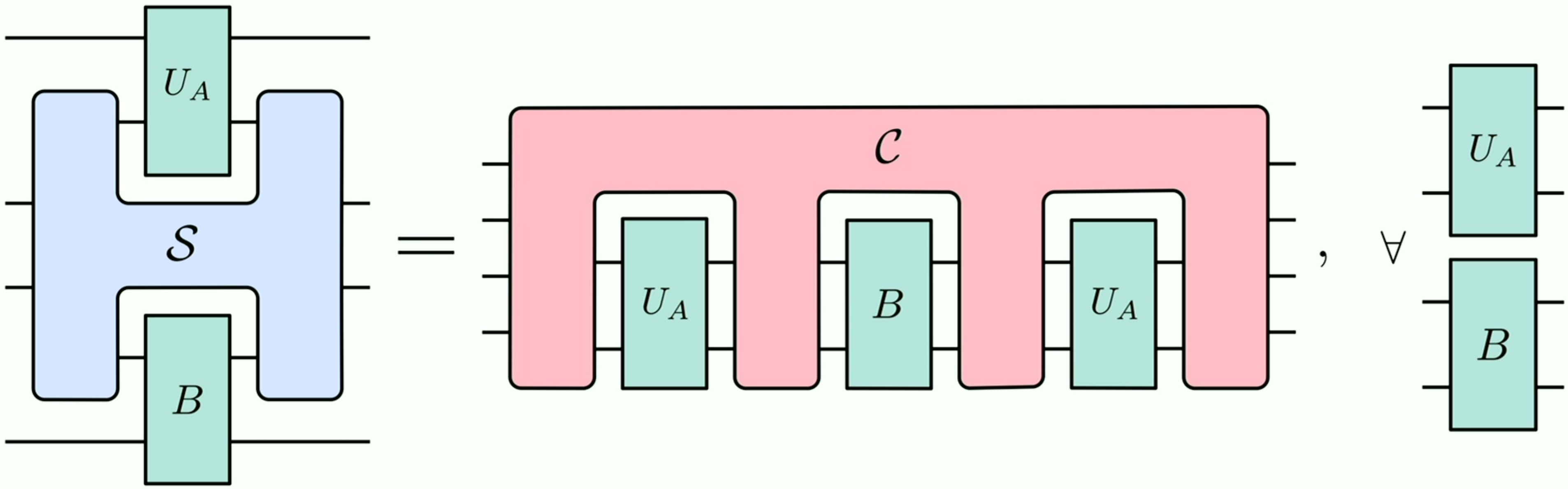
# First result: special-case simulation for extended channels




**For unitary channels A:** $A[\rho] = U_A[\rho] = \sum_i A_i \, \rho \, A_i^\dagger = A_0 \, \rho \, A_0^\dagger$

**For general channels B:** $B[\rho] = \sum_j B_j \, \rho \, B_j^\dagger$

# First result: special-case simulation for extended channels



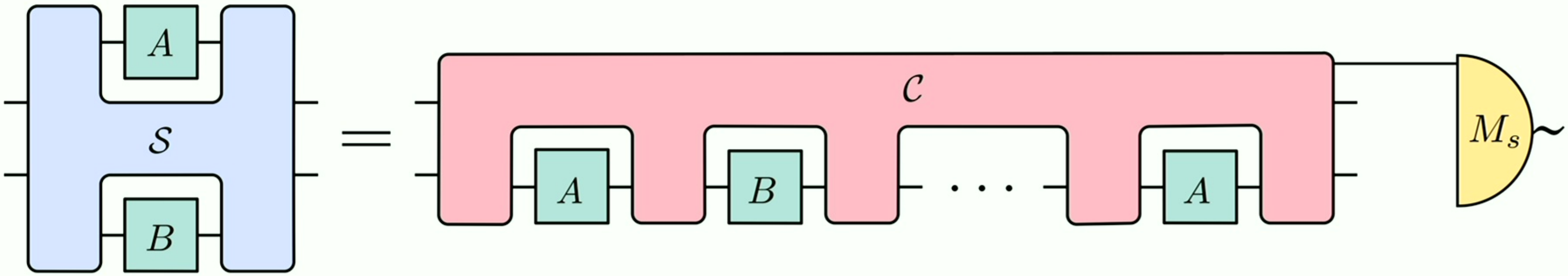
# Main result: no-go theorems

The action of the quantum switch cannot be simulated by a quantum circuit that has access to an extra call of each input operation.

$$(k_A, k_B) \in \{(2,1), (2,2), (3,1)\}$$

# Main result: no-go theorems

**Technique: Probabilistic simulation**



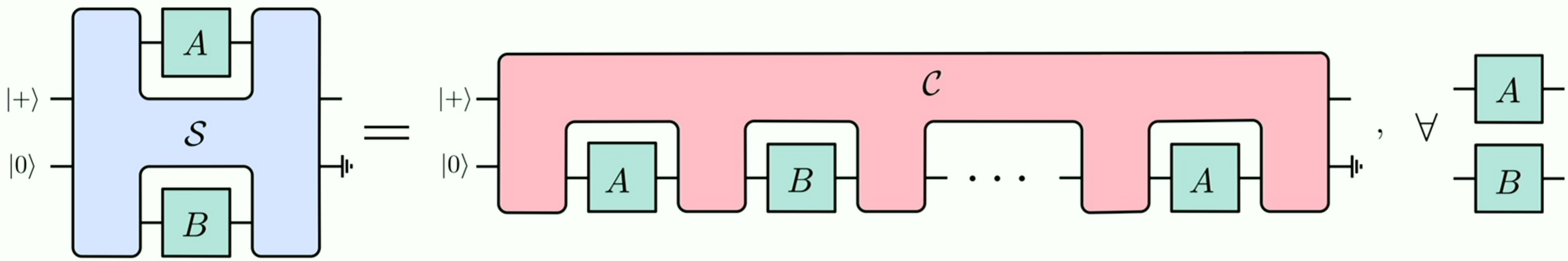

$$\mathcal{C}_s(A^{\otimes k_A}, B^{\otimes k_B}) = p\,\mathcal{S}(A, B) \quad \forall\ A, B$$

$$\mathcal{C} = \mathcal{C}_s + \mathcal{C}_f$$

$\max p < 1 \implies$ impossible simulation

# Main result: no-go theorems

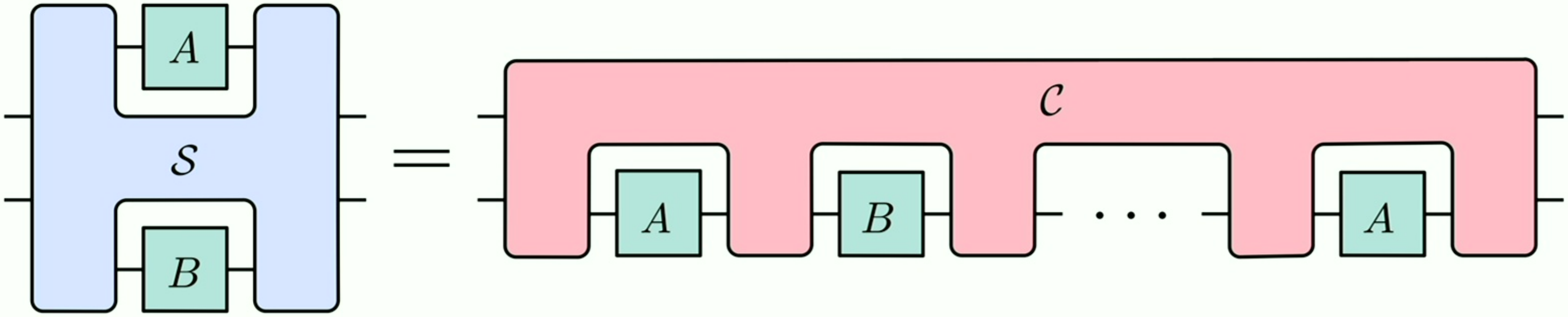
**Technique: Restricted simulation**

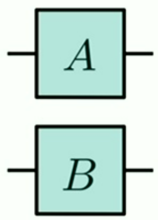



$$\mathrm{tr}_{t_O}\Big(\mathcal{C}(A^{\otimes k_A},\, B^{\otimes k_B})[\,|+\rangle\langle+|\otimes|0\rangle\langle 0|\,]\Big) = \mathrm{tr}_{t_O}\Big(\mathcal{S}(A,B)[\,|+\rangle\langle+|\otimes|0\rangle\langle 0|\,]\Big) \quad \forall\ A, B$$

# Main result: no-go theorems

**Technique: Input basis**




$\forall$ 


$\Longleftrightarrow$ $\forall$

$\{A_i\}_i$ that forms a basis for the space spanned by $k_A$ copies of a channel

$\{B_j\}_j$ that forms a basis for the space spanned by $k_B$ copies of a channel

# Main result: proof technique

**Technique: Semidefinite programming**

$$\begin{aligned}
\textbf{given} \quad & k_A, k_B \\
\textbf{max} \quad & p \\
\textbf{s.t.} \quad & \mathcal{C}_s(A^{\otimes k_A}, B^{\otimes k_B}) = p\,\mathcal{S}(A,B) \;\; \forall\, A, B \\
& \mathcal{C} = \mathcal{C}_s + \mathcal{C}_f \in \mathrm{COMBS}
\end{aligned}$$

# Main result: proof technique

**Technique: Semidefinite programming**

**given** $\{J_i^A\}_i, \{J_j^B\}_j, k_A, k_B$

**max** $p$

**s.t.** $C_s * [(J_i^A)^{\otimes k_A} \otimes (J_j^B)^{\otimes k_B}] = p\, S * (J_i^A \otimes J_j^B) \ \ \forall i, j$

$C_s \geq 0, \quad C - C_s \geq 0,$

$\mathbb{P}(C) = C, \quad \mathrm{tr}(C) = d_{c_I} d_{t_I} d_{A_O}^{k_A} d_{B_O}^{k_B},$

$\mathcal{M} \circ \mathcal{N}$

$\updownarrow$

$M * N$

# Main result: proof technique

**Technique: Semidefinite programming**

$$\begin{aligned}
\textbf{given} \quad & \{J_i^A\}_i, \{J_j^B\}_j, k_A, k_B \\
\textbf{min} \quad & \frac{1}{d_{A_I}^{k_A} d_{B_I}^{k_B} d_{c_O} d_{t_O}} \operatorname{tr}(\Gamma) \\
\textbf{s.t.} \quad & \sum_{i,j} \operatorname{tr}\big[R_{ij}\,(S * (J_i^A \otimes J_j^B))\big] = 1 \\
& \Gamma - \sum_{i,j} R_{ij} \otimes [(J_i^A)^{\otimes k_A} \otimes (J_j^B)^{\otimes k_B}]^T \geq 0 \\
& \Gamma \geq 0, \quad \overline{\mathbb{P}}(\Gamma) = \Gamma,
\end{aligned}$$

any feasible point that yields some
$p < 1$
constitutes a valid upper bound

# Main result: proof technique

## Technique: Computer-assisted proofs

***Algorithm:***

1. Construct symbolic non-floating point operators $\Gamma^{\mathtt{sym}}$ and $R_{ij}^{\mathtt{sym}}$ by truncating them and obtaining a symbolic operator with only rational numbers.

2. Force the operators $\Gamma^{\mathtt{sym}}$ and $R_{ij}^{\mathtt{sym}}$ to be self-adjoint by making use of the expression $(M + M^{\dagger})/2$, which is self-adjoint for any $M$.

3. Evaluate $t^{\mathtt{sym}} := \sum_{i,j} \mathrm{tr}\left[R_{ij}^{\mathtt{sym}} \left(S * (J_i^A \otimes J_j^B)\right)\right]$, where $S, J_i^A$, and $J_j^B$ are also symbolic operators. Define $R_{ij}^{\mathtt{ok}} := R_{ij}^{\mathtt{sym}}/t^{\mathtt{sym}}$ for all $i, j$.

4. Project $\Gamma^{\mathtt{sym}}$ onto the appropriate subspace to obtain $\overline{\mathbb{P}}(\Gamma^{\mathtt{sym}})$.

5. Find $\eta \in \mathbb{R}$ such that $\Gamma^{\mathtt{ok}} := \overline{\mathbb{P}}(\Gamma^{\mathtt{sym}}) + \eta \mathbb{1} \geq 0$ and $\Gamma^{\mathtt{ok}} - \sum_{i,j} R_{ij}^{\mathtt{ok}} \otimes ({J_i^A}^{\otimes k_A} \otimes {J_j^B}^{\otimes k_B})^T \geq 0$

6. Output the quantity $\mathrm{tr}(\Gamma^{\mathtt{ok}})/d_{c_I} d_{t_I} d_{A_O}^{k_A} d_{B_O}^{k_B}$, which is a rigorous upper bound of the primal problem.

# Restricted qubit simulations

| $(k_A, k_B)$ | order | probability |
|---|---|---|
| $(1,1)$ | AB | $p < \frac{4001}{10000}$ |
| $(2,1)$ | AAB | $p < \frac{5715}{10000}$ |
| | ABA | $p < \frac{4919}{10000}$ |
| | BAA | $p < \frac{5001}{10000}$ |
| $(2,2)$ | AABB | $p < \frac{8307}{10000}$ |
| | ABAB | $p < \frac{8484}{10000}$ |
| | ABBA | $p < \frac{8695}{10000}$ |
| $(3,1)$ | AAAB | $p < \frac{8373}{10000}$ |
| | AABA | $p < \frac{6909}{10000}$ |
| | ABAA | $p < \frac{7597}{10000}$ |
| | BAAA | $p < \frac{6845}{10000}$ |

## Restricted qubit simulations

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| | ABAA | $p < \frac{7597}{10000}$ |
| | BAAA | $p < \frac{6845}{10000}$ |

(identical channels)

| $k$ | order | probability |
|---|---|---|
| 2 | AA | $p < \frac{4001}{10000}$ |
| 3 | AAA | $p < \frac{6534}{10000}$ |
| 4 | AAAA | $p = 1$ (*) |

## Restricted qubit simulations

| $(k_A, k_B)$ | order | probability |
|---|---|---|
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| | BAA | $p < \frac{5001}{10000}$ |
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| | ABAB | $p < \frac{8484}{10000}$ |
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| | BAAA | $p < \frac{6845}{10000}$ |

(identical channels)

| $k$ | order | probability |
|---|---|---|
| 2 | AA | $p < \frac{4001}{10000}$ |
| 3 | AAA | $p < \frac{6534}{10000}$ |
| 4 | AAAA | $p = 1$ (*) |

Simulation with full output: $p < 1$

## Restricted qubit unitary simulations

| $(k_A, k_B)$ | order (unitary only) | probability |
|---|---|---|
| $(1,1)$ | AB | $p \approx 0.400$ |
| $(2,1)$ | AAB | $p \approx 0.596$ |
| | ABA | $\boldsymbol{p = 1}$ |
| | BAA | $p \approx 0.607$ |
| $(2,2)$ | AABB | $p = 1$ (*) |
| | ABAB | $\boldsymbol{p = 1}$ |
| | ABBA | $\boldsymbol{p = 1}$ |
| $(3,1)$ | AAAB | $p \approx 0.708$ |
| | AABA | $\boldsymbol{p = 1}$ |
| | ABAA | $\boldsymbol{p = 1}$ |
| | BAAA | $p = 1$ (*) |

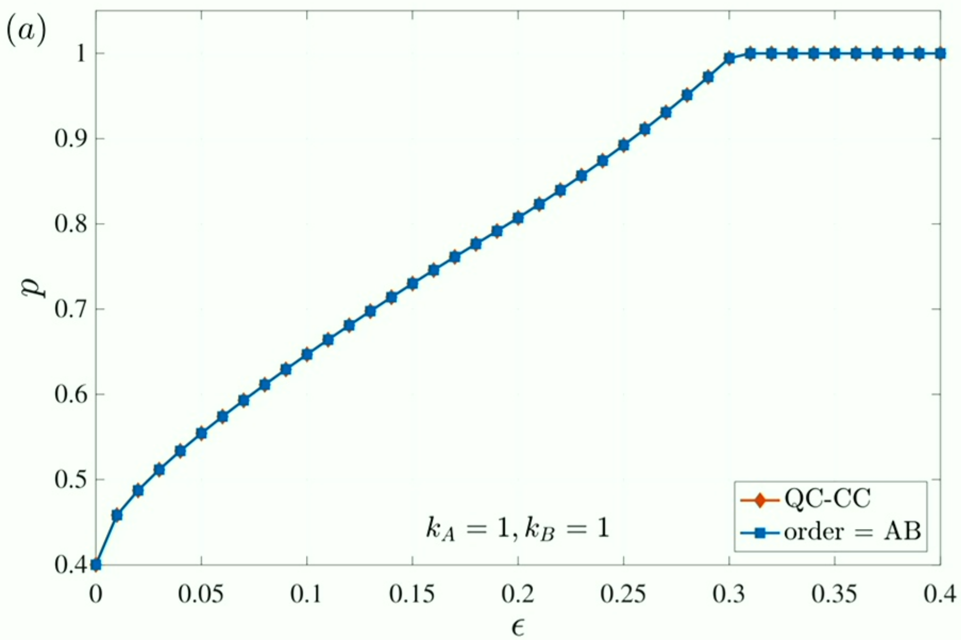
Simulation with full output:

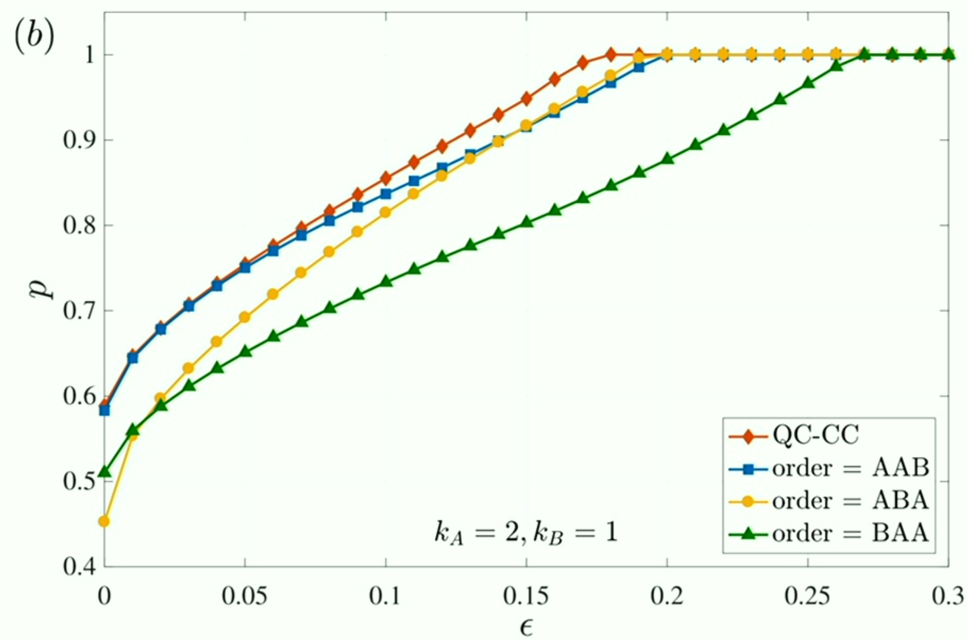
AABB: $p = 1$ (*) → $p < 1$

BAAA: $p = 1$ (*) → $p < 1$

# Main result: robustness

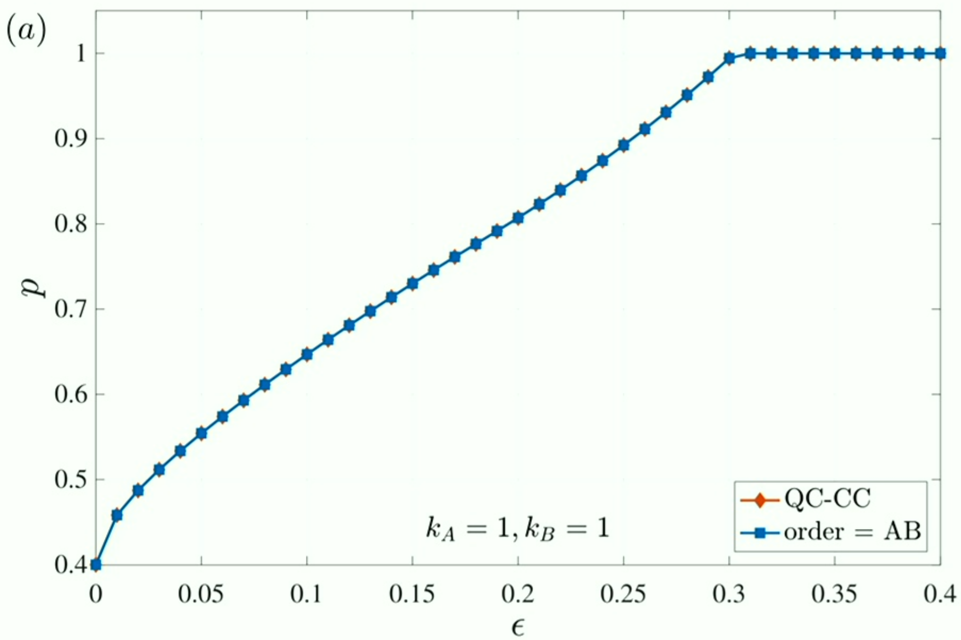
$$F(\mathcal{S}, \widetilde{\mathcal{S}}) \geq 1 - \epsilon$$

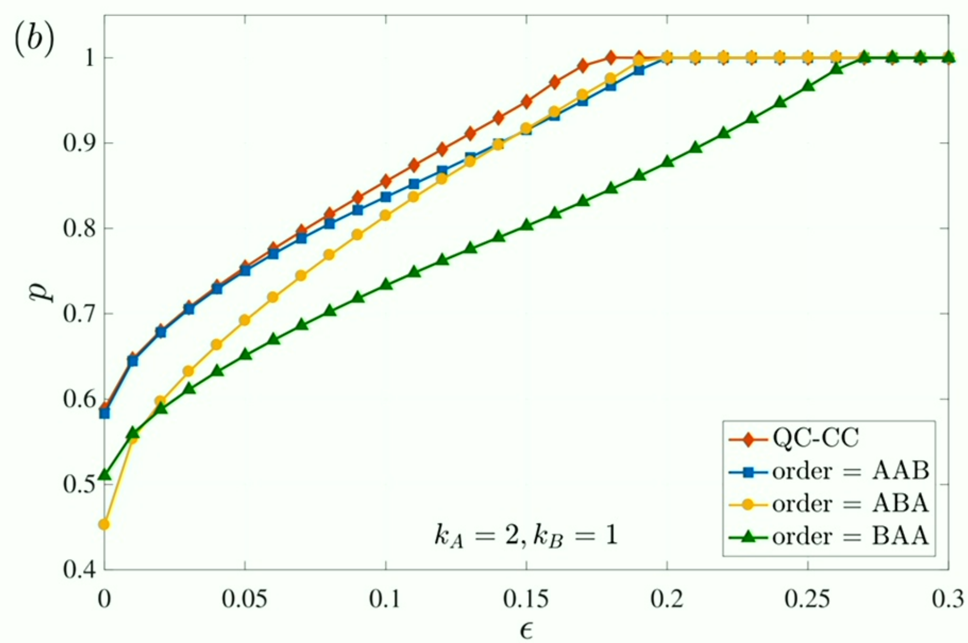


# Main result: robustness

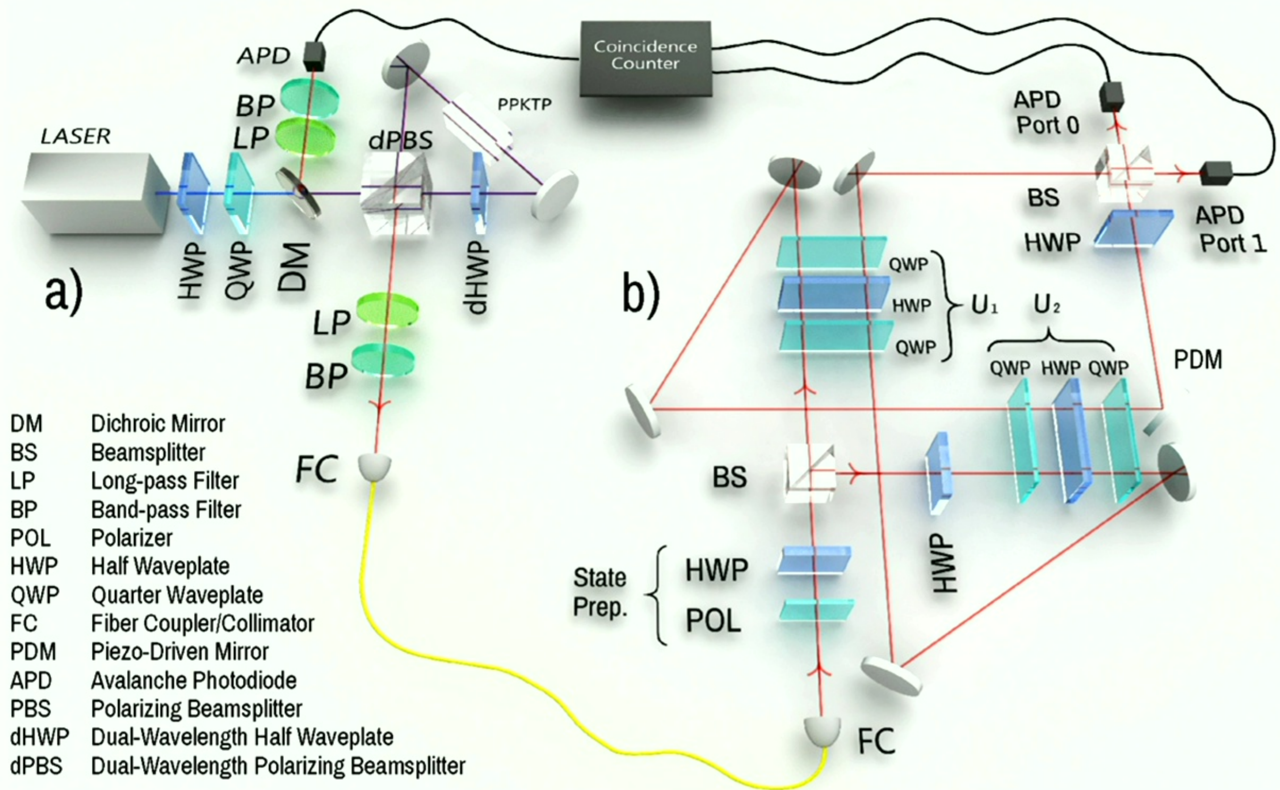
$$F(\mathcal{S}, \widetilde{\mathcal{S}}) \geq 1 - \epsilon$$




QC-CC definition: Wechs, Dourdent, Abbott, Branciard, PRX Quantum 2, 030335 (2021)

# Experimental implications




Procopio et al, Nat. Commun. 6, 7913 (2015). *Experimental superposition of orders of quantum gates*

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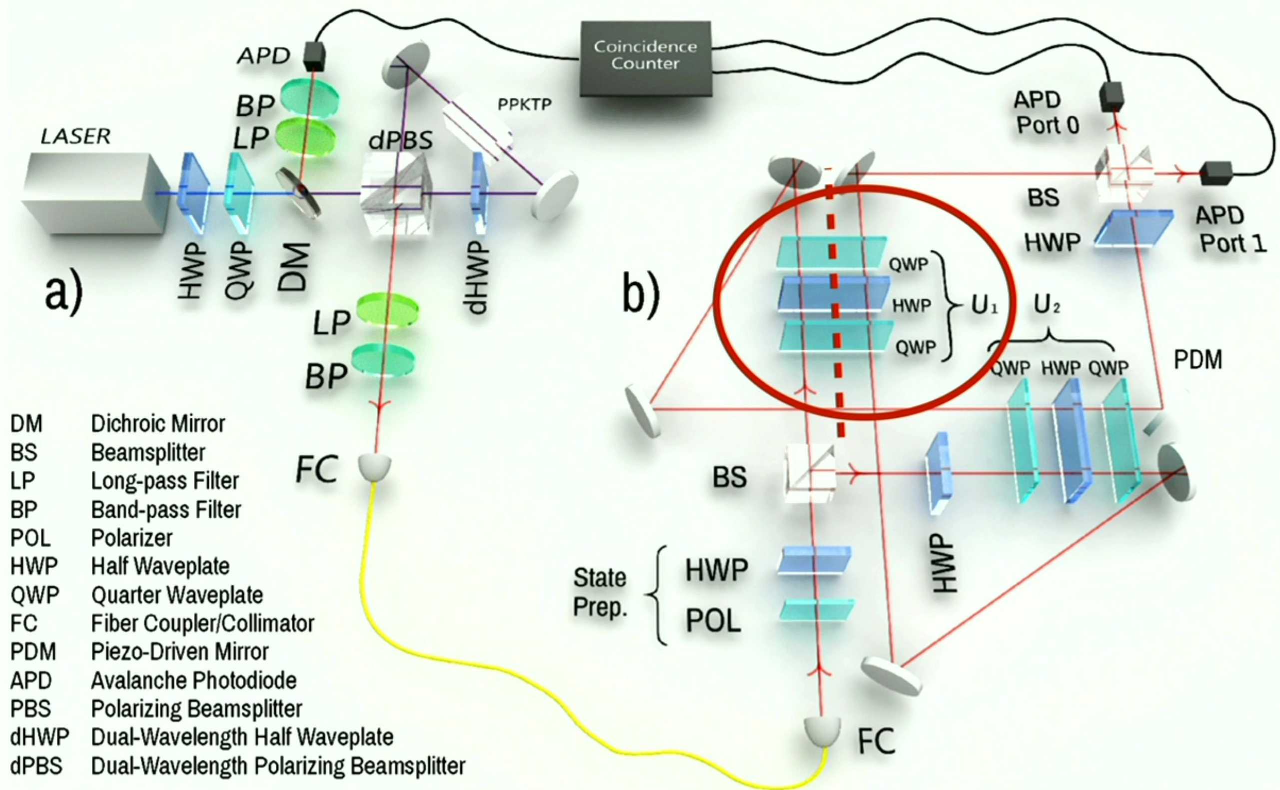


Procopio et al, Nat. Commun. 6, 7913 (2015). *Experimental superposition of orders of quantum gates*

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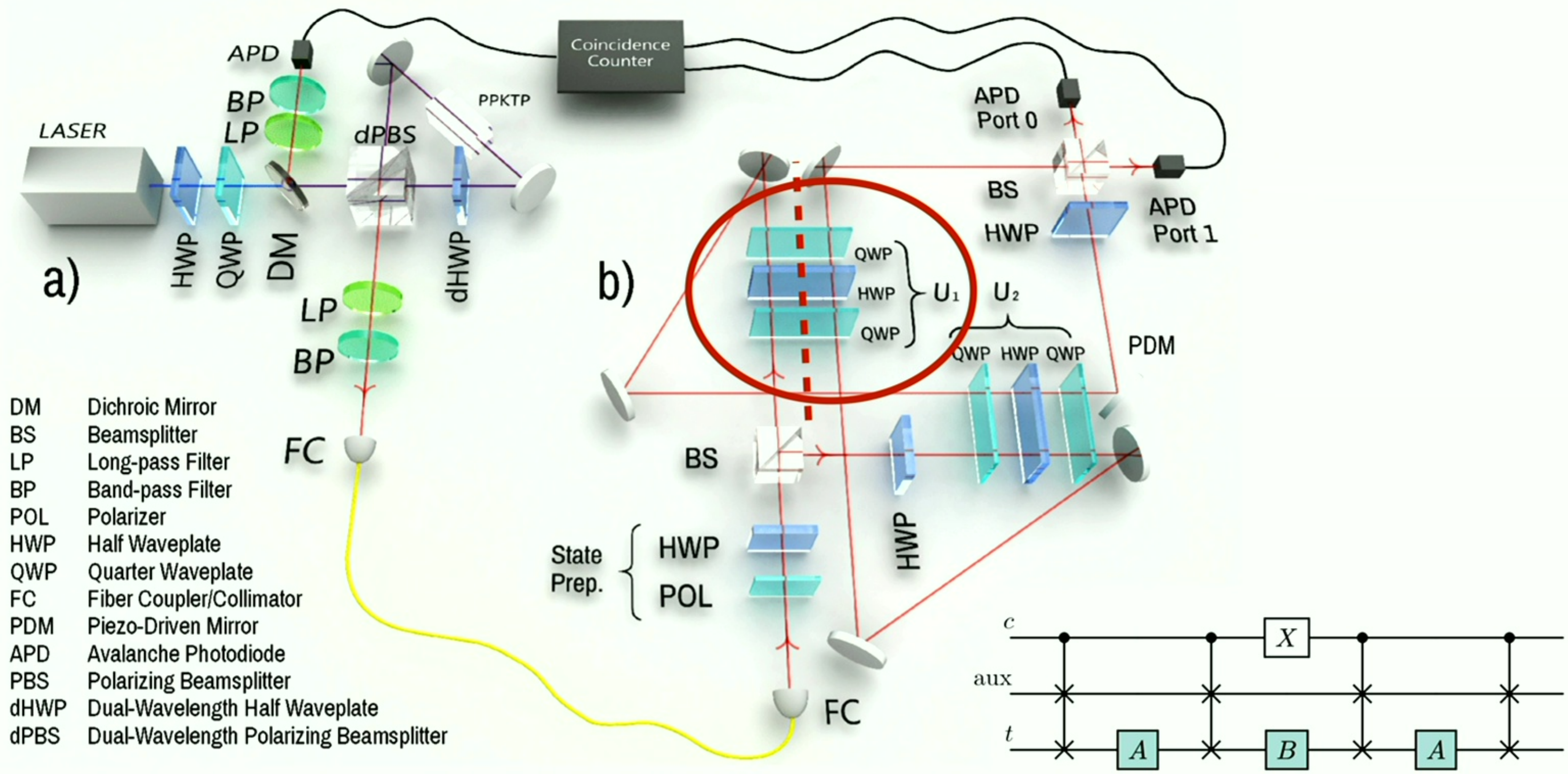
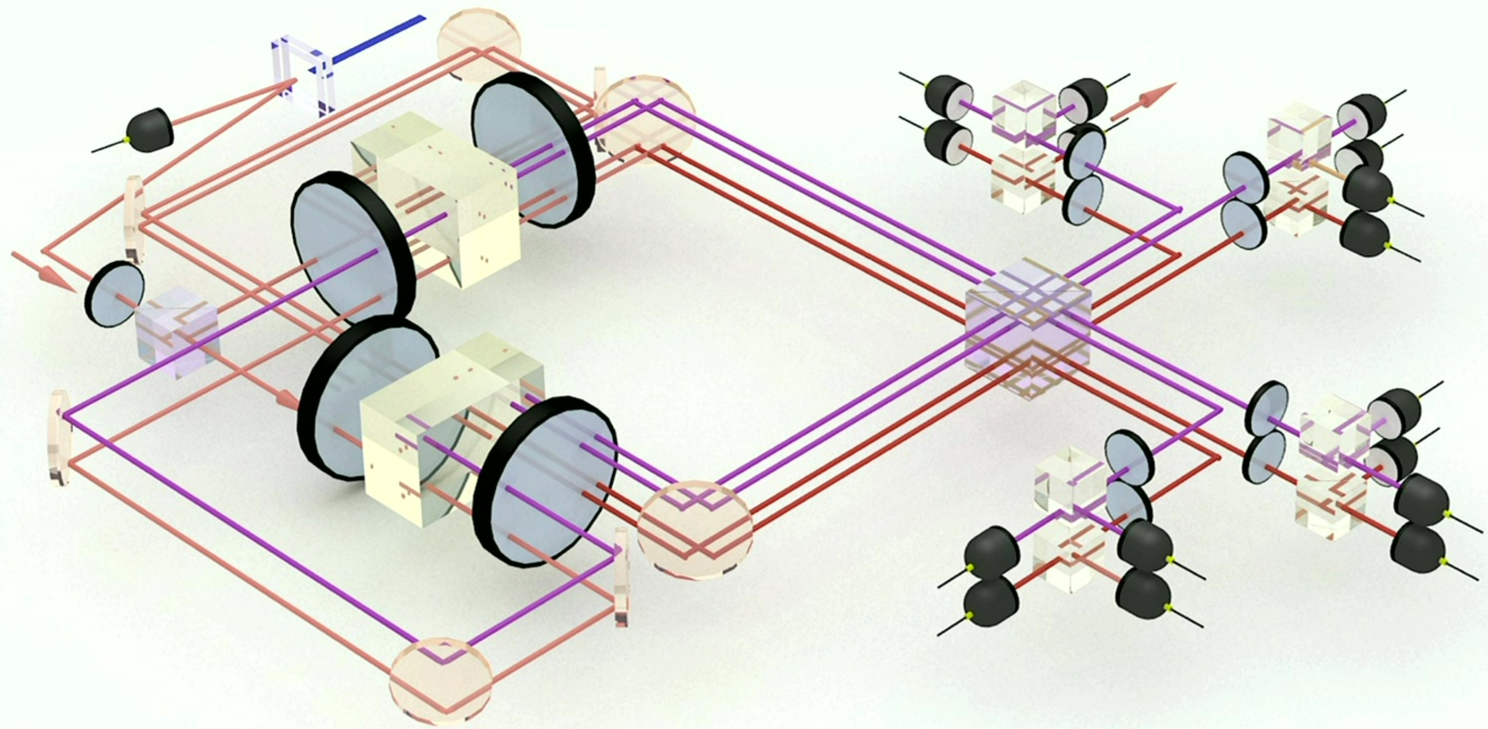


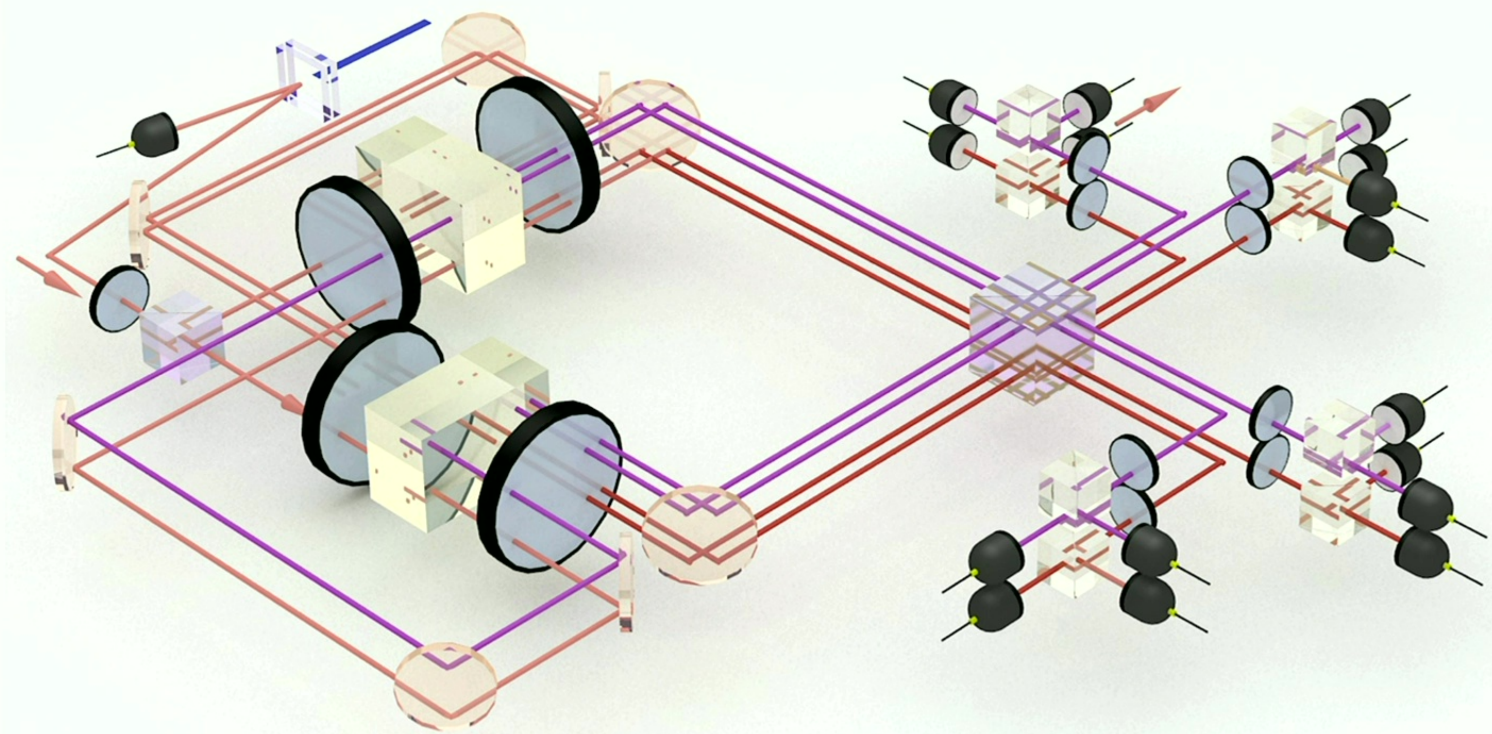

Procopio et al, Nat. Commun. 6, 7913 (2015). *Experimental superposition of orders of quantum gates*

# Experimental implications



Cao et al, Optica 10, 561 (2023), *Semi-device-independent certification of indefinite causal order in a photonic quantum switch*

# Experimental implications



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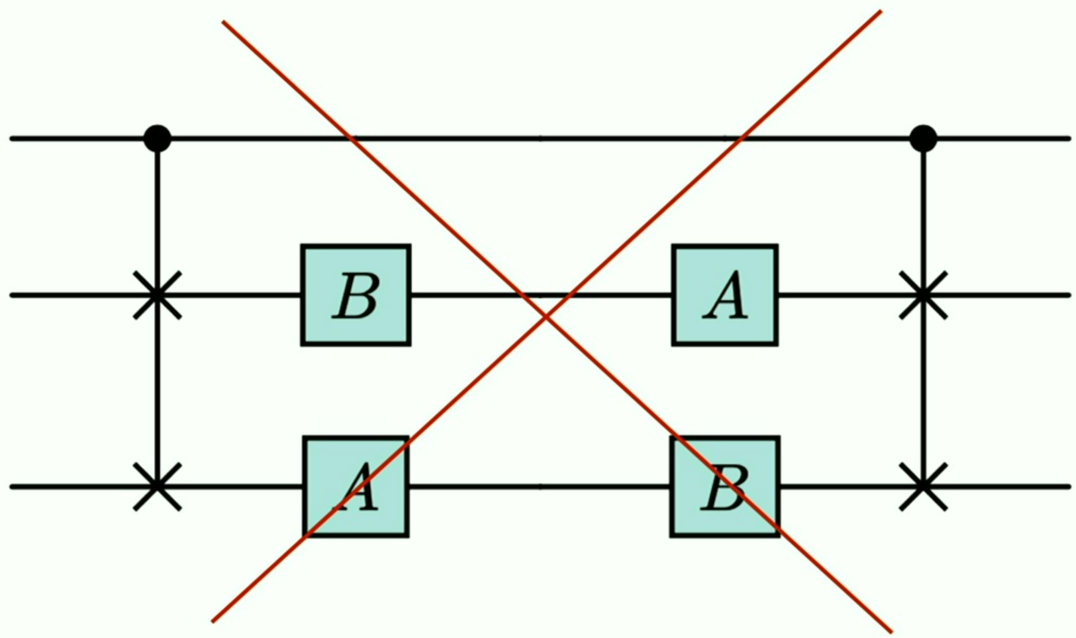
# Experimental implications



# Conclusions

**- Higher-order quantum operations**

- No-go theorem: the action of the quantum switch cannot be simulated by a quantum circuit that has access to (2,1), (2,2), or (3,1) queries of the input channels
- Robustness: probability $p < 1$ for $\epsilon > 0$

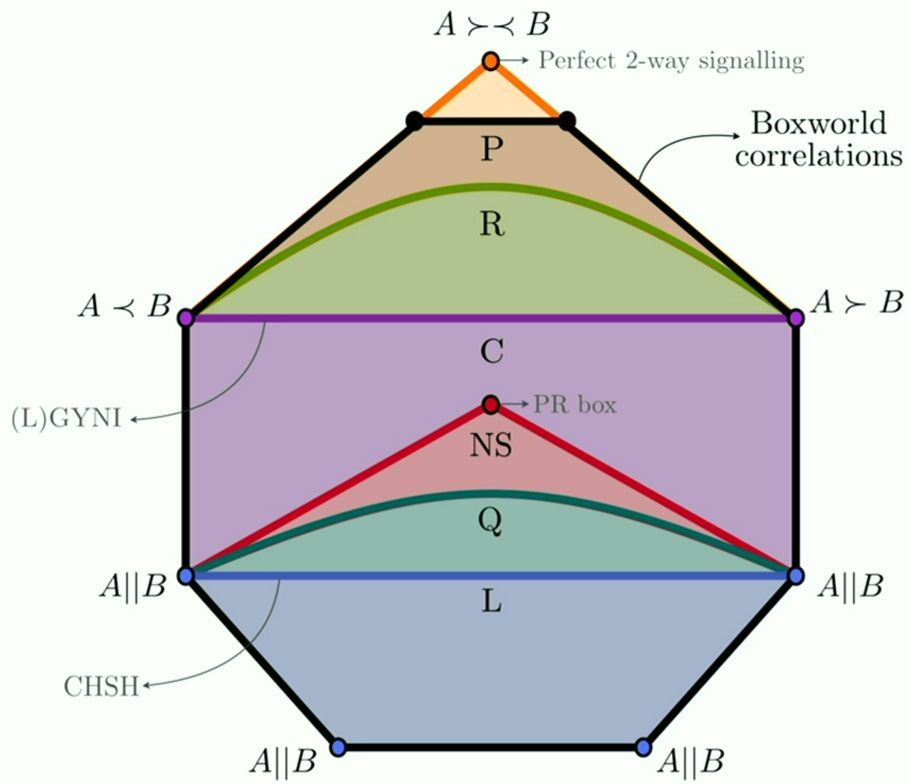
**- Open questions**

- Can the quantum switch be simulated with a finite number of queries?
- If yes, how is the scaling?
- What about other higher-order transformations with indefinite causal order?

# Advertisement

## Indefinite causal order in boxworld theories

Jessica Bavaresco, Ämin Baumeler, Yelena Guryanova, and Costantino Budroni



*Thank you!*